Algebra Qualifying Exam

January 2006

Do the following 7 problems. Show all your work and explain all steps in a proof or derivation.

1. Let $Q = \{ \pm 1, \pm i, \pm j, \pm k \}$ be the 8-element group generated by quaternionic units with the usual quaternionic relations:

   \[ i^2 = j^2 = k^2 = -1, \quad ij = -ji = k. \]

Let $D_4$ be the 8-element dihedral group generated by $a, b$ with relations

\[ a^4 = 1, \quad a^k \neq 1 \text{ if } 0 < k < 4, \quad b^2 = 1, \quad ba = a^{-1}b. \]

Is $D_4$ isomorphic to $Q$? Prove your answer. (10 pts)

2. Consider the system of equations

\[ x + y + z = 0, \quad x + 3y + 4z = 0. \]

Show that the integer solutions of this system form a group isomorphic to $\mathbb{Z}$. (5 pts)

3. Determine all Abelian groups of order $36$ up to isomorphism. (15 pts)

   a) Give the decomposition of each group in terms of invariant factors $m_1, \ldots, m_t$ satisfying $m_1 | m_2 | \cdots | m_t$ as

   \[ G = \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_t}. \]

   b) Give the decomposition of each group in terms of elementary divisors $p_1^{s_1}, \ldots, p_r^{s_r}$ with $p_i$ prime, as

   \[ G = \mathbb{Z}_{p_1^{s_1}} \oplus \cdots \oplus \mathbb{Z}_{p_r^{s_r}}. \]

   c) Give the isomorphism between the groups listed in a) with those listed in b).

4. Prove that any group of order 18 is solvable. (10 pts)

5. Let $V$ be a real finite-dimensional vector space with a positive definite inner product $\langle \cdot, \cdot \rangle$. Let $L : V \to \mathbb{R}$ be a linear functional on $V$. Show that (10 pts)

   \[ \exists \bar{h} \in V \text{ such that } L(\bar{x}) - \langle \bar{x}, \bar{h} \rangle, \quad \bar{x} \in V. \]

6. Let $R$ be a commutative ring with unity. Show that an element in $R$ is nilpotent if and only if it belongs to every prime ideal of $R$. (10 pts)

7. Let $E$ be a splitting field of the polynomial $x^5 - 2$ over the rationals $\mathbb{Q}$. Find the Galois group of $E/\mathbb{Q}$. (10 pts)