A MATRIX REPRESENTATION EXAMPLE

Example 1. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is the linear transformation defined by

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a \\ b+c \end{bmatrix}.$$

If $\mathcal{B}$ is the ordered basis $[\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3]$ and $\mathcal{C}$ is the ordered basis $[\mathbf{c}_1, \mathbf{c}_2]$, where

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{b}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{b}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

and

$$\mathbf{c}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ \mathbf{c}_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix},$$

what is the matrix representation of T with respect to $\mathcal{B}$ and $\mathcal{C}$?

We need to solve one equation for each basis vector in the domain V; one for each column of the transformation matrix A.

For Column 1: We must solve

$$r\begin{bmatrix} 2 \\ 1 \end{bmatrix} + s\begin{bmatrix} 3 \\ 0 \end{bmatrix} = T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)$$

which is

$$r\begin{bmatrix} 2 \\ 1 \end{bmatrix} + s\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

There can be only one solution (since $\mathcal{C}$ is a basis (!)) and this equation is easy enough to work in one's head:

$$1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + (-\frac{1}{3})\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The first column is thus determined as

$$A = \begin{bmatrix} 1 & * & * \\ -\frac{1}{3} & * & * \end{bmatrix}.$$

For Column 2: We must solve

$$r\begin{bmatrix} 2 \\ 1 \end{bmatrix} + s\begin{bmatrix} 3 \\ 0 \end{bmatrix} = T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We just worked this equation. The solution is

$$\begin{aligned} r &= 1 \\ s &= -\frac{1}{3} \end{aligned}$$

We have the second column:

$$A = \begin{bmatrix} 1 & 1 & * \\ -\frac{1}{3} & -\frac{1}{3} & * \end{bmatrix}.$$

For Column 3: We must solve

$$r \begin{bmatrix} 2 \\ 1 \end{bmatrix} + s \begin{bmatrix} 3 \\ 0 \end{bmatrix} = T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

As a linear system this is

$$\begin{array}{ccccc} 2r & + & 3s & = & 0 \\ r & & & = & 2 \end{array}.$$

The solution is

$$\begin{aligned} r &= 2 \\ s &= -\frac{4}{3}. \end{aligned}$$

We have the third column and so our answer:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{4}{3}. \end{bmatrix}.$$