NULL SPACE AND NULLITY

1. An Example

Recall that a system is homogeneous if it is of the form

$$A\mathbf{x} = \mathbf{0}.$$

The solution set here goes by the name "the null space of A," or $N(A)$. We can speed up the row operations a little if we notice that when doing row operations on

$$[A\,|\,\mathbf{0}]$$

the last column never changes. We can do operation just on A, as long as we remember when converting rows back to equations to put zeros "$= 0$" on the right.

Example 1. Find the null space of A, where

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 & 0 \\ 4 & 2 & 0 & 0 & 3 \\ 1 & 1 & 1 & -2 & 1 \\ 2 & 2 & 0 & 0 & 2 \\ 1 & 1 & 2 & -4 & 1 \end{bmatrix},$$

and find a basis for this null space.

We need to solve

$$A\mathbf{x} = \mathbf{0}.$$

Using just A, we use row ops to find reduced row echelon form:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 & 0 \\ 4 & 2 & 0 & 0 & 3 \\ 1 & 1 & 1 & -2 & 1 \\ 2 & 2 & 0 & 0 & 2 \\ 1 & 1 & 2 & -4 & 1 \end{bmatrix}$$

$$\begin{array}{c} \text{R3} - 4\text{R1} \to \text{R3} \\ \text{R4} - \text{R1} \to \text{R4} \\ \text{R5} - 2\text{R1} \to \text{R5} \\ \text{R6} - \text{R1} \to \text{R6} \end{array} \Bigg\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -4 & 0 \end{bmatrix}$$

$$\text{R2} \leftrightarrow \text{R3} \Bigg\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -4 & 0 \end{bmatrix}$$

$$\begin{array}{c} \text{R4} - \text{R3} \to \text{R4} \\ \text{R6} - 2\text{R3} \to \text{R6} \end{array} \Bigg\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -4 & 0 \end{bmatrix}$$

$$\begin{array}{c} \text{R4} - \text{R3} \to \text{R4} \\ \text{R6} - 2\text{R3} \to \text{R6} \end{array} \Bigg\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-\tfrac{1}{2}\text{R2} \to \text{R2} \Bigg\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \tfrac{1}{2} \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{R1} - \text{R2} \to \text{R1} \Bigg\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \tfrac{1}{2} \\ 0 & 1 & 0 & 0 & -\tfrac{1}{2} \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so we have the equations

$$(1) \qquad \begin{array}{rcl} x_1 + \tfrac{1}{2}x_5 &=& 0 \\ x_2 + \tfrac{1}{2}x_5 &=& 0 \\ x_3 - 2x_4 &=& 0 \end{array}$$

There are two free variables; we set $x_4 = r$ and $x_5 = s$ and find that $N(A)$ is the set of all $\mathbf{x}$ where

$$\mathbf{x} = \begin{bmatrix} -\frac{1}{2}s \\ -\frac{1}{2}s \\ 2r \\ r \\ s \end{bmatrix}.$$

To find a basis, we exand this formula to

$$\mathbf{x} = r \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

This shows that $N(A)$ is spanned by two vectors:

$$N(A) = \text{span} \left(\begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix} \right).$$

In fact, these two vectors are linearly independent; to see this, we assume

$$x \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Amoung the five equations are these two:

$$\begin{aligned} 1x + 0y &= 0 \\ 0x + 1y &= 0 \end{aligned}$$

and so the only solution is $(x, y) = (0, 0)$. In conclusion,

$$\begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for $N(A)$.

2. Null Space vs Nullity

Sometimes we only want to know how big the solution set is to
$$A\mathbf{x} = \mathbf{0}.$$

Definition 1. The *nullity* of a matrix A is the dimension of its null space:
$$\text{nullity}(A) = \dim(N(A)).$$

It is easier to find the nullity than to find the null space. This is because
The number of free variables (in the solved equations) equals the nullity of A.

3. Nullity vs Basis for Null Space

There is a general method to find a *basis for the null space*:
(a) Use row operations to reduced echelon form.
(b) Write out corresponding simplified equations for the null space.
(c) Set first free variable to 1, the others to 0. This solution $\mathbf{x}$ is a basis element.
(d) Repeat (b), so each free variable takes its trun being the only 1.

Here is our first method to find *just the nullity*:
(a) Use row operations *just to echelon form*.
(b) Count the non-pivot columns.

This distinction is important in some applications, as the nullity is therefore faster to compute.

Example 2. Find a basis for $N(A)$ where
$$A = \begin{bmatrix} 2 & 2 & -4 & 3 & -9 & 1 \\ 1 & 1 & -2 & -3 & 0 & 2 \\ 3 & 3 & -6 & -3 & -6 & 2 \end{bmatrix}.$$

Getting srarted with row ops:

$$\begin{bmatrix} 2 & 2 & -4 & 3 & -9 & 1 \\ 1 & 1 & -2 & -3 & 0 & 2 \\ 3 & 3 & -6 & -3 & -6 & 2 \end{bmatrix}$$
$$\downarrow R1 \leftrightarrow R2$$
$$\begin{bmatrix} 1 & 1 & -2 & -3 & 0 & 2 \\ 2 & 2 & -4 & 3 & -9 & 1 \\ 3 & 3 & -6 & -3 & -6 & 2 \end{bmatrix}$$
$$\begin{array}{l} R2 - 2R1 \to R2 \\ R3 - 3R1 \to R3 \end{array}$$
$$\begin{bmatrix} 1 & 1 & -2 & -3 & 0 & 2 \\ 0 & 0 & 0 & 9 & -9 & -3 \\ 0 & 0 & 0 & 6 & -6 & -4 \end{bmatrix}$$
$$\begin{array}{l} \frac{1}{3}R2 \to R2 \\ \frac{1}{3}R3 \to R3 \end{array}$$

$$\begin{bmatrix} 1 & 1 & -2 & -3 & 0 & 2 \\ 0 & 0 & 0 & 3 & -3 & -1 \\ 0 & 0 & 0 & 3 & -3 & -2 \end{bmatrix}$$
$$\downarrow R3 - R2 \to R3$$
$$\begin{bmatrix} 1 & 1 & -2 & -3 & 0 & 2 \\ 0 & 0 & 0 & 3 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

We know the nullity (it is 3) but the question asks us to really find the null space. We press on:

$$\begin{bmatrix} 1 & 1 & -2 & -3 & 0 & 2 \\ 0 & 0 & 0 & 3 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & -2 & 0 & -3 & 0 \\ 0 & 0 & 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\downarrow -\text{R3} \to \text{R3}$ $\qquad\qquad\qquad\qquad \downarrow \frac{1}{3}\text{R2} \to \text{R2}$

$$\begin{bmatrix} 1 & 1 & -2 & -3 & 0 & 2 \\ 0 & 0 & 0 & 3 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & -2 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\left.\begin{array}{c} \text{R1} - 2\text{R3} \to \text{R1} \\ \text{R2} + \text{R3} \to \text{R2} \end{array}\right\downarrow$

$$\begin{bmatrix} 1 & 1 & -2 & -3 & 0 & 0 \\ 0 & 0 & 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\downarrow \text{R1} + \text{R2} \to \text{R1}$

So we have three free variables, and the equations become

$$\begin{array}{ccccccccc} x_1 & + & x_2 & - & 2x_3 & & & - & 3x_5 & & = & 0 \\ & & & & & & x_4 & - & x_5 & & = & 0 \\ & & & & & & & & & x_6 & = & 0 \end{array}$$

and so the null space is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -r + 2s + 3t \\ r \\ s \\ t \\ t \\ 0 \end{bmatrix}.$$
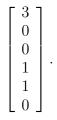
Set $r = 1$, $s = 0$ and $t = 0$ and we get

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Set $r = 0$, $s = 1$ and $t = 0$ and we get

$$\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Set $r = 0$, $s = 0$ and $t = 1$ and we get

$$\begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

A basis is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

4. More Finding Just the Nullity

Here is our second method to find *just the nullity*:

(a) Use row *and column* operations *just to echelon form.*
(b) Count the non-pivot columns.

This is similar to the situation with determinants. For now, you must trust me that this makes sense. We'll understand later why column ops can alter the null space but can't change its dimension.

N.B. The result after the column operations will be *in row echelon form* but will *not* always be in the *echelon form of A*. It will have the correct number of pivots.

Example 3. Find Nullity(A) where A is again the matrix

$$A = \begin{bmatrix} 2 & 2 & -4 & 3 & -9 & 1 \\ 1 & 1 & -2 & -3 & 0 & 2 \\ 3 & 3 & -6 & -3 & -6 & 2 \end{bmatrix}.$$

Now we can also do column operations, and can stop as soon as we see where the pivots will be.

$$\begin{bmatrix} 2 & 2 & -4 & 3 & -9 & 1 \\ 1 & 1 & -2 & -3 & 0 & 2 \\ 3 & 3 & -6 & -3 & -6 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & -4 & 3 & -9 & 2 \\ 0 & 0 & 0 & 18 & -18 & 0 \\ 0 & -1 & 2 & -9 & 12 & -1 \end{bmatrix}$$

$$\downarrow \begin{matrix} \text{R2} - 2\text{R1} \to \text{R2} \\ \text{R3} - 2\text{R1} \to \text{R3} \end{matrix} \qquad \downarrow \text{R2} \leftrightarrow \text{R3}$$

$$\begin{bmatrix} 2 & 2 & -4 & 3 & -9 & 1 \\ -3 & -3 & 6 & -9 & 18 & 0 \\ -1 & -1 & 2 & -9 & 12 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & -4 & 3 & -9 & 2 \\ 0 & -1 & 2 & -9 & 12 & -1 \\ 0 & 0 & 0 & 18 & -18 & 0 \end{bmatrix}$$

$$\downarrow \text{C1} \leftrightarrow \text{C6}$$

$$\begin{bmatrix} 1 & 2 & -4 & 3 & -9 & 2 \\ 0 & -3 & 6 & -9 & 18 & -3 \\ 0 & -1 & 2 & -9 & 12 & -1 \end{bmatrix}$$

$$\downarrow \text{R2} - 3\text{R3} \to \text{R2}$$

This is enough for us to see there are three non-pivot rows, so

$$\text{nullity}(A) = 3.$$

Notice that the pivots don't end up in the same locations as before, so we have lost track of $N(A)$.

There is often no advantage of being able to do column and row operations together; it is too complicated to figure out what to do next. In specific calculations, in in the theory of rank, it tells us a lot.