We start out by recalling Cotlar's Lemma and its simple proof.

Lemma: Suppose \( \{ T_j \} \) is a finite collection of bounded operators on \( L^2 \). Assume that we have \( \gamma(j), \; j \in \mathbb{Z}, \) with
\[
\sum_{j=-\infty}^{+\infty} \gamma(j) < +\infty
\]
and
\[
\| T_i^* T_j \| \leq \gamma(i-j)^2 \\
\| T_i T_j^* \| \leq \gamma(i-j)^2
\]
Then \( T = \sum T_j \) verifies
\[
\| T \| \leq A
\]
I briefly recall the proof: Recall that
\[
\| T \|^{2m} = \| (T^* T)^m \| \quad \text{We write}
\]
\[
(T^* T)^m = \sum_{i_n, \ldots, i_m} T_{i_n}^* T_{i_m} \ldots T_{i_m}
\]
We first associate the factors in each summand.
as \((T_i^1 T_i^2)\) \((T_{2m-1}^2 T_{2m})\) and
\[
\|T_{i_{11}}^1 - T_{i_{22}}^1\| \leq \gamma^2 (i_{11} - i_{22}). - \gamma^2 (i_{2m-1} - i_{2m})
\]

We can also associate the factors as
\[
T_{i_{11}}^1 (T_{i_{22}}^2 T_{i_{33}}^2) (T_{i_{2m-1}}^2 T_{i_{2m}}^2)
\]
so that, since \(\|T_{i_j}\| \leq \delta(0) \leq A\), we get
\[
\|T_{i_{11}}^1 T_{i_{22}}^1 - T_{i_{22}}^1 T_{i_{2m}}^2\| \leq A^2 \gamma^2 (i_{22} - i_{33}) - \gamma^2 (i_{2m-1} - i_{2m})
\]
Taking the geometric means gives
\[
\|(T^* T)^m\| \leq \sum_{i_1} A \delta(i_{11} - i_{22}) \delta(i_{22} - i_{33}) - \delta(i_{2m-1} - i_{2m})
\]

We then minimize, using \(\sum \delta(i_{11} - i_{22}) \leq A\), etc., to obtain
\[
\|(T^* T)^m\| \leq A^2 m \sum_{i_{11}} 1
\]

Suppose we have \(N\) rewards. We then obtain
\[
\|T\| \leq A N^{1/2} m, \text{ and let } m \to \infty
\]
proves the lemma.
(We can replace \(\mathbb{Z}\) by \(\mathbb{Z}^2\) with some proof).
Remarks. The original version (1953) of the lemma was proven by Citter for T-self-adjoint and mutually commuting. It was first used to obtain the $L^2$ boundedness of the "ergodic Hilbert transform", a situation where the Fourier transform was not available. The general version given was proved by Citter (67) and independently by E.M. Stein. It appears in a paper by Knapp-Stein from 1971, where they apply it to obtain the $L^2$ boundedness of intertwining operators given by singular integrals on nilpotent groups, in the context of representation theory for semi-simple groups.
I will now turn to pseudo-differential operators and some of the applications to pole. Thus, let \( a(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \) and consider the operator \( T_a f(x) = \int e^{ix \cdot \xi} a(x,\xi) \hat{f}(\xi) d\xi \).

This class of operators is a direct descendant of the variable coefficient singular integrals of Calderón-Zygmund and was first introduced by Kohn-Nirenberg (1965) and Unterberger-Boberzka (1964). The most common formulation we use today was introduced by Hörmander (1965, 1967). The function \( a(x,\xi) \) is the symbol and the "standard" class of symbols \( S^{m,\nu} \) is given by \( |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq A_{\alpha\beta} (1 + |\xi|^2)^{\nu-1/2} \).
This class of operators form a graded algebra, closed under adjoints. Moreover, for instance, the symbol of the composition is the product of the symbols, plus “laxer order”. Also, it is not difficult to show that $T_a$ is $L^2$ bounded if $a \in \Sigma^{0}$. In connection with hypoelliptic equations, Hörmander introduced the classes $\Sigma^m_{0,0}$, which are given by

$$|\partial^\alpha_x \partial^\beta_x a(x,\xi)| \leq A_{\alpha,\beta} (1 + |\xi|^2)^{m-|\alpha|+|\beta|}$$

when $0 < \delta < \delta \leq 1$, everything goes through as before. The case when $0 \leq \delta = \delta \leq 1$ was called the “exotic symbol class”. When $\delta = \delta = 1$, it was shown by example that operators in $\Sigma^0_{1,1}$ need not be $L^2$ bounded (the “forbidden symbol class”). The question of what happened for $\Sigma^0_{1,0}$...
$0 \leq \theta < 1$ remained open until it was answered by Calderón–Vaillancourt (1971-72).

**Theorem (Calderón–Vaillancourt)** If $a \in S^0_{\frac{\alpha}{\beta}, \theta} \ 0 \leq \theta < 1$, $Ta$ is $L^2$ bounded.

The importance of the issue was suggested by Mizohata, Huminou–Go and other Japanese mathematicians. The proof was by application of Cotlar's lemma. I will now give a sketch of the proof in the case $\beta = 0$. Note that the assumption on $a$ in this case is that its derivatives are bounded. The assumption is clearly symmetric in $x$ and $\zeta$, and we can thus consider (in view of Plancherel's theorem)

$$Sf(x) = \int a(x, \zeta) e^{i x \cdot \zeta} f(\zeta) d\zeta$$
If \( \sum_{i \in \mathbb{Z}^m} \phi(x-i) = 1 \), where \( \sum_{i \in \mathbb{Z}^m} \phi(\mathbf{x}; i) = 1 \),

and \( \alpha = (i, i^*) \in \mathbb{Z}^{2m} \), \( \beta = (j, j^*) \in \mathbb{Z}^{2m} \), then

we set \( a_{\alpha, \beta}(x, z) = \phi(x-i) \alpha(x, z) \phi(z-i^*) \).

One then verifies that

\[
\| S^*_\alpha S_\beta \| \leq A (1 + |\alpha - \beta|)^{-2N}
\]

\[
\| S^*_\alpha S_\beta \| \leq A (1 + |\alpha - \beta|)^{-2N}
\]

and Cotlar's lemma finishes the job. In order to check this, one uses that if

\[ Kf(x) = \int k(x, y) f(y) \, dy \]

and

\[ \sup_{x} \int |k(x, y)| \, dy \leq 1, \sup_{y} \int |k(x, y)| \, dx \leq 1 \]

then \( \| K \| \leq 1 \). One then observes that

\[
(S^*_\alpha S_\beta)(\phi)(z) = \int S_{\alpha, \beta}(x, z) \phi(x) \, dx , \quad \text{where}
\]

\[
S_{\alpha, \beta}(x, z) = \int a_{\alpha}(x, z) a_{\beta}(x, \eta^*) e^{-ix \cdot (\eta^* - \eta)} \, dx
\]
In this integral one integrates by parts, using

\[(I-Ax)^N e^{ix.(y-z)} - (1+iy-zi)^N e^{ix.(y-z)}\]

and the observation that \(A_x(x,z)\) and \(A_y(y,z)\)

have disjoint \(X\) support unless \(i-j\in\Omega_1\).

Thus, \(|A_{x,y}(z,y)| \leq A_N \frac{\phi(z-i') \phi(y-j')}{(1+|z-y|^2)^N}\)

\(i,j\in\Omega_1, 0\) elsewhere, which gives our bound.

The Calderón-Vaillancourt theorem and its proof find a remarkable application to

de in the work of R. Beals and C. Fefferman

(1974) on the local solvability of linear

partial differential operators of principal
type, which settled a long standing
conjecture of Nirenberg-Trèves.

I will now turn to more recent

applications to non-linear Schrödinger
equations. This has been a long-standing project, joint with G. Ponce and L. Vega (90-96).

The issue is the study of the local (in time) well-posedness of the initial value problem (Grassini-linear Schrödinger equations).

\[
\begin{align*}
\frac{\partial \mu}{\partial t} & = -i \sum_{k \geq 1} K(x, t, u, \bar{u}, \nabla u, \nabla \bar{u}) \frac{\partial}{\partial x_k} \mu + \\
& + \sum_{k \geq 1} \beta_k(x, t, u, \bar{u}, \nabla u, \nabla \bar{u}) \cdot \nabla u + \\
& + \sum_{k \geq 1} \beta_k(x, t, u, \bar{u}, \nabla u, \nabla \bar{u}) \cdot \nabla \bar{u} + \\
& + \sum_{k \geq 1} c_k(x, t, u, \bar{u}) u + \sum_{k \geq 1} c_k(x, t, u, \bar{u}) \bar{u} \\
\mu \big|_{t=0} & = \mu_0
\end{align*}
\]

In the nineties, our attention was focused in the so-called semi-linear case, when the second order operator is constant coefficient, i.e.,

\[
\sum_{j=1}^{k} \frac{\partial^2}{\partial x_j^2} - \sum_{i=k+1}^{n} \frac{\partial^2}{\partial x_i^2} = \delta_k, \quad 1 \leq k \leq n.
\]

(98)
In studying this problem, we were forced to understand the linear Cauchy problem (in simplified form)

\[ \partial_t \mu = iL \mu + \int_0^1 G(x) \partial_x \mu + G_1 \mu \]

\[ (\text{LCP}) \]

\[ \mu|_{t=0} = \mu_0 \]

and understand the issue of its well-posedness in (say) \( L^2 \). It turns out that there was previous work in this problem, which goes back to Mizohata and his former student Takeuchi \((80, 81)\). Let \( \mathbf{z} = (z_1, z_k, \ldots, z_k^n, -z_m) \), which is a bi-characteristic for \( L \). Then, the Mizohata result is that a necessary condition for the \( L^2 \) well-posedness of \( (\text{LCP}) \) is:
$$|\text{Im} \int_0^t \vec{b}_1(x + \lambda \vec{w}) \cdot \omega \, ds| \leq C$$

for all \( \omega \in \mathbb{R}^{n-1}, x \in \mathbb{R}^n \setminus \{0\} \). Thus, for instance \( \vec{b}_1(x) = i.e. \) does not work!

To illustrate our approach to (LCP), let's assume that \( \vec{b}_1(x) \in C_c^\infty(B_1), C \in C_c^\infty(\mathbb{R}^n) \).

Our idea is to apply the "energy method" to the equation verified by \( \vec{v} = C\vec{u} \), where \( C \) is an invertible operator, with symbol in \( S^{0,0}_{\rho_0} \). Thus, let us calculate

\[
\frac{d}{dt} \langle C\vec{u}, C\vec{u} \rangle = \frac{d}{dt} \int C\vec{u} \cdot C\vec{u} \, dx =
\]

\[
= \langle Ci\vec{u}, C\vec{u} \rangle + \langle C\vec{b}, \vec{v}, C\vec{u} \rangle + \langle C\vec{c}, \vec{w}, C\vec{u} \rangle + \langle \vec{u}, Ci\vec{u} \rangle + \langle C\vec{u}, C\vec{b}, \vec{v} \rangle + \langle C\vec{u}, C\vec{c}, \vec{w} \rangle
\]
Integration by parts gives

\[ \langle i\dot{\mathbf{u}}, \mathbf{u} \rangle + \langle \mathbf{u}, i\dot{\mathbf{u}} \rangle = 0 \]

Thus, we can rewrite this as

\[
\frac{d}{dt} \langle \mathbf{u}, \mathbf{u} \rangle = 2 \text{Re} \langle i [\mathbf{u} \times \mathbf{u}] \mathbf{u}, \mathbf{u} \rangle + \\
+ 2 \text{Re} \langle \mathbf{u}^2 \mathbf{u}, \mathbf{u} \rangle + \\
+ O(\|\mathbf{u}(t)\|^2_{L^2}) , \text{ since} \\
C \text{ is } L^2 \text{ bounded (we hope). The idea} \\
is to construct } C \text{ so that the first 2} \\
terms "almost" cancel, } C \text{ is } L^2 \text{ bounded} \\
and invertible, which then gives,} \]

for \( \mathbf{v} = C \mathbf{u} \),

\[
\frac{d}{dt} \langle \mathbf{v}, \mathbf{v} \rangle = O(\|\mathbf{v}\|^2_{L^2}) , \text{ which yields} \\
\text{control of } \mathbf{v} \text{ fast short time, and hence} \\
\]
In order to achieve the above conclusion one writes things down at the symbol level. If $C(x, \bar{z})$ is the symbol of $C$, a calculation shows that the symbol of $\bar{z} \left[ C \bar{z} - L C \right] = -2 \bar{z} \cdot \nabla_x C(x, \bar{z})$ while the symbol of $C \bar{z} \nabla \bar{z}$ is $i C(x, \bar{z}) \bar{b}_1(x) \cdot \bar{z}$. One then has to make "small"

$$-2 \bar{z} \cdot \nabla_x C(x, \bar{z}) + i C(x, \bar{z}) \bar{b}_1(x) \cdot \bar{z}$$

Since we need $C$ to be invertible, $C$ has to be non-zero, so we write $C(x, \bar{z}) = \exp(\lambda(x, \bar{z}))$. One then has in the expression above

$$C(x, \bar{z}) \cdot -2 \bar{z} \cdot \nabla_x C(x, \bar{z}) + i \bar{b}_1(x) \cdot \bar{z}$$

which needs to be small (preferably 0).
But, the equation can be integrated exactly, and the choice
\[ \tilde{\chi}(x, \frac{3}{2}) = -i \frac{1}{2} \int_{\|x\|}^{\infty} \frac{b_1(x + \frac{3}{2})}{\|x\|^{\frac{3}{2}}} \, ds \]
makes the bracket 0. (This explains the Mizohata condition mentioned before.)

We seem to be done, but what symbol bounds does \( \tilde{\chi} \) verify? They are
\[ |\partial_s^a \partial_x^b \tilde{\chi}(x, \frac{3}{2})| \leq A_{a, b} \left( \frac{1 + |x|}{1 + |\frac{3}{2}|} \right)^{1 + | \frac{3}{2} |} \]
and this is optimal. Unfortunately, this symbol class is too large and the operators \( c(\cdot, \frac{3}{2}) = \exp(\tilde{\chi}) \), with \( \tilde{\chi} \) verifying these bounds need not be \( L^2 \)-bounded!

What to do? We may define, for \( R \) large
\[ \chi(x, \frac{3}{2}) = \Theta(\frac{3}{2}, R) \cdot \psi \left( \frac{R - x}{\|x\|} \right) \tilde{\chi}(x, \frac{3}{2}), \]
where $\Theta = 1$ for $|\lambda|$ large, $= 0$ for $|\lambda|$ small and

$\Psi = 1 - \Theta$. We can now check that $\Psi$ and $c \in S_{0,0}$ and for large will
be invertible and the above procedure can be
carried out thanks to Calderón-Vaillancourt
and Aker's Lemma.

In the general quasi-linear case, which we
traveled in 2004–06 (also joint with my
former Phd student C. Rohonyi), it turns
out that this trick does not suffice
and we are forced to face the issue
of whether $\exp(\Psi(x,\lambda))$ with $\Psi$
defined above gives rise to a bounded
$L^2$-operator. The answer turns
out to be yes. The proof is a
delicate application of (once more!) Cotler's lemma. One also needs to develop an "almost calculus" for such operators, again by the use of Cotler's lemma. In the end, one is able to obtain the local well-posedness of "general" quasilinear Shrödinger equations.