THINGS THAT ARE/AREN'T VECTOR SPACES

1. What isn't

Suppose we must make a mathematical model in a lab where the professor is obsessed with volleyball. He does not what his 6-vectors drawn with the numbers in a row. Instead, he wants them to put them is a two by three formation, with arrows drawn on the outside; one up, one down:

$$\uparrow \begin{vmatrix} 1 & 3 \\ 0 & 4 \\ 5 & 6 \end{vmatrix} \downarrow$$

This looks ok, but she wants the elements to "rotate like before a serve" before any operation is performed. These are so much not like vectors, we insist she call them "volltors."

To clarify things, we lab assistance decide on the precise definitions

$$\uparrow \begin{vmatrix} a_1 & a_2 \\ a_6 & a_3 \\ a_5 & a_4 \end{vmatrix} \downarrow + \uparrow \begin{vmatrix} b_1 & b_2 \\ b_6 & b_3 \\ b_5 & b_4 \end{vmatrix} \downarrow = \uparrow \begin{vmatrix} a_6 + b_6 & a_1 + b_1 \\ a_5 + b_5 & a_2 + b_2 \\ a_4 + b_4 & a_3 + b_3 \end{vmatrix} \downarrow$$

and

$$r \uparrow \begin{vmatrix} a_1 & a_2 \\ a_6 & a_3 \\ a_5 & a_4 \end{vmatrix} \downarrow = \uparrow \begin{vmatrix} ra_1 & ra_2 \\ ra_6 & ra_3 \\ ra_5 & ra_4 \end{vmatrix} \downarrow$$

Have we created a vector space?

Certainly, we are ok with closure (C1 and C2) since we stick with six real numbers and put the opposing arrows around them. The trouble will be in the axioms.

What about A1? We need

$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$$

to hold for all volltors $\mathbf{v}$ an $\mathbf{w}$. We can gather evidence about A1 by picking specific values for the numbers in $\mathbf{v}$ and $\mathbf{w}$, but if we want to prove the axiom is true we will eventually need to work with generic volltors. The best start if often just to give labels to all the parts: let

$$\mathbf{v} = \uparrow \begin{vmatrix} a_1 & a_2 \\ a_6 & a_3 \\ a_5 & a_4 \end{vmatrix} \downarrow$$

and

$$\mathbf{w} = \uparrow \begin{vmatrix} b_1 & b_2 \\ b_6 & b_3 \\ b_5 & b_4 \end{vmatrix} \downarrow .$$

Then

$$\mathbf{v} + \mathbf{w} = \left| \uparrow \begin{array}{cc} a_6 + b_6 & a_1 + b_1 \\ a_5 + b_5 & a_2 + b_2 \\ a_4 + b_4 & a_3 + b_3 \end{array} \downarrow \right|$$

and

$$\mathbf{w} + \mathbf{v} = \left| \uparrow \begin{array}{cc} b_6 + a_6 & b_1 + a_1 \\ b_5 + a_5 & b_2 + a_2 \\ b_4 + a_4 & b_3 + a_3 \end{array} \downarrow \right| .$$

These are equal, since the a_j and b_j are just reals, so

$$a_j + b_j = b_j + a_j.$$

So

$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$$

for all volltors $\mathbf{v}$ and $\mathbf{w}$.

On to A2. If we a expecting this one to hold as well (the last one held, and this professor always says "every trend starts with a single instance, so assume every single instance ends with a trend") we go for the generic elements. For this axiom, we need vollters $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{q}$, so let

$$\mathbf{v} = \left| \uparrow \begin{array}{cc} a_1 & a_2 \\ a_6 & a_3 \\ a_5 & a_4 \end{array} \downarrow \right| ,$$

$$\mathbf{w} = \left| \uparrow \begin{array}{cc} b_1 & b_2 \\ b_6 & b_3 \\ b_5 & b_4 \end{array} \downarrow \right|$$

and

$$\mathbf{q} = \left| \uparrow \begin{array}{cc} c_1 & c_2 \\ c_6 & c_3 \\ c_5 & c_4 \end{array} \downarrow \right| .$$

We compute

$$(\mathbf{v} + \mathbf{w}) + \mathbf{q} = \left| \uparrow \begin{array}{cc} a_6 + b_6 & a_1 + b_1 \\ a_5 + b_5 & a_2 + b_2 \\ a_4 + b_4 & a_3 + b_3 \end{array} \downarrow \right| + \left| \uparrow \begin{array}{cc} c_1 & c_2 \\ c_6 & c_3 \\ c_5 & c_4 \end{array} \downarrow \right|$$

$$= \left| \uparrow \begin{array}{cc} a_5 + b_5 + c_6 & a_6 + b_6 + c_1 \\ a_4 + b_4 + c_5 & a_1 + b_1 + c_2 \\ a_3 + b_3 + c_4 & a_2 + b_2 + c_3 \end{array} \downarrow \right|$$

and

$$\mathbf{v} + (\mathbf{w} + \mathbf{q}) = \left| \uparrow \begin{array}{cc} a_1 & a_2 \\ a_6 & a_3 \\ a_5 & a_4 \end{array} \downarrow \right| + \left| \uparrow \begin{array}{cc} b_6 + c_6 & b_1 + c_1 \\ b_5 + c_5 & b_2 + c_2 \\ b_4 + c_4 & b_3 + c_3 \end{array} \downarrow \right|$$

$$= \left| \uparrow \begin{array}{cc} a_6 + b_5 + c_5 & a_1 + b_6 + c_6 \\ a_5 + b_4 + c_4 & a_2 + b_1 + c_1 \\ a_4 + b_3 + c_3 & a_3 + b_2 + c_2 \end{array} \downarrow \right| .$$

These came out with different formulas, but do the represent different numbers? Not always, but in many cases. It only takes one case as a counter-example the shows A2 false. So let's plug in some numbers until we have verified that these can come out different.

$$\left(\begin{bmatrix} 1 & 2 \\ 6 & 3 \\ 5 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}\right) + \begin{bmatrix} 1 & 6 \\ 2 & 5 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 8 \\ 8 & 8 \\ 8 & 8 \end{bmatrix}$$

while

$$\begin{bmatrix} 1 & 2 \\ 6 & 3 \\ 5 & 4 \end{bmatrix} + \left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 6 \\ 2 & 5 \\ 3 & 4 \end{bmatrix}\right) = \begin{bmatrix} 10 & 4 \\ 10 & 4 \\ 10 & 10 \end{bmatrix}.$$

The set of volltors, with these operations, does not form a vector space.

2. WHAT ISN'T, REVISITED

In the last section we could have worked with 3 by 2 matrices, scalar multiplication and an alternative "plus." We could denote this $\oplus$:

$$\begin{bmatrix} a_1 & a_2 \\ a_6 & a_3 \\ a_5 & a_4 \end{bmatrix} \oplus \begin{bmatrix} b_1 & b_2 \\ b_6 & b_3 \\ b_5 & b_4 \end{bmatrix} = \begin{bmatrix} a_6 + b_6 & a_1 + b_1 \\ a_5 + b_5 & a_2 + b_2 \\ a_4 + b_4 & a_3 + b_3 \end{bmatrix}.$$

With this notation, what we did above showed that for any two such matrices A and B,

$$A \oplus B = B \oplus A,$$

but that for some specific matrices A_1, B_2 and C_3,

$$(A_1 \oplus B_2) \oplus C_3 \neq A_1 \oplus (B_2 \oplus C_3).$$

Since A2 fails, we don't really care about A1 or the other axioms. Just showing that A2 fails is enough to prove that $\mathbb{R}^{3,2}$ with $\oplus$ and $\otimes$.

When we say $\mathbb{R}^{3,2}$ is a vector space, we are not being very precise. What we mean to say is that $\mathbb{R}^{3,2}$ becomes a vector space when you equip it the the standard addition and standard scalar multiplication.

3. WHAT IS

An important vector space is that consisting of all sequences of real numbers. For example, there is

$$2, 3, 4, \ldots$$

given by the formula

$$a_n = n + 1, \quad (n \geq 1).$$

(Let's assume sequences start at $n = 1$.) If we are refering to this entire sequence, not any specific term in it, we typically call the sequence a. A better way to show this is

$$a = (2, 3, 4, \ldots).$$

Sequences add in essentially the same way as do matrices. So

$$(1, 0, 1, 0, \ldots) + (1, 1, 1, 1 \ldots) = (2, 1, 2, 1, \ldots).$$

The general rule is

$$(a_1, a_2, a_3, \ldots) + (b_1, b_2, b_3, \ldots) = (a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots)$$

Sequences can can be scaled, which is like turning up or down the volume if working with a positive scalar:

$$ra = (ra_1, ra_2, ra_3, \ldots).$$

We can write $r \cdot a$ instead of ra, if we are so moved. Putting all this together, we have what looks rather similar to row vectors. We have such equations as

$$2(1, 0, 1, 0, \ldots) + 3(1, 1, 1, 1 \ldots) = (5, 3, 5, 3, \ldots).$$

Problem15, page 123, leaves it to the dear reader to verify that the set of all sequences, taken with these two operations, is a vector space. Here I will work out just a little of this.

The author is using $\{\}$ to denote a sequence. I hate this, but there you are. So $\{n^2\}$ is shorthand for the sequence of squares,

$$(1, 4, 9, 25, \cdots).$$

Here is a proof that A6 holds. For any sequence a, and any scalars r and s, we can compute

$$\begin{aligned}
(r + s)a &= (r + s)(a_1, a_2, a_3, \ldots) \\
&= ((r + s)a_1, (r + s)a_2, (r + s)a_3, \ldots)
\end{aligned}$$

and

$$\begin{aligned}
ra + sa &= r(a_1, a_2, a_3, \ldots) + s(a_1, a_2, a_3, \ldots) \\
&= (ra_1, ra_2, ra_3, \ldots) + (sa_1, sa_2, sa_3, \ldots) \\
&= (ra_1 + sa_1, ra_2 + sa_2, ra_3 + sa_3, \ldots) \\
&= ((r + s)a_1, (r + s)a_2, (r + s)a_3, \ldots)
\end{aligned}$$

and so

$$(r + s)a = ra + sa.$$

I admit this is dull, and it can be hard to keep straight distinctions such as the term a_n in a sequence versus the entire sequence a. You must admit, however, that streams of real numbers get added together and scaled all the time. So this example is at the heart of a lot of science.

Scientists call a sequence a "time series" just to annoy mathematicians.