

# ALGEBRA QUALIFIER EXAM

There are 10 problems. Each problem is worth 10 points.

1. Let  $G$  be a subgroup of the additive group  $\mathbf{R}$  of all real numbers. Assume  $G$  contains no positive element less than  $10^{-2002}$

2. Let  $G$  and  $H$  be finite groups of coprime orders. Prove that any homomorphism from  $G$  to  $H$  is trivial.

3. Prove that the symmetric group  $S_4$  is solvable.

4. Compute the center of the group of all invertible  $n \times n$  matrices with real coefficients.

5. Show that the group defined by generators  $a, b$  and relations  $a^2 = b^3 = e, ab = b^2a$  has 6 elements.

6. Prove that if  $M$  is a maximal ideal in a ring  $R$  then  $R/M$  is a field.

7. Prove that if  $R_1$  and  $R_2$  are two rings and  $P$  is a prime ideal in  $R_1 \times R_2$  then either  $P = P_1 \times R_2$  or  $P = R_1 \times P_2$ , where  $P_i$  is a prime ideal in  $R_i$ .

8. Prove that  $\mathbf{Z}[\sqrt{-1}]$  is a unique factorisation domain.

9. Prove that if  $p$  is a prime number of the form  $p = 4k + 1$  then the polynomial  $t^2 + 1$  is reducible in  $\mathbf{F}_p[t]$ .

10. Prove that the polynomial  $t^4 + t + 1$  is irreducible in  $\mathbf{Q}[t]$ .