

# Algebra Qualifying Exam

August 2005

Do the following 8 problems. Show all your work and explain all steps in a proof or derivation.

1. Let  $p$  be a prime and let  $G$  be a group with order  $|G| = p^n$ . Prove that the center of  $G$  is non-trivial, i.e. prove that there is an element  $z \in G$  with  $z \neq e$  and such that  $gz = zg$  for all  $g \in G$ .

2. Let  $R$  be a ring and  $I \subset R$  an ideal. Suppose that  $a \equiv b \pmod{I}$  and  $c \equiv d \pmod{I}$ .

(i) Show that  $a + c \equiv (b + d) \pmod{I}$ .

(ii) Show that  $ac \equiv bd \pmod{I}$ .

3. Show that the matrices

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are similar over the rationals  $\mathbb{Q}$ .

4. Let  $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$  be an exact sequence of modules over a commutative ring  $R$ . Show that if  $C$  is a free  $R$  module, then the exact sequence splits.

5. Let  $A, B$  be any two endomorphisms of a vector space  $V$  over  $\mathbb{R}$ , such that

$$A \circ B - B \circ A = Id,$$

where  $Id$  is the identity endomorphism. Show that  $V$  is infinite dimensional.

6. Show that no group of order 48 is simple.

7. Consider the ring  $\mathbb{Z}[\sqrt{-5}]$ .

(i) Find all the units in  $\mathbb{Z}[\sqrt{-5}]$ .

(ii) Show that  $\mathbb{Z}[\sqrt{-5}]$  is an integral domain but not a Unique Factorization Domain (UFD).

8. Let  $\mathbb{Z}$  be the ring of integers and  $\mathbb{Q}$  the field of rational numbers. Prove that

$$(\mathbb{Z}/7\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} = 0.$$