ONE DETERMINANT, MANY COMPUTATIONS

Let's compute det(A) is several ways.

$$A = \begin{bmatrix} 0 & 1 & 4 & 32 \\ 3 & 1 & 4 & 40 \\ 6 & 0 & 2 & 32 \\ 3 & 1 & 6 & 60 \end{bmatrix}$$

1. Row op to Row Echelon

Our first method is to follow Gaussian Elimination algorithm until we have row echelon form. As long as we find four pivots we just take the product of the factors.

$$\begin{bmatrix} 0 & 1 & 4 & 32 \\ 3 & 1 & 4 & 40 \\ 6 & 0 & 2 & 32 \\ 3 & 1 & 6 & 60 \end{bmatrix}$$
$$\Big\downarrow \text{R1} \leftrightarrow \text{R3} \qquad \text{factor: } -1$$
$$\begin{bmatrix} 6 & 0 & 2 & 32 \\ 3 & 1 & 4 & 40 \\ 0 & 1 & 4 & 32 \\ 3 & 1 & 6 & 60 \end{bmatrix}$$
$$\Big\downarrow \tfrac{1}{6}\text{R1} \rightarrow \text{R1} \qquad \text{factor: } 6$$
$$\begin{bmatrix} 1 & 0 & \frac{1}{3} & \frac{16}{3} \\ 3 & 1 & 4 & 40 \\ 0 & 1 & 4 & 32 \\ 3 & 1 & 6 & 60 \end{bmatrix}$$
$$\Big\downarrow \begin{matrix} \text{R2} - 3\text{R1} \rightarrow \text{R2} & \text{factor: } 1 \\ \text{R4} - 3\text{R1} \rightarrow \text{R4} & \text{factor: } 1 \end{matrix}$$
$$\begin{bmatrix} 1 & 0 & \frac{1}{3} & \frac{16}{3} \\ 0 & 1 & 3 & 24 \\ 0 & 1 & 4 & 32 \\ 0 & 1 & 5 & 44 \end{bmatrix}$$
$$\Big\downarrow \begin{matrix} \text{R3} - \text{R2} \rightarrow \text{R3} & \text{factor: } 1 \\ \text{R4} - \text{R2} \rightarrow \text{R4} & \text{factor: } 1 \end{matrix}$$
$$\begin{bmatrix} 1 & 0 & \frac{1}{3} & \frac{16}{3} \\ 0 & 1 & 3 & 24 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 2 & 20 \end{bmatrix}$$
$$\Big\downarrow \text{R4} - 2\text{R3} \rightarrow \text{R4} \qquad \text{factor: } 1$$
$$\begin{bmatrix} 1 & 0 & \frac{1}{3} & \frac{16}{3} \\ 0 & 1 & 3 & 24 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$
$$\Big\downarrow \tfrac{1}{4}\text{R4} \rightarrow \text{R4} \qquad \text{factor: } 4$$
$$\begin{bmatrix} 1 & 0 & \frac{1}{3} & \frac{16}{3} \\ 0 & 1 & 3 & 24 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

so

$$\begin{aligned} \det(A) &= (-1) \times 6 \times 4 \\ &= -24. \end{aligned}$$

2. Row op to Upper Triangular

It is easier to settle for upper triangular. In this case, it means we can avoid fractions:

$$\left[\begin{array}{cccc} 0 & 1 & 4 & 32 \\ 3 & 1 & 4 & 40 \\ 6 & 0 & 2 & 32 \\ 3 & 1 & 6 & 60 \end{array}\right]$$

$$\Big\downarrow \mathrm{R1} \leftrightarrow \mathrm{R3} \qquad \text{factor: } -1$$

$$\left[\begin{array}{cccc} 6 & 0 & 2 & 32 \\ 3 & 1 & 4 & 40 \\ 0 & 1 & 4 & 32 \\ 3 & 1 & 6 & 60 \end{array}\right]$$

$$\Big\downarrow \tfrac{1}{2}\mathrm{R1} \rightarrow \mathrm{R1} \qquad \text{factor: } 2$$

$$\left[\begin{array}{cccc} 3 & 0 & 1 & 16 \\ 3 & 1 & 4 & 40 \\ 0 & 1 & 4 & 32 \\ 3 & 1 & 6 & 60 \end{array}\right]$$

$$\Big\downarrow \begin{array}{ll} \mathrm{R2} - \mathrm{R1} \rightarrow \mathrm{R2} & \text{factor: } 1 \\ \mathrm{R4} - \mathrm{R1} \rightarrow \mathrm{R4} & \text{factor: } 1 \end{array}$$

$$\left[\begin{array}{cccc} 3 & 0 & 1 & 16 \\ 0 & 1 & 3 & 24 \\ 0 & 1 & 4 & 32 \\ 0 & 1 & 5 & 44 \end{array}\right]$$

$$\Big\downarrow \begin{array}{ll} \mathrm{R3} - \mathrm{R2} \rightarrow \mathrm{R3} & \text{factor: } 1 \\ \mathrm{R4} - \mathrm{R2} \rightarrow \mathrm{R4} & \text{factor: } 1 \end{array}$$

$$\left[\begin{array}{cccc} 3 & 0 & 1 & 16 \\ 0 & 1 & 3 & 24 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 2 & 20 \end{array}\right]$$

$$\Big\downarrow \mathrm{R4} - 2\mathrm{R3} \rightarrow \mathrm{R4} \qquad \text{factor: } 1$$

$$\left[\begin{array}{cccc} 3 & 0 & 1 & 16 \\ 0 & 1 & 3 & 24 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

so

$$\begin{aligned}\det(A) &= (-1)\times 2\times 3\times 4\\ &= -24.\end{aligned}$$

3. Do-over: Row op to Upper Triangular

There is no rule that says you must write down each row operation. By not doing so, you shorten your write-up. The trade-off is that it is harder to follow.

Let's redo the last computation, so with exactly the same row operations. I am not going to list the factors of the row ops. Instead, I'll keep a constant in from so we have equality of determinant at every stage.

$$\begin{aligned}\begin{vmatrix}0&1&4&32\\3&1&4&40\\6&0&2&32\\3&1&6&60\end{vmatrix} &= -\begin{vmatrix}6&0&2&32\\3&1&4&40\\0&1&4&32\\3&1&6&60\end{vmatrix}\\ &= -2\begin{vmatrix}3&0&1&16\\3&1&4&40\\0&1&4&32\\3&1&6&60\end{vmatrix}\\ &= -2\begin{vmatrix}3&0&1&16\\0&1&3&24\\0&1&4&32\\0&1&5&44\end{vmatrix}\\ &= -2\begin{vmatrix}3&0&1&16\\0&1&3&24\\0&0&1&8\\0&0&2&20\end{vmatrix}\\ &= -2\begin{vmatrix}3&0&1&16\\0&1&3&24\\0&0&1&8\\0&0&0&4\end{vmatrix}\\ &= -2\times 3\times 1\times 1\times 4\\ &= -24.\end{aligned}$$

4. Expanding Along a Row

We can expand on the second row, at each step:

$$
\begin{aligned}
&\begin{vmatrix} 0 & 1 & 4 & 32 \\ 3 & 1 & 4 & 40 \\ 6 & 0 & 2 & 32 \\ 3 & 1 & 6 & 60 \end{vmatrix} \\
&= -3 \begin{vmatrix} \cancel{0} & 1 & 4 & 32 \\ \cancel{3} & \cancel{1} & \cancel{4} & \cancel{40} \\ \cancel{6} & 0 & 2 & 32 \\ \cancel{3} & 1 & 6 & 60 \end{vmatrix} + \begin{vmatrix} 0 & \cancel{1} & 4 & 32 \\ \cancel{3} & \cancel{1} & \cancel{4} & \cancel{40} \\ 6 & \cancel{0} & 2 & 32 \\ 3 & \cancel{1} & 6 & 60 \end{vmatrix} - 4 \begin{vmatrix} 0 & 1 & \cancel{4} & 32 \\ \cancel{3} & \cancel{1} & \cancel{4} & \cancel{40} \\ 6 & 0 & \cancel{2} & 32 \\ 3 & 1 & \cancel{6} & 60 \end{vmatrix} + 40 \begin{vmatrix} 0 & 1 & 4 & \cancel{32} \\ \cancel{3} & \cancel{1} & \cancel{4} & \cancel{40} \\ 6 & 0 & 2 & \cancel{32} \\ 3 & 1 & 6 & \cancel{60} \end{vmatrix} \\
&= -3 \begin{vmatrix} 1 & 4 & 32 \\ 0 & 2 & 32 \\ 1 & 6 & 60 \end{vmatrix} + \begin{vmatrix} 0 & 4 & 32 \\ 6 & 2 & 32 \\ 3 & 6 & 60 \end{vmatrix} - 4 \begin{vmatrix} 0 & 1 & 32 \\ 6 & 0 & 32 \\ 3 & 1 & 60 \end{vmatrix} + 40 \begin{vmatrix} 0 & 1 & 4 \\ 6 & 0 & 2 \\ 3 & 1 & 6 \end{vmatrix} \\
&= -3 \left(2 \begin{vmatrix} \cancel{1} & 4 & 32 \\ \cancel{0} & \cancel{2} & \cancel{32} \\ \cancel{1} & 6 & 60 \end{vmatrix} - 32 \begin{vmatrix} 1 & 4 & \cancel{32} \\ \cancel{0} & \cancel{2} & \cancel{32} \\ 1 & 6 & \cancel{60} \end{vmatrix} \right) \\
&\quad + \left(-6 \begin{vmatrix} \cancel{0} & 4 & 32 \\ \cancel{6} & \cancel{2} & \cancel{32} \\ \cancel{3} & 6 & 60 \end{vmatrix} + 2 \begin{vmatrix} 0 & \cancel{4} & 32 \\ \cancel{6} & \cancel{2} & \cancel{32} \\ 3 & \cancel{6} & 60 \end{vmatrix} - 32 \begin{vmatrix} 0 & 4 & \cancel{32} \\ \cancel{6} & \cancel{2} & \cancel{32} \\ 3 & 6 & \cancel{60} \end{vmatrix} \right) \\
&\quad -4 \left(-6 \begin{vmatrix} \cancel{0} & 1 & 32 \\ \cancel{6} & \cancel{0} & \cancel{32} \\ \cancel{3} & 1 & 60 \end{vmatrix} - 32 \begin{vmatrix} 0 & 1 & \cancel{32} \\ \cancel{6} & \cancel{0} & \cancel{32} \\ 3 & 1 & \cancel{60} \end{vmatrix} \right) \\
&\quad +40 \left(-6 \begin{vmatrix} \cancel{0} & 1 & 4 \\ \cancel{6} & \cancel{0} & \cancel{2} \\ \cancel{3} & 1 & 6 \end{vmatrix} - 2 \begin{vmatrix} 0 & 1 & \cancel{4} \\ \cancel{6} & \cancel{0} & \cancel{2} \\ 3 & 1 & \cancel{6} \end{vmatrix} \right) \\
&= -3 \left(2 \begin{vmatrix} 1 & 32 \\ 1 & 60 \end{vmatrix} - 32 \begin{vmatrix} 1 & 4 \\ 1 & 6 \end{vmatrix} \right) + \left(-6 \begin{vmatrix} 4 & 32 \\ 6 & 60 \end{vmatrix} + 2 \begin{vmatrix} 0 & 32 \\ 3 & 60 \end{vmatrix} - 32 \begin{vmatrix} 0 & 4 \\ 3 & 6 \end{vmatrix} \right) \\
&\quad -4 \left(-6 \begin{vmatrix} 1 & 32 \\ 1 & 60 \end{vmatrix} - 32 \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} \right) + 40 \left(-6 \begin{vmatrix} 1 & 4 \\ 1 & 6 \end{vmatrix} - 2 \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} \right) \\
&= -3\,(2(28) - 32(2)) + (-6(48) + 2(-96) - 32(-12)) \\
&\quad -4\,(-6(28) - 32(-3)) + 40\,(-6(2) - 2(-3)) \\
&= -3\,(-8) + (-96) - 4\,(-72) + 40\,(-6) \\
&= -24
\end{aligned}
$$

It is not customary to write down the matrices with the crossed out row and column. I just thought one complete example would help you.

5. Mixing Row and Column Operations with Expansion

Column operations work just like row operations for determinants. So if all you want is the determinant, and you see patterns in the columns, take advantage. The idea is to create lots of zeros so expanding is not so painful.

In this computation, I do:

- a type II column operation ($\frac{1}{3}C1 \rightarrow$ C1)
- a type III row operation
- type III column operation
- expand along the second row
- expand along the second row:

$$\begin{aligned}
\begin{vmatrix} 0 & 1 & 4 & 32 \\ 3 & 1 & 4 & 40 \\ 6 & 0 & 2 & 32 \\ 3 & 1 & 6 & 60 \end{vmatrix} &= 3\begin{vmatrix} 0 & 1 & 4 & 32 \\ 1 & 1 & 4 & 40 \\ 2 & 0 & 2 & 32 \\ 1 & 1 & 6 & 60 \end{vmatrix} \\
&= 3\begin{vmatrix} 0 & 1 & 4 & 32 \\ 1 & 0 & 0 & 8 \\ 2 & 0 & 2 & 32 \\ 1 & 1 & 6 & 60 \end{vmatrix} \\
&= 3\begin{vmatrix} 0 & 1 & 4 & 32 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 2 & 16 \\ 1 & 1 & 6 & 52 \end{vmatrix} \\
&= -3\begin{vmatrix} 1 & 4 & 32 \\ 0 & 2 & 16 \\ 1 & 6 & 52 \end{vmatrix} \\
&= -3\left(\begin{vmatrix} 2 & 16 \\ 6 & 52 \end{vmatrix} + \begin{vmatrix} 4 & 32 \\ 2 & 16 \end{vmatrix}\right) \\
&= -3\,(8-0)) \\
&= -24
\end{aligned}$$