Regularizations of Singular Integral Operators (joint work with C. Liaw)

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Hilbert and Cauchy Transforms:

- Hilbert transform:

$$Tf(x) = (\pi)^{-1} \int_{\mathbb{R}} \frac{f(y)}{x-y} dy$$

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- The case with μ on $\mathbb{C}$ (Cauchy Transform) appears in problems related to analytic capacity. The measure μ is "one-dimensional"

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and one is interested in boundedness of such operators in $L^2(\mu)$.
- But sometimes we are interested in boundedness $L^2(\mu) \to L^2(\nu)$, and there is no apriori restrictions on μ and ν.

An example:

- Let $A := M_t$ in $L^2(\mu)$, and let $A_\alpha = A + \alpha(\,\cdot\,, b)b$, where b is cyclic for A.
  WLOG can assume that $b \equiv 1$.
- By the spectral theorem A_α unitarily equivalent to M_s in $L^2(\mu_\alpha)$.

$$M_s U_\alpha = U_\alpha A_\alpha, \qquad U_\alpha : L^2(\mu) \to L^2(\mu_\alpha) \text{ unitary.}$$

$U_\alpha b$ is cyclic, so WLOG can assume $U_\alpha b = U_\alpha \mathbf{1} = \mathbf{1}$.

Informal reasoning gives us:

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Indeed,

$$\begin{aligned} sK(s,t) &= U_\alpha M_t + \alpha(\,\cdot\,, b)U_\alpha b \\ &= K(s,t)t + \alpha\mathbf{1} \end{aligned}$$

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Theorem (C. Liaw, S.Treil, 2009)

$$U_\alpha f(s) = f(s) - \alpha \int_{\mathbb{R}} \frac{f(t) - f(s)}{t - s} d\mu(t), \qquad \forall f \in C^1_{\mathrm{c}}.$$

Rigidity theorem

Theorem (C. Liaw–S. Treil, 2009)

Let an operator V defined on C_c^1 by

$$Vf(s) = f(s) - \alpha \int_{\mathbb{R}} \frac{f(t) - f(s)}{t - s} d\mu(s)$$

extends to a bounded operator $L^2(\mu) \to L^2(\nu)$, and let $\ker V = \{0\}$. Then there exists $h \geq 0$ such that $1/h \in L^\infty(\nu)$, and $M_h V$ is a unitary operator from $L^2(\mu) \to L^2(\nu)$.

Moreover, the unitary operator $U := M_h V$ gives the spectral representation of the operator $A_\alpha := M_t + \alpha(\,\cdot\,, b)b$, $b \equiv \mathbf{1}$, in $L^2(\mu)$, namely $UA_\alpha = M_s U$, where M_s is the multiplication by the independent variable s in $L^2(\nu)$.
In other words, all operators of such type appear from rank one perturbations.

Calderón–Zygmund operators

A *Calderón–Zygmund operator* of dimension d in $\mathbb{R}^N$, $d \leq N$, is an integral operator, bounded in L^2 and with kernel K satisfying the following growth and smoothness conditions

1. $|K(x,y)| \leq \dfrac{C_{\mathrm{cz}}}{|x-y|^d}$ for all $x, y \in \mathbb{R}^N$, $x \neq y$.
2. There exists $\alpha > 0$ such that

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq C_{\mathrm{cz}} \frac{|x-x'|^\alpha}{|x-y|^{d+\alpha}}$$

for all $x, x', y \in \mathbb{R}^N$ such that $|x - x'| < |x-y|/2$.

Has singularity at the diagonal $x = y$. The measure is at best d-dimensional, $\mu(B_{x,r}) \leq Cr^d$, so integral is not well defined.

How to interpret?

Interpreting CZO: regularization

- Let m be a function in $\mathbb{R}^N$, $m \equiv 0$ in a neighborhood of 0, and $m(x) \equiv 1$ in a neighborhood of ∞.
- Kernels $K_\varepsilon(x,y) = K(x-y)m((x-y)/\varepsilon)$ do not have singularity near the diagonal $x = y$, so the integral

$$T_\varepsilon f(x) = \int K_\varepsilon(x,y) f(y) d\mu(y)$$

  is well defined for $f \in L^\infty_{\mathrm{c}}$.
- We say that T is bounded if T_ε are uniformly (in ε bounded).
- If $m(x) = \mathbf{1}_{[1,\infty)}(|x|)$ we get the classical truncation. Smooth truncations are also used (and are more natural from my point of view).

Does boundedness depend on regularization?

Interpreting CZO: axiomatic approach

- We assume that bilinear form $\langle Tf, g\rangle$ is defined on a dense set; in classical situations when one works with Lebesgue measure in $\mathbb{R}^N$ or on a smooth manifold it is often $C_{\mathrm{c}}^{\infty} \times C_{\mathrm{c}}^{\infty}$.
- The fact that T is an integral operators means simply that
$$\langle Tf, g\rangle = \iint K(x,y) f(y) g(x) d\mu(y) d\nu(x)$$
for compactly supported f and g with separated supports.
- A multiplication operator M_φ, $M_\varphi f = \varphi f$ is an operator with kernel $K(x,y) \equiv 0$.
- If $\mu = \nu$, $\mu(B_{x,r}) \le Cr^d$, the generalization of Cotlar inequality implies that the truncations T_ε are uniformly bounded for all reasonable truncating functions m.

Nothing was known in the two-weight case!

Setup

- Radon measures μ and ν in $\mathbb{R}^n$ are fixed.
- μ and ν do not have common atoms; no other restriction is assumed.
- A singular kernel $K(x,y)$ is a function in $L^2_{\mathrm{loc}}(\mu \times \nu)$ off the diagonal $x = y$ (can blow-up at the diagonal).

Definition (Restricted boundedness)

A kernel K is called L^p restrictedly bounded if

$$|\langle Tf, g\rangle| = \left| \iint K(x,y) f(y) g(x) d\mu(y) d\nu(x) \right| \leq C \|f\|_{L^p(\mu)} \|g\|_{L^{p'}(\nu)}$$

for all $f, g \in L^\infty_c$, $\operatorname{supp} f \cap \operatorname{supp} g = \varnothing$.

Here $1/p + 1/p' = 1$.
The best constant C is called the restricted norm .

Main result

Recall:

- m is a cut-off function, $m \equiv 0$ in a neighborhood of 0 and $m \equiv 1$ in a neighborhood of ∞.
- T_ε are "truncated operators" with kernels K_ε, $K_\varepsilon(x,y) = K(x,y)m((x-y)/\varepsilon)$;
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Theorem (C. Liaw, S. Treil)

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Let K be an L^p restrictedly bounded singular kernel, and let $m \in C^\infty$. Then the resularized operators T_ε are uniformly (in ε) bounded.

In fact it is sufficient to assume that $1-m$ is in the Sobolev space H^s, $s > N/2$.

Recall: $f \in H^s \iff \int_{\mathbb{R}^N} (1+|x|)^{2s} |\widehat{f}(x)|^2 dx < \infty$.

Classical truncations

For interesting operators the result holds with classical truncations ($m(x) = \mathbf{1}_{[1,\infty)}(|x|)$) as well!

Interesting operator: convolution kernel, $K(x,y) = \widetilde{K}(y-x)$,

$$\widetilde{K}(x) = A(|x|)B(x/|x|),$$

where $A(r) \geq 0$ for all $r > 0$ and B is a function (with values in some $\mathbb{R}^m$) in the Sobolev space H^k, $k > N/2$ on the unit sphere $\mathbb{S}_{N-1}$ in $\mathbb{R}^N$.

Examples

- $\widetilde{K}(z) = 1/z$, $z \in \mathbb{C}$ — Cauchy Transform.
- $\widetilde{K}(z) = 1/z^2$, $z \in \mathbb{C}$ — Beurling–Ahlfors Transform;
- $\widetilde{K}(x) = x/|x|^{d+1}$, $x \in \mathbb{R}^N$ — Riesz Transform of order d in R^N.

Recall: Riesz Transform of order d is convolution integral operator with kernel $K(x,y) = (x-y)/|x-y|^{d+1}$.

Theorem (C. Liaw, S. Treil)

For Riesz Transform of order d the restricted L^p boundedness, $1 < p < \infty$, implies the following generalized two-weight Muckenhoupt A_p^d condition of order d;

$$\sup_B (\operatorname{diam} B)^{-d} \mu(B)^{1/p'} \nu(B)^{1/p} < \infty;$$

here the supremum is taken over all balls in $\mathbb{R}^N$.

In particular, if $\mu = \nu$ then μ is d-dimensional, $\mu(B_{x,r}) \le Cr^d$.
It was known before, but the proofs were ugly.

If $d = N$ then $\mu_{\mathrm{s}} \perp \nu_{\mathrm{s}}$.

A simple idea: Schur multipliers

- Replacing $K(x,y)$ by $K(x,y)e^{-iax}e^{iay}$, $a \in \mathbb{R}$ does not change its restricted norm.
- So if $\rho \in L^1(\mathbb{R}^N)$, then for the kernel

$$\int_{\mathbb{R}} K(x,y)e^{-iax}e^{iay}\rho(a)da = K(x,y)\widehat{\rho}(x-y)$$

  its restricted norm increases at most $\|\rho\|_1$ times.
- Functions m in the Wiener Algebra $W = \mathcal{F}(L^1) + c$ are Schur multipliers (multipliers with respect to restricted norm).
- If $m = \widehat{\rho}$, $\rho \in L^1$, then $m(s/\varepsilon) = \mathcal{F}\{x \mapsto \varepsilon^N \rho(\varepsilon x)\}(s)$, and

$$\int_{\mathbb{R}^N} |\varepsilon^N \rho(\varepsilon x)|dx = \int_{\mathbb{R}^N} |\rho(x)|dx = \|\rho\|_1.$$

  so the Schur multipliers $m((x-y)/\varepsilon)$ are uniformly (in ε) bounded.

Schur multipliers, continued

- If $\varphi \in H^k$, $k > N/2$, then $\varphi \in \mathcal{F}(L^1)$ (easy exercise in Cauchy–Schwartz).
  Note that the rescaling $\varphi \mapsto \varphi_\varepsilon$, $m_\varepsilon(x) = m(x/\varepsilon)$ does change Sobolev norm, but the norm in the Wiener Algebra W is preserved.
- So, if $1 - m \in C_c^\infty \subset W^k$, $k > N/2$, then the kernels K_ε,

$$K_\varepsilon(x, y) m((x - y)/\varepsilon)$$

  are uniformly (in ε) restrictedly bounded.

Schur multipliers: an example

- Let $\rho(x) = e^{-x}\mathbf{1}_{[0,\infty)}$, and let

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- Then $m(s/\varepsilon) = \dfrac{s}{s - i\varepsilon}$.
- Then for $K(x,y) = 1/(x-y)$ we have

$$K_\varepsilon(x,y) = \frac{1}{x-y}\frac{x-y}{x-y-i\varepsilon} = \frac{1}{x+i\varepsilon-y}$$

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- We used Schur multipliers before in complex analysis, probably without noticing it!

From restricted boundedness to boundedness

Proposition

Let $K \in L^2_{\mathrm{loc}}(\mu \times \nu)$ (everywhere, not just off the diagonal $x = y$), and let K be restrictedly bounded with restricted norm C. Then the integral operator T with kernel K is bounded, $\|T\| \leq 2C$.

Note that T is well defined on $f \in L^\infty_{\mathrm{c}}$.
Idea of the proof:

- Restricting everything to a compact set assume WLOG that T is compact $L^2(\mu) \to L^2(\nu)$. Take f and g.
- Considering f only on "almost half" of the support and g on "almost the other half", we get the sequences f_n, g_n with separated supports such that $f_n \to \frac{1}{2}f$, $g_n \to \frac{1}{2}g$ weakly.
- $\langle Tf_n, g_n \rangle$ can be estimated because of restricted boundedness, and $\langle Tf_n, g_n \rangle \to \frac{1}{4}\langle Tf, g \rangle$

Lemma (One weight version)

Let σ be a Radon measure without atoms in $\mathbb{R}^N$. There exist Borel sets E_n^1, E_n^2, $n \in \mathbb{N}$ such that $\operatorname{dist}(E_n^1, E_n^2) > 0$ $\forall n$.

1. *The operators P_n^k, $P_n^k f := \mathbf{1}_{E_n^k} f$, $k = 1, 2$ converge to $\frac{1}{2} I$ in weak operator topology in $L^2(\sigma)$.*
2. *For any $p \in [1, \infty)$ and for $k = 1, 2$*

$$\lim_{n \to \infty} \|\mathbf{1}_{E_n^k} f\|_{L^p(\sigma)} = 2^{-1/p} \|f\|_{L^p(\sigma)}, \qquad \forall f \in L^p(\sigma).$$

- Trivially to do for Lebesgue measure
- More work needed for general case.

One can get a two weigh version from this Lemma

Lemma (Two weight version)

Let μ and ν be Radon measures in $\mathbb{R}^N$ without common atoms. here exist Borel sets E_n^1, E_n^2, $n \in \mathbb{N}$ such that $\operatorname{dist}(E_n^1, E_n^2) > 0$ $\forall n$.

1. *The operators P_n^1 and P_n^2, given by $P_n^1 f := \mathbf{1}_{E_n^1}\left(f_{\mu_c} + \frac{1}{2} f_{\mu_a}\right)$, $P_n^2 g := \mathbf{1}_{E_n^2}\left(g_{\nu_c} + \frac{1}{2} g_{\nu_a}\right)$ converge to $\frac{1}{2} I$ in weak operator topology of $L^2(\mu)$ and $L^2(\nu)$ respectively.*
2. *For any $p \in [1, \infty)$ and for any $f \in L^p(\mu)$, $g \in L^p(\nu)$*

$$\lim_{n\to\infty} \|\mathbf{1}_{E_n^1} f\|_{L^p(\mu)} \le 2^{-1/p} \|f\|_{L^p(\mu)},$$
$$\lim_{n\to\infty} \|\mathbf{1}_{E_n^2} g\|_{L^p(\nu)} \le 2^{-1/p} \|g\|_{L^p(\nu)}.$$

- If μ and ν do not have atoms, apply one weight lemma to $\sigma = \mu + \nu$: the sets $E_n^{1,2}$ from this lemma would do a trick.
- Atoms: add one atom to each set at each step, and remove small neighborhoods of added atoms from the other set.

Proof of the one-weight lemma

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Proof of the one-weight lemma

- Since P_n are always contractions, sufficiently to check everything on dense sets.
- Sufficient to check everything only on $\mathbf{1}_Q$ for standard dyadic cubes in $\mathbb{R}^N$, i.e. on cubes of form $Q = 2^k([0,1)^N + \mathbf{j})$, $k \in \mathbb{Z}$, $\mathbf{j} \in \mathbb{Z}^N$.
- Sufficient to show that given $\varepsilon > 0$ and the size 2^k we can split all the cubes of this size almost in half with relative error ε:

$$\left|\sigma(Q \cap E^{1,2}) - \frac{1}{2}\sigma(Q)\right| \leq \varepsilon\sigma(Q).$$

Note that the estimate for size 2^k implies the same estimate for bigger dyadic cubes.

Proof of the one-weight lemma: splitting a cube

Want to split cubes with relative error ε,

$$\left|\sigma(Q \cap E^{1,2}) - \frac{1}{2}\sigma(Q)\right| \leq \varepsilon\sigma(Q).$$

Let us construct the disjoint sets $E^{1,2}$, we then shrink them to make separated.

- Pick a size $\delta = 2^{-m}$ such that $\sigma(R) \leq \frac{1}{2}\varepsilon\sigma(Q)$ for every dyadic cube $R \subset Q$ of this size. Follows from continuity of the measure.
- Split cubes Q into dyadic cubes R of this size, and order these cubes.

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- Put R_1 to E^1 and R_2 to E^2.

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- Then always put the next cube to the set of smaller measure σ.

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- Split cubes Q into dyadic cubes R of this size, and order these cubes.
- Put R_1 to E^1 and R_2 to E^2.
- Then always put the next cube to the set of smaller measure σ.
- We end up with error $\leq \frac{1}{2}\varepsilon\sigma(Q)$.

Proof of the one-weight lemma: shrinking the cubes

We constructed disjoint sets $E^{1,2}$, now we want to make them separated.

- Replace each small cube R by rR where $r < 1$ and close to 1;
- rR means dilation with respect to the the corner, not the center: $r[0,1)^N = [0,r)^N$, i.e. we dilate this cube with respect to the origin.
- With such dilation, $R = \bigcup_{r\in(0,1)} rR$, and thus it follows from countable additivity that

$$\sigma(R) = \lim_{r\to 1-} \sigma(rR).$$

Remarks

- Assumption about no common atoms is not really restriction:
  Interaction between μ_{c} and ν and the interaction between μ and ν_{c} is given by the main result.
  Interaction between μ_{a} and ν_{a} is described by a matrix.
- The result about two weight Muckenhoupt condition is obtained by multiplying K by an appropriate Schur multiplier to get he kernel satisfying
  $$K(x,y) \geq r^{-d} \qquad \text{for } |x-y| \leq r.$$
  The modified kernel is positive in some direction.
- The result about classical truncations is obtained similarly: we use an appropriate Schur multiplier to get the positive kernel dominating the difference between the classical and smooth truncation.

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