ALGEBRA QUALIFIER EXAM

There are 10 problems. Each problem is worth 10 points.
1. Let $G$ be a subgroup of the additive group $\mathbb{R}$ of all real numbers and let $\epsilon$ be a positive real number. Assume $G$ contains no positive element less than $\epsilon$. Show that $G$ is cyclic.
2. Give an example of a Galois extensions $K/\mathbb{Q}$ whose Galois group is $\mathbb{Z}/5\mathbb{Z}$. 
3. Prove that the symmetric group $S_4$ is solvable.
4. Give an example of a ring with exactly two prime ideals.
5. Show that the free group on a set of 2 elements is not solvable.
6. Show that it is possible to embed \( \mathbb{C} \) in \( \mathbb{R} \) as an abelian group but not as a field.
7. Prove that the ideal \((2, 1 + \sqrt{-5})\) in \(\mathbb{Z}[\sqrt{-5}]\) is not principal.
8. Prove that the polynomial $t^3 + 2t + 1$ is irreducible in $\mathbb{Q}[t]$. Compute its Galois group.
9. Prove that if $\mathbb{Q}(t)$ is the field of rational functions in the variable $t$ then the field extension $\mathbb{Q}(t^3 + 1) \subset \mathbb{Q}(t)$ is finite. Compute its degree.
10. Prove that there exist infinitely many field automorphisms \( \phi : \mathbb{C} \rightarrow \mathbb{C} \) of order two.