ALGEBRA QUALIFIER EXAM

There are 10 problems. Each problem is worth 10 points.
1. Let $G$ be a subgroup of the additive group $\mathbb{R}$ of all real numbers. Assume $G$ contains no positive element less than $10^{-2002}$. 
2. Let $G$ and $H$ be finite groups of coprime orders. Prove that any homomorphism from $G$ to $H$ is trivial.
3. Prove that the symmetric group $S_4$ is solvable.
4. Compute the center of the group of all invertible $n \times n$ matrices with real coefficients.
5. Show that the group defined by generators $a, b$ and relations $a^2 = b^3 = e$, $ab = b^2a$ has 6 elements.
6. Prove that if $M$ is a maximal ideal in a ring $R$ then $R/M$ is a field.
7. Prove that if $R_1$ and $R_2$ are two rings and $P$ is a prime ideal in $R_1 \times R_2$ then either $P = P_1 \times R_2$ or $P = R_1 \times P_2$, where $P_i$ is a prime ideal in $R_i$. 
8. Prove that $\mathbb{Z}[\sqrt{-1}]$ is a unique factorisation domain.
9. Prove that if \( p \) is a prime number of the form \( p = 4k + 1 \) then the polynomial \( t^2 + 1 \) is reducible in \( \mathbf{F}_p[t] \).
10. Prove that the polynomial $t^4 + t + 1$ is irreducible in $\mathbb{Q}[t]$. 