QUALIFYING EXAM - ALGEBRA 2005

1. Let G be a group of order pq where p and q are primes. Let n be an integer such that $p < n < q$. Show that any action of G on a set with n elements has a fixed point. Show that there always exists an action of G on a set of p elements having no fixed point.

2. Let F be a free group on 2 generators and let $[F, F]$ be its commutator subgroup. Show that $F/[F, F] \simeq \mathbf{Z} \times \mathbf{Z}$.

3. Compute the center of the diehedral group D_{2n} where n is even.

4. Show that if M and N are two distinct maximal ideals in a commutative ring R then $M^3 + N^3 = R$.

5. Prove the the ring $\mathbf{Z}\left[\frac{-1+\sqrt{-3}}{2}\right]$ is Euclidean. Prove that the ring $\mathbf{Z}[\sqrt{-3}]$ is not principal.

6. Prove that an ideal in an integral domain R is a free $R-$module if and only if it is principal.

7. Prove that any field extension of degree 2 is normal.

8. Let $K \subset L$ be a field extension and let $\alpha, \beta \in L$ be two elements of degrees m and n over K. Assume m and n are coprime. Prove that $K(\alpha, \beta)$ has degree mn over K.

9. Show that the only automorphism of the field $\mathbf{R}$ of real numbers is the identity. Prove that the complex field has uncountably many automorphisms.

10. Show that the Galois group of $x^5 - 7$ over $\mathbf{Q}$ is non-abelian.