

# Algebra Qualifying Exam

August 1999

Do the following 8 problems. Show all your work and explain all steps in a proof or derivation.

1. Let  $p$  be a prime and let  $G$  be a group with order  $|G| = p^n$ . Prove that the center of  $G$  is non-trivial, i.e. prove that there is an element  $z \in G$  with  $z \neq e$  and such that  $gz = zg$  for all  $g \in G$ .

2. Let  $R$  be a commutative ring with multiplicative identity 1. Show that  $R$  satisfies the ascending chain condition on ideals (i.e. whenever  $I_1 \subset I_2 \subset \dots$  is a nested sequence of ideals in  $R$ , there is an  $n$  such that  $I_n = I_{n+1} = \dots$ ) if and only if every ideal is finitely generated.

3. Prove that the polynomial ring  $\mathbf{Z}[x]$  with integer coefficients is not a principal ideal domain.

4. Show that the matrices

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are similar over the rationals  $\mathbf{Q}$ .

5. Let  $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$  be an exact sequence of modules over a commutative ring  $R$ . Show that for any  $R$ -module  $D$ , the induced sequence

$$0 \rightarrow \text{Hom}(D, A) \xrightarrow{f_*} \text{Hom}(D, B) \xrightarrow{g_*} \text{Hom}(D, C)$$

is exact.

6. Show that every group  $G$  of order 56 has a nontrivial normal subgroup.

7. Let  $F_q$  denote the finite field with  $q$  elements, and let  $f(x) \in F_q[x]$  be irreducible. Show that  $f(x)$  divides  $x^{q^n} - x$  if and only if the degree of  $f$  divides  $n$ .

8. Let  $\zeta$  be a primitive  $n$ th root of unity in the complex number field  $\mathbf{C}$ . Show that  $\mathbf{Q}(\zeta + \zeta^{-1})$  is Galois over  $\mathbf{Q}$ . Hint: First show by induction that if  $\sigma$  is an automorphism of  $\mathbf{Q}(\zeta)$  leaving  $\mathbf{Q}(\zeta + \zeta^{-1})$  fixed then for any integer  $k$  we have  $\sigma(\zeta^k + \zeta^{-k}) = \zeta^k + \zeta^{-k}$ .