Bilinear Operators in Analysis and PDEs

Part II – Bilinear pesudodifferential operators: beyond the Coifman-Meyer's theory

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Multipliers

$$Tf(x) = k * f(x) \text{ or } \widehat{Tf}(\xi) = m(\xi)\hat{f}(\xi)$$

$$T : L^2 \to L^2 \iff m \in L^\infty$$

Hörmander-Mihlin's condition:

$$|\partial^\alpha m(\xi)| \leq C_\alpha |\xi|^{-|\alpha|} \Rightarrow \begin{cases} T : L^p \to L^p, \quad 1 < p < \infty \\ T : L^1 \to L^{1,\infty} \\ T : L^\infty \to BMO \end{cases}$$

$$|\partial^\alpha m(\xi)| \leq C_\alpha |\xi|^{-|\alpha|} \Rightarrow |\partial^\alpha k(x)| \leq C_\alpha |x|^{-(n+|\alpha|)}$$

Classical Calderón-Zygmund singular integrals:

$$|\partial^\alpha k(x)| \leq C_\alpha |x|^{-(n+|\alpha|)}, \quad |\alpha| \leq 1$$

plus some *cancellation*, e.g.

$$\left| \int_{r<|x|<R} k(x)\,dx \right| \leq C$$

(essentially $T(1) = 0$), also give all the L^p results.

(Linear) T(1) – Theorem

David-Journé (1984)

$$T : \mathcal{S} \to \mathcal{S}', \quad \langle T(f), g \rangle = \langle K, f \otimes g \rangle$$

$$|\partial^\alpha K(x, y)| \leq C|x - y|^{-(n+|\alpha|)}, \quad |\alpha| \leq 1$$

Then,

$$T : L^2 \to L^2 \iff \begin{cases} T \in WBP \\ T(1), T^*(1) \in BMO \end{cases}$$

T has the WBP if

$$|\langle T(\varphi_{y,R}), \psi_{z,R} \rangle| \lesssim R^n,$$

for all *smooth bumps* φ and ψ supported on the unit ball and such that $\|\partial^\alpha \varphi\|_{L^\infty}, \|\partial^\alpha \psi\|_{L^\infty} \leq 1$, for all $|\alpha| \leq N$ (N sufficiently large).

Here $\varphi_{y,R}(x) = \varphi(R^{-1}(x - y))$; same with ψ.

The WBP is a *translation and dilation invariance condition* (even though T is not translation or dilation invariant).

$T(1)$ and $T^*(1)$ are a priori defined as distributions (modulo constants):

For φ with mean zero,

$$\langle T(1), \varphi \rangle = \lim_{j \to \infty} \langle T(\Phi_j), \varphi \rangle,$$

where $\Phi_j \to 1$ are smooth cutoff functions.

Similarly with $T^*(1)$.

$T(1), T^*(1) \in BMO$ are *cancellation conditions*.

In fact, a first step in the proof of the T(1)-Theorem is to reduce things to the case

$$T(1) = T^*(1) = 0$$

Essentially,

$$\int K(x, y)\, dy = 0 \qquad \int K(x, y)\, dx = 0$$

The proof of the theorem uses Cotlar's lemma

Equivalent formulations:

$T : L^2 \to L^2$

$\iff T : L^p \to L^p$, $1 < p < \infty$
(classical)

$\iff \sup_\xi (\|T(e^{ix\cdot\xi})\|_{BMO} + \|T^*(e^{ix\cdot\xi})\|_{BMO}) \leq C$
(David-Journé, 1984)

$\iff \|T(\varphi_{z,R})\|_{L^2} + \|T^*(\varphi_{z,R})\|_{L^2} \leq CR^{n/2}$
(Stein, 1993)

$\iff \|T_\epsilon(\chi_B)\|_{L^2} + \|T_\epsilon^*(\chi_B)\|_{L^2} \leq C|B|^{1/2}$
(Nazarov-Treil-Volberg, 1998) (T_ϵ are the usual truncated integrals)

$\iff |\langle K, \varphi_{z,R} \otimes f\rangle| + |\langle K, f \otimes \varphi_{z,R}\rangle| \leq CR^{n/2}\|f\|_{L^2}$
for all f with supp$f \subset B(z, R)$
(folklore?, BDNTTV)

Coifman-Meyer's bilinear operators

We saw in the first lecture operators (of order zero)

$$T(f,g)(x) = \int m(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix(\xi+\eta)}\,d\xi d\eta$$

$$|\partial^\alpha m(\xi,\eta)| \leq C_\alpha(|\xi|+|\eta|)^{-|\alpha|}$$

Then, formally,

$$T(f,g)(x) = \int K(x-y,x-z)f(y)g(z)\,dydz$$

where K is a Calderón-Zygmund kernel in $\mathbb{R}^{2n}$

$$|\partial^\alpha K(y,z)| \leq C_\alpha(|y|+|z|)^{-(2n+|\alpha|)}$$

We have

$$\|T(f,g)\|_{L^r} \leq C\|f\|_{L^p}\|g\|_{L^q}$$

for $1/p+1/q=1/r$, $1<p.q<\infty$ and $1/2<r<\infty$ (as well as a appropriate end-point results)

(Coifman-Meyer, Kenig-Stein, Grafakos-T.)

Example

The Riesz transforms in $\mathbb{R}^2$ can be seen as bilinear operators on $\mathbb{R} \times \mathbb{R}$, e.g.

$$R_1(f,g)(x) = \text{p.v.} \int_{\mathbb{R}^2} \frac{x-y}{|(x-y, x-z)|^3} f(y)g(z)\, dydz$$

$$R_1 : L^p(\mathbb{R}) \times L^q(\mathbb{R}) \to L^r(\mathbb{R})$$

for $1/p + 1/q = 1/r < 2$

The next step is to consider kernels of the form

$$K(x,y,z)$$

and operators formally given by

$$T(f,g)(x) = \int_{\mathbb{R}^{2n}} K(x,y,z) f(y) g(z)\, dydz$$

T(1) Theorem for bilinear Calderón-Zygmund Operators

$$T : \mathcal{S} \times \mathcal{S} \to \mathcal{S}'$$

$$\langle T(f_1, f_2), f_3 \rangle = \langle K, f_1 \otimes f_2 \otimes f_3 \rangle$$

$$|\partial^\alpha K(y_0, y_1, y_2)| \lesssim (\sum |y_j - y_k|)^{2n+|\alpha|}, \quad |\alpha| \leq 1$$

Christ-Journé (1987):

$$|\langle K, f_1 \otimes f_2 \otimes f_3 \rangle| \lesssim \|f_j\|_\infty \, \|f_k\|_2 \, \|f_l\|_2$$

iff K satisfies a multilinear WBP and the three distributions $T(1,1)$, $T^{*1}(1,1)$, $T^{*2}(1,1)$ are in BMO.

Grafakos-T. (2002):

$T : L^4 \times L^4 \to L^2$

$\iff \; T : L^p \times L^q \to L^r, \;\; 1/r = 1/p + 1/q < 2$

$\iff \; \sup_{\xi_1, \xi_2} (\|T^{*j}(e^{ix\cdot\xi_1}, e^{ix\cdot\xi_2})\|_{BMO} \leq C$

($T^{*0} = T$, T^{*1}, T^{*2} are the transposes of T)

Example

$$T_\sigma(f,g)(x) = \int \sigma(x,\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix\cdot(\xi+\eta)}d\xi d\eta.$$

with σ in $BS^0_{1,0}$ (the Coifman-Meyer class)

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \leq C_{\alpha\beta\gamma}(1+|\xi|+|\eta|)^{-(|\beta|+|\gamma|)}$$

If $\sigma \in BS^0_{1,0}$ then T_σ has a Calderón-Zygmund kernel

$$|\partial_x^\alpha K(x,y_1,y_2)| \leq C_\alpha(|x-y_1|+|x-y_2|)^{-(2n+|\alpha|)},$$

$$T(e^{i\xi\cdot}, e^{i\eta\cdot})(x) = \sigma(x,\xi,\eta)e^{ix\cdot(\xi+\eta)}$$

which is in L^∞ (uniformly in all ξ, η).

The same applies to the transposes of T. This follows from a symbolic calculus for the transposes.

By the bilinear T(1)-Theorem

$$T_\sigma : L^p \times L^q \to L^r$$

$1/r = 1/p + 1/q, \ r > 1/2$

Coiman and Meyer proved this for $r > 1$ using Littlewood-Paley theory.

More general bilinear pseudodifferential operators

$$T_\sigma(f,g)(x) = \int \sigma(x,\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix\cdot(\xi+\eta)}d\xi d\eta$$

For $0 \leq \delta \leq \rho \leq 1$, we say that $\sigma \in BS^m_{\rho,\delta}$ if

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \leq$$

$$C_{\alpha\beta\gamma}(1+|\xi|+|\eta|)^{m+\delta|\alpha|-\rho(|\beta|+|\gamma|)}$$

These are the bilinear analog of the Hörmander classes in the linear case; $\sigma \in S^m_{\rho,\delta}$ if

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \leq C_{\alpha\beta}(1+|\xi|)^{m+\delta|\alpha|-\rho|\beta|}$$

Most of the L^p results in these talks have the property that are symmetric with respect to transposition. This plays an important role in the proofs too.

Symbolic calculus for the transposes

If $\sigma(x,\xi,\eta) = \sigma(\xi,\eta)$ then the transposes of T_σ are very easy to compute

$$T_\sigma^{*1} = T_{\sigma^{*1}} \quad \text{and} \quad T_\sigma^{*2} = T_{\sigma^{*2}}$$

$$\sigma^{*1}(\xi,\eta) = \sigma(-\xi-\eta,\eta) \quad \text{and} \quad \sigma^{*2}(\xi,\eta) = \sigma(\xi,-\xi-\eta)$$

For x-dependent symbols the situation is more complicated but the following calculus holds.

For example if $\sigma \in BS^0_{1,0}$, then σ^{*1} and σ^{*2} are in $BS^0_{1,0}$ and

$$\sigma^{*1}(x,\xi,\eta) = \sum_\alpha \frac{i^\alpha}{\alpha!} \partial_x^\alpha \partial_\xi^\alpha \sigma(x,-\xi-\eta,\eta)$$

in the sense that for every $N > 0$,

$$\sigma^{*1}(x,\xi,\eta) - \sum_{|\alpha|<N} \frac{i^\alpha}{\alpha!} \partial_x^\alpha \partial_\xi^\alpha \sigma(x,-\xi-\eta,\eta) \in BS^{-N}_{1,0}$$

and similarly with σ^{*2}.

(Benyi-T. 2003)

The class $BS^0_{1,1}$ is the largest class that produces operators with bilinear Calderón-Zygmund kernels. But this class does not produce in general bounded operators on L^p spaces because it is not closed by transposition.

Nevertheless $BS^0_{1,1}$ is bounded on Sobolev spaces of positive smoothness.

If $\sigma \in BS^m_{1,1}$, $m \geq 0$, $s > 0$, then T_σ has a bounded extension from $L^p_{m+s} \times L^q_{m+s}$ into L^r_s,

$$\|T_\sigma(f,g)\|_{L^r_s} \leq C(\|f\|_{L^p_{m+s}}\|g\|_{L^q} + \|f\|_{L^p}\|g\|_{L^q_{m+s}}).$$

$1/p + 1/q = 1/r$, $1 < p, q, r < \infty$

(Bényi-T. 2003; Bényi-Nahmod-T. 2006)

The above results are analogous to the linear ones, but not everything holds the same in the bilinear setting.

For example, Calderón-Vaillancourt's theorem fails.

In the linear case $\sigma \in S^0_{0,0} \Rightarrow T_\sigma : L^2 \to L^2$, but

$$\sigma \in BS^0_{0,0} \not\Rightarrow T_\sigma : L^2 \times L^2 \to L^1$$

(or any $L^p \times L^q \to L^r$, $1 \leq p, q, r < \infty$)

There is a substitute result:

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x, \xi, \eta)| \leq C_{\alpha\beta\gamma}$$

$$\sup_x \int \left(\int |\partial_\eta^\gamma \sigma(x, \xi, \eta)|^2 \, d\xi \right)^{1/2} d\eta \leq C_\gamma$$

$$\sup_x \int \left(\int |\partial_\xi^\gamma \sigma(x, \xi, \eta)|^2 \, d\eta \right)^{1/2} d\xi \leq C_\gamma$$

$$\Rightarrow T : L^2 \times L^2 \to L^1$$

The proof uses a bilinear version of *Cotlar's lemma* (Bényi-T. 2004)

Let H be a Hilbert space and V a normed space of functions. If

$$T_j : V \times H \to H, j \in \mathbf{Z},$$

is a sequence of bounded bilinear operators and $\{a(j)\}_{j\in\mathbf{Z}}$ is a sequence of positive real numbers such that

$$||T_i(f, T_j^{*2}(\bar{f}, g))||_H + ||T_i^{*2}(\bar{f}, T_j(f, g))||_H \leq a(i-j),$$

for all $f \in V, g \in H, ||f||_V = ||g||_H = 1$ and for all $i, j \in \mathbf{Z}$, then

$$||\sum_{j=n}^{m} T_j|| \leq \sum_{i=-\infty}^{\infty} \sqrt{a(i)},\ n, m \in \mathbf{Z}, n \leq m.$$

This lemma can be sometimes used to prove estimates of the form

$$L^\infty \times L^2 \to L^2$$

(and then by duality $L^2 \times L^2 \to L^1$)

Composition of pseudodifferential operators

Let $J^m = (I - \Delta)^{m/2}$, then if $\sigma \in BS^m_{1,0}$, $m \geq 0$, then

$$T_\sigma(f,g) = T_{\sigma_1}(J^m f, g) + T_{\sigma_2}(f, J^m g).$$

for some σ_1 and σ_2 in $BS^0_{1,0}$ In particular,

$$T_\sigma : L^p_m \times L^q_m \to L^r$$

$1/p + 1/q = 1/r$, $1 < p, q < \infty$.

However, if $T_\sigma \in \mathrm{Op}BS^0_{1,0}$ and $L_a \in \mathrm{Op}S^m_{1,0}$, then in general $L_a T_\sigma \notin \mathrm{Op}BS^m_{1,0}$

$L_a T_\sigma$ has a symbol that satisfies estimates in terms of $|\xi + \eta|$. This provides another motivation to look at more general classes

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \leq$$

$$C_{\alpha\beta\gamma}(1 + |\eta - \xi \tan\theta|)^{m+\delta|\alpha|-\rho(|\beta|+|\gamma|)}$$

for $\theta \in (-\pi/2, \pi/2]$ (with the convention that $\theta = \pi/2$ corresponds to estimates in terms of $1 + |\xi|$).

In the one-dimensional case

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \leq$$

$$C_{\alpha\beta\gamma}(1 + dist((\xi,\eta);\Gamma_\theta))^{m+\delta|\alpha|-\rho(|\beta|+|\gamma|)}$$

where Γ_θ is the line at angle θ with respect to the axis $\eta = 0$.

Note that $\theta = -\pi/4, 0, \pi/2$ are the degenerate directions for the bilinear Hilbert transform.

There exits a calculus for the composition with linear operators and for the transposes.

In particular:

1) If $T_\sigma \in \text{Op}BS^0_{1,0}$ and $L_a \in \text{Op}S^m_{1,0}$, then $L_a T_\sigma \in \text{Op}BS^m_{1,0;-\pi/4}$

2) $\{BS^0_{1,0;\,\theta}\}_\theta$ is closed under transposition

(Bényi-Nahmod-T. 2006)

The symbols of the transposes can be computed explicitly and it holds that

$$\sigma_\theta \rightsquigarrow \sigma_{\theta^{*1}},\ \sigma_{\theta^{*2}}$$

where for $\theta \neq 0, \pi/2, -\pi/4$

$$\cot\theta + \cot\theta^{*1} = -1$$

$$\tan\theta + \tan\theta^{*2} = -1.$$

In the degenerate directions

$$\{0, \pi/2, -\pi/4\}^{*1} = \{0, -\pi/4, \pi/2\}$$

$$\{0, \pi/2, -\pi/4\}^{*2} = \{-\pi/4, \pi/2, 0\}$$

Modulation invariant operators

We will concentrate now on bilinear operators in 1-dimension. We saw in the first talk the proof of the boundedness of the bilinear Hilbert transform (Lacey-Thiele, 1997-1999)

$$T(f,g)(x) = \int \text{sign}(\xi-\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix\cdot(\xi+\eta)}d\xi d\eta$$

and of the operators (Gilbert-Nahmod, 2000-2002)

$$T_m(f,g)(x) = \int m(\xi-\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix\cdot(\xi+\eta)}d\xi d\eta$$

$$|d^\alpha m(z)| \leq C_\alpha |z|^{-\alpha}.$$

We will look at the L^p-boundedness of variable coefficient operators with a modulation invariance

$$T_\sigma(f,g)(x) = \int \sigma(x,\xi-\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix\cdot(\xi+\eta)}d\xi d\eta$$

where $|\partial_x^\beta \partial_{\xi,\eta}^\alpha \sigma(x,\xi-\eta)| \leq C_\alpha(1+|\xi-\eta|)^{-|\alpha|}$.

They satisfy the modulation invariance

$$\langle T(f_1,f_2),f_3\rangle = \langle T(e^{iz\cdot}f_1, e^{iz\cdot}f_2), e^{-i2z\cdot}f_3\rangle$$

(other directions can be considered too)

Let's compute the kernel of these operators.

Assume $\sigma(x,u)$ is in $S^0_{1,0}(\mathbb{R})$ and consider

$$T(f_1,f_2)(x) = \int_{\mathbb{R}^2} \sigma(x,\xi-\eta)\widehat{f}_1(\xi)\widehat{f}_2(\eta)e^{ix(\xi+\eta)}d\xi d\eta$$

Undoing the Fourier transforms,

$$= \int_{\mathbb{R}^2} k(x,x-y)\delta(z-2x+y)f_1(y)f_2(z)\,dydz$$

$$= \int_{\mathbb{R}} k(x,t)f_1(x-t)f_2(x+t)\,dt$$

(valid at least for functions with disjoint support)

$$k(x,x-y) = (\mathcal{F}^{-1}\sigma)(x,x-y)$$

(the inverse Fourier transform is taken in the second variable). Such a k is a (linear) Calderón-Zygmund kernel, but the Schwartz kernel of T is

$$K(x,y,z) = k(x,x-y)\delta(z-2x+y)$$

which is too singular to fall under the scope of the previous multilinear T(1)-Theorems. Note also that the BHT is obtained with $k(x,t) = 1/t$.

From the Hilbert transform to the variable coeff. BHT

(linear) Hilbert Transform

$$\mathrm{sign}(\xi)$$

(linear) C-Z Sing. Int. / H-M multipliers

$$|\partial^\alpha m(\xi)| \leq C_\alpha |\xi|^{-\alpha}$$

(linear) Classical PDOs ($S^0_{1,0}$)

$$|\partial_x^\beta \partial_\xi^\alpha \sigma(x,\xi)| \leq C_{\alpha\beta}(1+|\xi|)^{-\alpha}$$

Bilinear Hilbert Transform

$$\mathrm{sign}(\xi-\eta)$$

Bilinear Singular Multipliers

$$|\partial^\alpha m(\xi-\eta)| \leq C_\alpha |\xi-\eta|^{-\alpha}$$

Bilinear Coifman–Meyer's PDOs ($BS^0_{1,0}$)

$$|\partial_x^\beta \partial_{\xi,\eta}^\alpha \sigma(x,\xi,\eta)| \leq C_{\alpha\beta}(1+|\xi|+|\eta|)^{-|\alpha|}$$

Variable Coeff. Bilinear Hilbert Transforms

$$|\partial_x^\beta \partial_{\xi,\eta}^\alpha \sigma(x,\xi-\eta)| \leq C_{\alpha\beta}(1+|\xi-\eta|)^{-|\alpha|}$$

For symmetry purposes, we will look from now on to the trilinear form

$$\Lambda(f_1, f_2, f_3) = \langle T(f_1, f_2), f_3 \rangle$$
$$= \langle T^{*1}(f_2, f_3), f_1 \rangle = \langle T^{*2}(f_1, f_3), f_2 \rangle$$

For the rest of this talk we will assume that all Calderón-Zygmund kernels considered satisfy

$$|\partial^\alpha k(x,t)| \leq C|t|^{-|\alpha|} \quad t \neq 0, |\alpha| \leq 1$$

A trilinear form Λ on $\mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$, is said to be associated with a Calderón-Zygmund kernel k if

$$\Lambda(f_1, f_2, f_3) = \int_{\mathbb{R}^2} \prod_{j=1}^{3} f_j(x + \beta_j t) k(x,t)\, dx\, dt \tag{1}$$

for some $\beta = (\beta_1, \beta_2, \beta_3)$ and all functions f_1, f_2, f_3 in $\mathcal{S}(\mathbb{R})$ such that the intersection of their supports is empty.

We will assume that $\beta_1, \beta_2, \beta_3$ are different (otherwise Λ reduces to a combination of a pointwise product and a bilinear form).

We will also assume β to be of unit length and perpendicular to $\alpha = (1, 1, 1)$. Let $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ be a unit vector perpendicular to α and β; then $\gamma_j \neq 0$.

We impose the *modulation symmetry along the direction of* γ:

$$\Lambda(f_1, f_2, f_3) = \Lambda(M_{\gamma_1\xi} f_1, M_{\gamma_2\xi} f_2, M_{\gamma_3\xi} f_3) \tag{2}$$

for all $\xi \in \mathbb{R}$, where $M_\eta f(x) = e^{i\eta x} f(x)$.

From (1) this trivially holds for functions with disjoint support, but we want the modulation symmetry to hold for all Schwartz functions even when the representation formula (1) is not valid as an absolutely convergent integral.

Modulation invariant T(1,1)-Theorem

Bényi, Demeter, Nahmod, Thiele, T., Villarroya

Assume Λ *is a trilinear form on* $\mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$ *associated with a kernel* k *as in* (1) *and with modulation symmetry* (2) *in the direction of* γ*. Then,*

$$|\Lambda(f_1, f_2, f_3)| \lesssim \prod_{j=1}^{3} \|f_j\|_{p_j}$$

for all exponents $2 \leq p_1, p_2, p_3 \leq \infty$ *with*

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$$

if and only if for all intervals I*, all* L^2*-normalized bump functions* ϕ_I *and* ψ_I *supported in* I*, and all* f *in* $\mathcal{S}$

$$|\Lambda(\phi_I, \psi_I, f)| \lesssim |I|^{-1/2} \|f\|_2, \tag{3}$$

$$|\Lambda(\phi_I, f, \psi_I)| \lesssim |I|^{-1/2} \|f\|_2,$$

$$|\Lambda(f, \phi_I, \psi_I)| \lesssim |I|^{-1/2} \|f\|_2.$$

Moreover, in such a case T satisfies

$$\|T(f_1, f_2)\|_{L^r} \lesssim \|f_1\|_{L^p} \|f_1\|_{L^q}$$

$1/p + 1/q = 1/r < 3/2$

Note that this is the same range as for the BHT.

The restricted boundedness conditions (3) can be assumed also for all f smooth and supported in, say, $100I$.

Recall that an L^2-normalized bump adapted to an interval I satisfies

$$|\partial^\alpha \varphi(x)| \le C|I|^{-1/2-\alpha} \left(1 + \left|\frac{x - c(I)}{|I|}\right|^2\right)^{-N/2}$$

for all $0 \le \alpha \le N$.

Example

Consider again the trilinear form

$$\Lambda(f_1, f_2, f_3) = \int_{\mathbb{R}} T_3(f_1, f_2)(x) f_3(x)\, dx =$$

$$\int_{\mathbb{R}^3} \sigma(x, \xi - \eta) \widehat{f}_1(\xi) \widehat{f}_2(\eta) f_3(x) e^{ix(\xi+\eta)} d\xi d\eta dx$$

with σ in $S^0_{1,0}$.

Note that this form has modulation symmetry in the direction $\gamma = (1, 1, -2)/\sqrt{6}$ for all triples f_1, f_2, f_3, not just the ones with disjoint supports.

Assuming f is supported in CI

$$|\Lambda(\phi_I, \psi_I, f)| \lesssim \|\widehat{\phi}_I\|_{L^1} \|\widehat{\psi}_I\|_{L^1} \|f\|_{L^1}$$

$$\lesssim \|\widehat{\phi}_I\|_{L^1} \|\widehat{\psi}_I\|_{L^1} |I|^{1/2} \|f\|_{L^2}$$

$$\lesssim |I|^{-1/2} \|f\|_{L^2}.$$

Here, we have used that $\widehat{\phi}_I$ and $\widehat{\psi}_I$ are L^2-normalized and adapted to intervals of length $|I|^{-1}$.

More precisely,

$$\phi_I(x) = |I|^{-1/2}\phi_0((x - x_0)/|I|)$$

where ϕ_0 is adapted to and supported in the unit interval centered at the origin

$$\|\widehat{\phi}_I\|_{L^1} = |I|^{-1/2}\|\widehat{\phi}_0\|_{L^1} \leq C|I|^{-1/2}$$

where C depends only on finitely many derivatives of ϕ_0. The same estimate applies to ψ_I.

To obtain the other restricted boundedness conditions, write

$$\Lambda(f, \phi_I, \psi_I) = \int_{\mathbb{R}} T^{*1}(\psi_I, \phi_I)(x) f(x)\, dx,$$

and using the symbolic calculus

$$T^{*1}(g, h)(x) = \int_{\mathbb{R}^2} \sigma_1(x, \xi, \eta)\widehat{g}(\xi)\widehat{h}(\eta)e^{ix(\xi+\eta)}d\xi d\eta$$

where

$$|\partial_x^\mu \partial_{\xi,\eta}^\alpha \sigma_1(x, \xi, \eta)| \lesssim (1 + |\xi + 2\eta|)^{-|\alpha|}.$$

The computations done with T can now be repeated with T^{*1}. Similarly with T^{*2}.

Plan of the proof

It is immediate that the boundedness of Λ implies the conditions (3). Actually, it is enough to have, say

$$T,\, T^{*1},\, T^{*2} : L^4 \times L^4 \to L^2$$

to obtain (3).

First, the conditions (3) are used to show that

$$\begin{cases} T(1,1), T^{*1}(1,1), T^{*2}(1,1) \in BMO \\ |\Lambda(\phi)| \lesssim |I|^{-1/2} \end{cases}$$

where $\phi(x, y, z)$ is a smooth bump L^2-normalized and adapted to $I \times I \times I$.

The theorem is then reduced to the case

$$\begin{cases} T(1,1) = T^{*1}(1,1) = T^{*2}(1,1) = 0 \\ |\Lambda(\phi)| \lesssim |I|^{-1/2} \end{cases}$$

using some modulation invariant paraproducts. This last set of conditions is used to discretize the operator.

Then the fun begins!!!

More about $T(1,1)$ and BMO

$$\langle T(1,1), f\rangle = \lim_{k\to\infty} \langle T(\Phi_k, \Phi_k), f\rangle =$$

$$\lim_{k\to\infty} \Lambda(\Phi_k \otimes \Phi_k \otimes f)$$

exist for f with mean zero. Here $\Phi_k(x) = \Phi(2^{-k}x)$ with Φ a smooth version of the characteristic function of the interval $(-1/2, 1/2)$.

To show $T(1,1) \in BMO$ one uses the conditions (3) to show that $T(\Phi_k, \Phi_k)$ are uniformly in BMO. This is analogous to the linear $T(1)$ *a la Stein.*

For all this there are always two typical Calderón-Zygmund theory estimates. One far away from the support of a function, where the regularity of the kernel is used; and another local one, where some a priori bound is needed.

One trilinear form $\rightsquigarrow$ three bilinear forms

By the Schwartz kernel theorem, the trilinear form Λ can be represented by a tempered distribution in $\mathbb{R}^3$, which we shall also denote by Λ, so that

$$\Lambda(\phi_1, \phi_2, \phi_3) = \Lambda(\phi_1 \otimes \phi_2 \otimes \phi_3).$$

(In this way, $\Lambda(\phi)$ has a meaning for any $\phi \in \mathcal{S}(\mathbb{R}^3)$ not necessarily a tensor product.)

The kernel representation of Λ continues to hold when a test function ϕ has support disjoint from the span of $(1, 1, 1)$, namely

$$\Lambda(\phi) = \int \phi(x\alpha + t\beta)\, k(x, t)\, dxdt$$

$$= \int \phi(x + \beta_1 t, x + \beta_2 t, x + \beta_3 t)\, k(x, t)\, dxdt$$

Using the modulation invariance, one can show that

$$\Lambda(f) = \Lambda_*(f|_{\gamma^\perp}).$$

where $\Lambda_* \in \mathcal{S}'(\gamma^\perp)$.

There are three distinct bilinear forms that we can consider

$$\Lambda_1(\phi_2, \phi_3) = \Lambda_*((1 \otimes \phi_2 \otimes \phi_3)|_{\gamma^\perp})$$

$$\Lambda_2(\phi_1, \phi_3) = \Lambda_*((\phi_1 \otimes 1 \otimes \phi_3)|_{\gamma^\perp})$$

$$\Lambda_3(\phi_1, \phi_2) = \Lambda_*((\phi_1 \otimes \phi_2 \otimes 1)|_{\gamma^\perp})$$

(the functions on the right hand side are in $\mathcal{S}(\gamma^\perp)$, since none of the components of γ is zero).

For ϕ_2, ϕ_3 with disjoint support,

$$\Lambda_1(\phi_2, \phi_3) = \Lambda_*((1 \otimes \phi_2 \otimes \phi_3)|_{\gamma^\perp}) =$$

$$\int \phi_2(x + \beta_2 t)\phi_3(x + \beta_3 t)k(x,t)\, dxdt.$$

Similarly for Λ_2 and Λ_3.

The Λ_j's are bilinear Calderón-Zygmund forms!!!!

Moreover, if Λ satisfies the trilinear WBP,

$$|\Lambda(\phi)| \lesssim |I|^{-1/2},$$

where $\phi(x, y, z)$ is a smooth bump L^2-normalized and adapted to $I \times I \times I$, then the forms Λ_j, $j = 1, 2, 3$, satisfy the bilinear weak boundedness property.

Also, the identities

$$\Lambda_1(1, \cdot) = \Lambda(1, 1, \cdot) = \Lambda_2(1, \cdot),$$

$$\Lambda_2(\cdot, 1) = \Lambda(\cdot, 1, 1) = \Lambda_3(\cdot, 1),$$

$$\Lambda_3(1, \cdot) = \Lambda(1, \cdot, 1) = \Lambda_1(\cdot, 1),$$

hold when the terms are interpreted as tempered distributions modulo constants.

This will be important when estimating the action of Λ on bump functions.

(almost orthogonality estimates)

The reduction to $T(1,1) = 0$

The idea is to construct for

$$T(1,1) = f,\ T^{*2}(1,1) = g,\ T^{*1}(1,1) = h\ \in BMO,$$

modulation invariant forms Λ_f, Λ_g, Λ_h, such that

$$\Lambda_f(1,1,.) = f,\ \Lambda_f(1,.,1) = 0,\ \Lambda_f(.,1,1) = 0$$

$$\Lambda_g(1,1,.) = 0,\ \Lambda_g(1,.,1) = g,\ \Lambda_g(.,1,1) = 0$$

$$\Lambda_h(1,1,.) = 0,\ \Lambda_h(1,.,1) = 0,\ \Lambda_h(.,1,1) = h$$

Then, $\Lambda - \Lambda_f - \Lambda_g - \Lambda_h$ will satisfy

$$T(1,1) = T^{*2}(1,1) = T^{*1}(1,1) = 0$$

With Φ as before (smooth version of $\chi_{(-1/2,1/2)}$), define

$$\widehat{\phi_1} = \widehat{\phi_2} = \Phi$$

$$\widehat{\phi_3} = \tau_{10}\Phi + \tau_{-10}\Phi$$

$$\psi(x) =$$

$$\int \phi_1(x+(\beta_1-\beta_3)t)\phi_2(x+(\beta_2-\beta_3)t)\phi_3(x)\,dt$$

(ψ has mean zero)

By Calderón's reproducing formula,

$$f = c\int_0^{+\infty} f * \psi_t * \psi_t \,\frac{dt}{t}$$

i.e.

$$f = \lim_{\epsilon\longrightarrow 0} c\int_\epsilon^{1/\epsilon} f * \psi_t * \psi_t \,\frac{dt}{t}$$

at least in the distributional sense when tested against bump functions with mean zero.

Defining

$$\psi_{k,n}(x) := 2^{-k/2}\psi(2^{-k}x - n),$$

a simple change of coordinates gives

$$f = \lim_{\kappa \to +\infty} c \int_{|k| \leq \kappa} \int_{\mathbb{R}} \left\langle f, \psi_{k,n} \right\rangle \psi_{k,n} \, dk \, dn \quad (4)$$

with the equality holding in the sense of distributions when tested again functions with mean zero.

If we define $\phi_{i,k,n}$ by translation and dilation in the analogous manner,

$$\psi_{k,n}(x) =$$

$$\int \phi_{1,k,n}(x+(\beta_1-\beta_3)t)\phi_{2,k,n}(x+(\beta_2-\beta_3)t)\phi_{3,k,n}(x) \, dt$$

Set

$$c_{k,n} := c \left\langle f, \psi_{k,n} \right\rangle$$

Then, for each $\kappa \in \mathbb{R}$ and each $f_i \in \mathcal{S}(\mathbb{R})$ we define

$$\Lambda_\kappa(f_1, f_2, f_3) = \int_{|k|\le\kappa}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} c_{k,n} \times$$

$$\times \left[\prod_{i=1}^{3} f_i(x+\beta_i t)\phi_{i,k,n}(x+\beta_i t)\right] dx\,dt\,dn\,dk$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \prod_{i=1}^{3} f_i(x+\beta_i t) K_\kappa(x,t) dt dx, \quad (5)$$

$$K_\kappa(x,t) = \int_{|k|\le\kappa}\int_{\mathbb{R}} c_{k,n} \prod_{i=1}^{3} \phi_{i,k,n}(x+\beta_i t)\,dn\,dk.$$

Discretizing, K_κ can be seen as an average of sums

$$\tilde{K_\kappa}(x,t) = \sum_{\substack{2^{-\kappa}\le|I|\le 2^\kappa \\ I \text{ dyadic}}} c_I \prod_{i=1}^{3} \phi_{i,I}(x+\beta_i t),$$

with $\phi_{i,I}$ being L^2-adapted to I and $|c_I| \lesssim |I|^{1/2}$.

$\tilde{K_\kappa}$ is a Calderón-Zygmund kernel!

Define now for each $f_i \in \mathcal{S}(\mathbb{R})$

$$\Lambda(f_1, f_2, f_3) = \lim_{\kappa \to +\infty} \Lambda_\kappa(f_1, f_2, f_3).$$

Λ has the required properties!!!

Since the function

$$\phi_1(x)\phi_2(x + (\beta_2 - \beta_1)t)\phi_3(x + (\beta_3 - \beta_1)t)$$

has mean zero in t for every fixed x, it is easy to verify that $\Lambda_\kappa(., 1, 1) = 0$. We conclude that $\Lambda(., 1, 1) = 0$. Likewise we see $\Lambda(1, ., 1) = 0$.

To see that $\Lambda(1, 1, .) = f$, we replace the integration variable x by $y = x + \beta_3 t$ and integrate in t to obtain for $f_3 \in \mathcal{S}(\mathbb{R})$ with mean zero

$$\Lambda_\kappa(1, 1, f_3) = \int_{\mathbb{R}} \left[\int_{|k| \le \kappa} \int_{\mathbb{R}} c_{k,n} \psi_{k,n}(y) \, dn \, dk \right] f_3(y) dy$$

The limit on the right hand side is $\int_{\mathbb{R}} f(y) f_3(y) dy$, and so we conclude that $\Lambda(1, 1, .) = f$ as tempered distributions modulo constants.

It remains to prove that Λ is bounded. Up to some universal constant, $\Lambda(f_1, f_2, f_3)$ coincides with

$$\lim_{\kappa\to+\infty} \int_{|k|\leq\kappa} \int_{\mathbb{R}^2} c_{k,n} 2^{-k} \times$$

$$\left[\prod_{i=1}^{3} \int f_i(x)\phi_{i,k,n}(x) e^{2\pi i \gamma_i 2^{-k} l x}\, dx\right] dn\, dl\, dk.$$

Hence Λ is up to a universal constant an average of forms of the type

$$\sum_{k,n,l\in\mathbb{Z}} c_{k,n} 2^{-k} \prod_{i=1}^{3} \left\langle f_i, \phi_{i,k,n,l} \right\rangle, \qquad (6)$$

where for some $k_0, n_0, l_0 \in [0,1]$

$$\phi_{i,k,n,l}(x) = \overline{\phi_{i,k+k_0,n+n_0}(x)} e^{-2\pi i \gamma_i 2^{-(k+k_0)}(l+l_0)x}$$

The forms (6) are the basic model that we have to estimate to prove the boundedness of Λ. The function $\phi_{i,k,n,l}$ is an L^2- normalized bump function adapted to the interval

$$[2^{k+k_0}(n+n_0)), 2^{k+k_0}(n+n_0+1))$$

By changing the bump function constants mildly, we can assume that the function is adapted to the dyadic interval $I_{k,n} = [2^k n, 2^k (n+1))$.

The proof of the boundedness of the models is similar to the one for the models for the BHT

If you followed the first talk closely, you can try to do it...

:)

Note that now there is a lack of separation in the tiles used but this can be compensated by the estimates we now have for the coefficients $c_{k,n}$:

Since f is in BMO, the coefficients $c_{k,n}$ satisfy a Carleson sequence condition

$$\sum_{I_{k,n} \subset J} |c_{k,n}|^2 \leq C_f |J|.$$

OK fine, but why $T(1,1) = 0$?

The proof of the theorem in the reduced case uses a phase-space analysis similar to the one used for the BHT.

In particular we use a Whitney decomposition in frequency with respect to a bad set given by three planes through the line $<\gamma>$. This decomposition is done in terms of *tubes* or rectangular boxes with square cross sections (not in terms of cubes as in the case of the BHT).

The conditions

$$T(1,1) = T^{*1}(1,1) = T^{*2}(1,1) = 0$$

are used to control Λ on the frequency side near the bad set.

Heuristic

Assume $k(t)$ is an x independent Calderón-Zygmund kernel

$$\Lambda(f_1, f_2, f_3) =$$

$$\text{p.v} \int f_1(x+\beta_1 t) f_2(x+\beta_2 t) f_3(x+\beta_3 t) k(t) dx\, dt.$$

One reduces the boundedness of $|\Lambda(f_1, f_2, f_3)|$ to that of model operators

$$\sum_{p \in \mathbf{P}} a_p \, |I_p|^{-1/2} |\langle f_1, \phi_{p_1} \rangle \langle f_2, \phi_{p_2} \rangle \langle f_3, \phi_{p_3} \rangle|,$$

where $\mathbf{P}$ is the collection of all multi-tiles in phase space $p = p_1 \times p_2 \times p_3, \quad p_i = I_{p_i} \times \omega_{p_i}$ with $|I_p| := |I_{p_1}| = |I_{p_2}| = |I_{p_3}|$ and ω_{p_i} pairwise disjoint for each fixed p. The ϕ_{p_i} are wave packets L^2- adapted to the time-frequency tile p_i and $\{a_p\} \in \ell^\infty(\mathbf{P})$.

$\mathbf{P}$ is a one parameter family of multi-tiles: each ω_{p_i} determines uniquely the other two ω_{p_j}.

In the case k depends on both x and t we proceed differently. We can write

$$\Lambda(f_1, f_2, f_3) = \langle k(x,t) | f_1 \otimes f_2 \otimes f_3(x\alpha + t\beta) \rangle$$

Applying the Fourier inversion formula to each f_j

$$\Lambda(f_1, f_2, f_3) = \langle \widehat{k}(-\alpha\cdot\xi, -\beta\cdot\xi) | \widehat{f}_1 \otimes \widehat{f}_2 \otimes \widehat{f}_3(\xi) \rangle. \tag{7}$$

Suppose that

$$|\partial^\alpha k(x,t)| \lesssim |t|^{-(1+|\alpha|)},$$

for all $|\alpha| \geq 0$. These conditions essentially imply

$$|\partial_v^m \widehat{k}(u,v)| \lesssim |v|^{-m}, \tag{8}$$

for $v \neq 0$ and $m \geq 1$. The estimates say that the derivatives $\partial_v^m \widehat{k}$ are only singular for $v = 0$.

So we may expect trouble when

$$-\beta \cdot \xi = 0$$

That is on on the frequency side we expect the form Λ to blow up on $\beta^\perp$

The representation (7) is not unique; it depends on β. We can write

$$\Lambda(f_1, f_2, f_3) = \langle K_3(x,t) | f_1 \otimes f_2 \otimes f_3(x\alpha + t\beta^3) \rangle$$
$$= \langle T(f_1, f_2) | f_3 \rangle$$
$$\Lambda(f_1, f_2, f_3) = \langle K_1(x,t) | f_1 \otimes f_2 \otimes f_3(x\alpha + t\beta^1) \rangle$$
$$= \langle T^{*1}(f_2, f_3) | f_1 \rangle$$
$$\Lambda(f_1, f_2, f_3) = \langle K_2(x,t) | f_1 \otimes f_2 \otimes f_3(x\alpha + t\beta^2) \rangle$$
$$= \langle T^{*2}(f_1, f_3) | f_2 \rangle$$

where the vectors β^j are still perpendicular to γ and satisfy that the component $\beta^j_j = 0$ (each K_j is related to k by a change of variable).

We obtain then three frequency representations

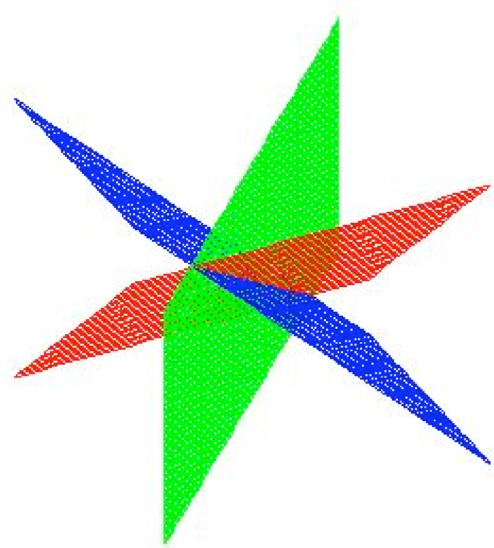
$$\Lambda(f_1, f_2, f_3) = \langle \widehat{K}_j(-\alpha \cdot \xi, -\beta^j \cdot \xi) | \widehat{f}_1 \otimes \widehat{f}_2 \otimes \widehat{f}_3(\xi) \rangle. \qquad (9)$$

In each of them, away from a plane through γ defined by

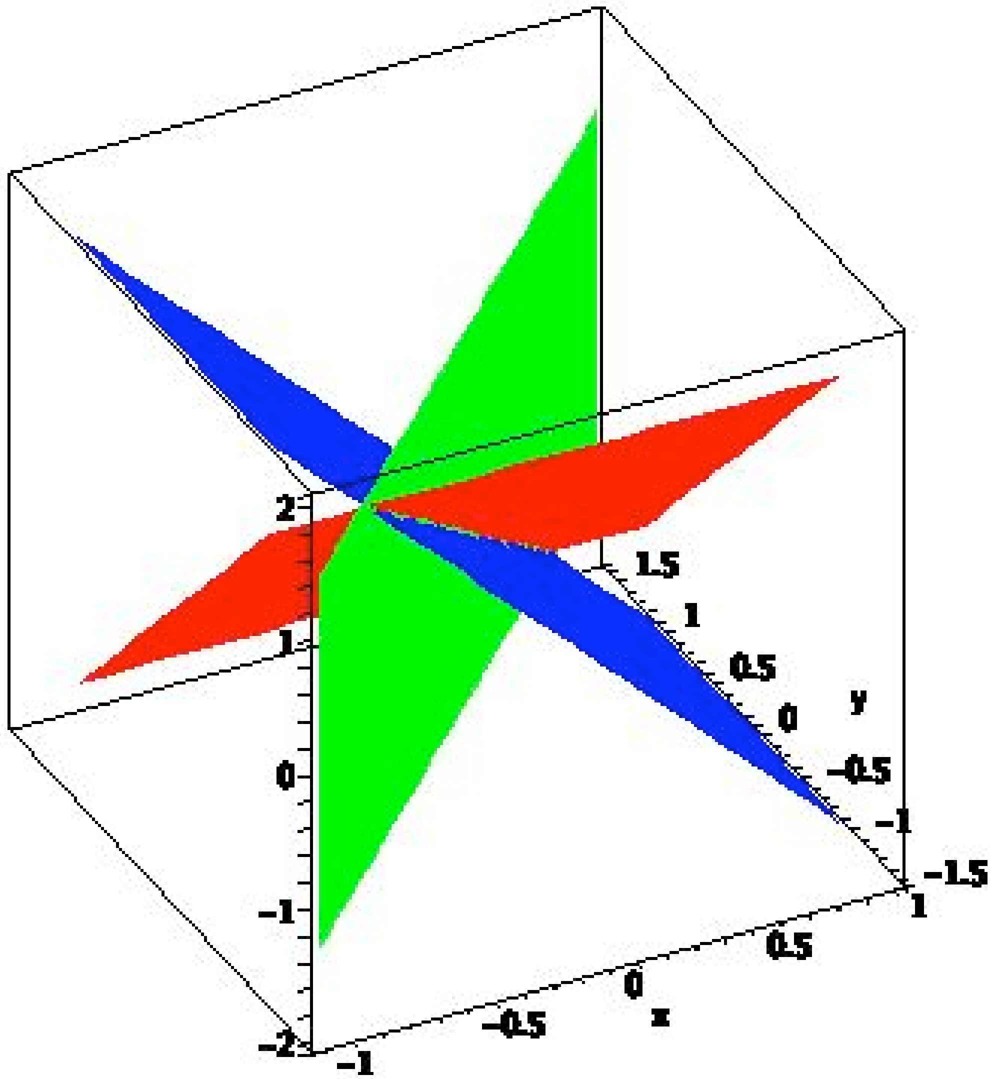
$$P_j := (\beta^j)^\perp = \mathsf{span}(e_j, \gamma),$$

the form Λ is given by a symbol whose derivatives blow up according to (8) when ξ approaches P_j.

We use a Whitney decomposition that simultaneously resolves all of the three singular sets P_j independently of the representation we are using.



The bad set



(Pictures courtesy of E. Gavosto)

The conditions

$$T(1,1) = T^{*1}(1,1) = T^{*2}(1,1) = 0 \quad (10)$$

are needed to control the behavior on the bad set

$$S := \bigcup_{i=1}^{3} P_j$$

and eliminate the potential singularities of $\widehat{\Lambda}$ on it.

At least formally, the conditions (10) translate into

$$\langle \widehat{K}_j(-\xi_j, 0) | \widehat{f}_j(\xi_j) \rangle = 0,$$

and hence $\widehat{K}_j(u, 0) = 0$, so $\widehat{\Lambda}$ vanishes in some sense on S.

In the case of the BHT (7) takes the simpler form

$$\Lambda(f_1, f_2, f_3) = \langle \delta(\alpha \cdot \xi) \widehat{K}(-\beta \cdot \xi) | \widehat{f}_1 \otimes \widehat{f}_2 \otimes \widehat{f}_3(\xi) \rangle,$$

which vanishes if $\widehat{f}_1 \otimes \widehat{f}_2 \otimes \widehat{f}_3$ is supported away from $\alpha^{\perp}$ (which also contains $\langle \gamma \rangle$).

In the x-independent case, $\widehat{\Lambda}$ is supported on $\alpha^\perp$ and is possibly singular only on the line $\langle\gamma\rangle$. A one-parameter family of boxes, i.e. cubes, is then used for the BHT.

In the x-dependent case, however, a two-parameter family of boxes in $\mathbb{R}^3$ is used to decompose the complement of the bad set S.

We then estimate $\Lambda(f_1, f_2, f_3)$ by

$$\sum_{\omega\in\Omega} \sum_{I_1,I_2,I_3} |\Lambda(\phi_{I_1,\omega_1}, \phi_{I_2,\omega_2}, \phi_{I_3,\omega_3})| \prod_{i=1}^{3} |\langle f_i, \phi_{I_i,\omega_i}\rangle|$$

where we need to obtain good (almost orthogonal) estimates on the coeff.

$$|\Lambda(\phi_{I_1,\omega_1}, \phi_{I_2,\omega_2}, \phi_{I_3,\omega_3})|$$

etc...

(Do not miss the next talk...:)

Thanks for your attention!