A BASIS MAKES AN ISOMORPHISM

Here are some critical facts rather buried in the book.

- An isomorphism is a linear transformation $T : V \to W$ for which $T(\mathbf{v}) = \mathbf{w}$ has a unique solution in $\mathbf{v}$ for any given $\mathbf{w}$.
- Given an isomorphism $T : V \to W$, every linear algebra question in V can be translated into a question in W by applying T to all the vectors in the question.
- A basis $\mathcal{B}$ for V determines an isomorphism
$$S : V \to \mathbb{R}^n$$
by "taking coordinates"
$$S(\mathbf{v}) = [\mathbf{v}]_{\mathcal{B}}.$$

If you can find "easy" coordinates based on an "easy" basis V, you should consider translating every question in V into a question in $\mathbb{R}^n$ by taking coordinates.

Next, an example of what I mean by the second point. You might just ignore the proof. The overall idea is that an isormorhism $T : V \to W$ means that W is "just like" V in the context of any question involving addition and scalar multiplication.

Lemma 1. *Suppose there is an isomorhism $T : V \to W$. The vector $\mathbf{v}_0$ is in the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ if and only if $\mathbf{w}_0$ is in the span of $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ where $\mathbf{w}_j = T(\mathbf{v}_j)$.*

Proof. If
$$\mathbf{v_0} = r\mathbf{v}_1 + s\mathbf{v}_2 + t\mathbf{v}_3$$
then
$$\begin{aligned}\mathbf{w_0} &= T(\mathbf{v}_0)\\ &= T(r\mathbf{v}_1 + s\mathbf{v}_2 + t\mathbf{v}_3)\\ &= rT(\mathbf{v}_1) + sT(\mathbf{v}) + tT(\mathbf{v}_3)\\ &= \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3.\end{aligned}$$
If
$$\mathbf{w_0} = r\mathbf{w}_1 + s\mathbf{w}_2 + t\mathbf{w}_3$$
then
$$T(\mathbf{v}_0) = w_0$$
and
$$\begin{aligned}T(r\mathbf{w}_1 + s\mathbf{w}_2 + t\mathbf{w}_3) &= rT(\mathbf{v}_1) + sT(\mathbf{v}) + tT(\mathbf{v}_3)\\ &= r\mathbf{w}_1 + s\mathbf{w}_2 + t\mathbf{w}_3\\ &= w_0.\end{aligned}$$
The equation $T(\mathbf{x}) = \mathbf{w}_0$ has only one solution so
$$\mathbf{v}_0 = r\mathbf{w}_1 + s\mathbf{w}_2 + t\mathbf{w}_3.$$

□

More "translation" lemmas.

Lemma 2. *Suppose there is an isomorhism $T : V \to W$. The vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ form a basis for V if and only if $T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_n)$ form a basis for W.*

Suppose there is an isomorhism $T : V \to W$. For vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ in V,

$$dim(span(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)) = dim(span(T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_n))).$$

1. Using an Easy Basis

Example 1. What is the dimension of the span of

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}?$$

Ouch. The standard casting out will take forever. Here's a better way.

An easy ordered basis for the vector space of upper-triangular 3-by-3 matrices is

$$\mathcal{B} : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let's use coordinates with respect to $\mathcal{B}$. Since

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

will have coordinate vector

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

and so forth, we need only settle the translated question: what is the dimension of the span of

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \\ 1 \\ 5 \\ 0 \end{bmatrix}?$$

So let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 2 & 1 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 5 & 5 \\ 1 & 1 & 0 & 1 & 6 & 0 \end{bmatrix}.$$

According to Matlab, the row reduced form of A is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

So we have six pivots, and the answer is 6.