Algebra Qualifying Exam

August 2005

Do the following 8 problems. Show all your work and explain all steps in a proof or derivation.

1. Let $p$ be a prime and let $G$ be a group with order $G = p^n$. Prove that the center of $G$ is non-trivial, i.e., prove that there is an element $z \in G$ with $z \neq e$ and such that $gz = zg$ for all $g \in G$.

2. Let $R$ be a ring and $I \subset R$ an ideal. Suppose that $a = b \text{ mod } I$ and $c = d \text{ mod } I$.

   (i) Show that $a + c \equiv b + d \text{ mod } I$.

   (ii) Show that $ac - bd \text{ mod } I$.

3. Show that the matrices

   $$
   \begin{pmatrix}
   1 & 1 & 0 \\
   0 & 1 & 0 \\
   0 & 1 & 1
   \end{pmatrix}
   \quad \text{and} \quad
   \begin{pmatrix}
   1 & 1 & 0 \\
   0 & 1 & 0 \\
   0 & 0 & 1
   \end{pmatrix}
   $$

   are similar over the rationals $\mathbb{Q}$.

4. Let $0 \longrightarrow A \overset{f}{\longrightarrow} B \overset{g}{\longrightarrow} C \longrightarrow 0$ be an exact sequence of modules over a commutative ring $R$. Show that if $C$ is a free $R$ module, then the exact sequence splits.

5. Let $A, B$ be any two endomorphisms of a vector space $V$ over $\mathbb{R}$, such that

   $$A \circ B - B \circ A = Id,$$

   where $Id$ is the identity endomorphism. Show that $V$ is infinite dimensional.

6. Show that no group of order 48 is simple.

7. Consider the ring $\mathbb{Z}[\sqrt{-5}]$.

   (i) Find all the units in $\mathbb{Z}[\sqrt{-5}]$.

   (ii) Show that $\mathbb{Z}[\sqrt{-5}]$ is an integral domain but not a Unique Factorization Domain (UFD).

8. Let $\mathbb{Z}$ be the ring of integers and $\mathbb{Q}$ the field of rational numbers. Prove that

   $$\left( \mathbb{Z}/7\mathbb{Z}, \otimes_{\mathbb{Z}} \mathbb{Q} - 0 \right).$$