Instruction: Complete all problems.

1a) State Rouché's theorem and use it to show that all zeros of the polynomial

\[ p(z) = z^4 + 6z + 3 \]

lie in the circle \(|z| < 2\).

b) How many zeros of \(p(z)\) lie in the annulus \(1 < |z| < 2\)?

2) Classify the singularities in \(\mathbb{C}\) of the functions

\[ f(z) = \frac{z - \sin z}{z^4} \]

and

\[ g(z) = \frac{1}{z^2(z + 1)} + \sin\left(\frac{1}{z}\right). \]

3) Let

\[ f(z) = \frac{1}{z^2(e^z - e^{-z})}, \quad 0 < |z| < \pi. \]

Compute the first three non-zero terms of the Laurent expansion of \(f(z)\) in \(0 < |z| < \pi\).

4) Let \(f(z)\) and \(g(z)\) be entire functions satisfying

\[ |f(z)| \leq 10|g(z)| \quad \text{for all} \quad z \in \mathbb{C}. \]

Does it follow that there exists \(\lambda \in \mathbb{C}\) with

\[ f(z) = \lambda g(z) \quad \text{for all} \quad z \in \mathbb{C}? \]

Give a proof or a counterexample.

5) Let \(f(z)\) be an entire function for which the real part

\[ \text{Re} f(z + iy) = u(x, y) \]

is a bounded function. Does it follow that \(f(z)\) is a constant function? Give a proof or a counterexample.

6) Evaluate
\[ \int_0^\pi \frac{dt}{5 + 4 \cos t}. \]

7) Evaluate
\[ \int_0^\infty \frac{x \sin x}{(x^2 + 1)(x^2 + 4)} \, dx. \]

8) Let \( a \in \mathbb{C}, |a| > 1, \) and let \( f(t) \) denote the \( 2\pi \)-periodic function
\[ f(t) = \frac{1}{a + e^{it}}, \quad t \in \mathbb{R}. \]
Write \( f(t) \) as a Fourier series,
\[ f(t) = \sum_{k=-\infty}^{\infty} \tilde{f}(k)e^{ikt}. \]
Determine the Fourier coefficients.
Hint: Use the geometric sum formula.