CASTING OUT, SLOW AND FAST

1. Slow but general

Leon barely covers the standard "casting out" method. It is an algorithm that works in any vector space V.

Input: A finite set of vectors $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$ contained in V
Ouput: A subset B_m of B with

$$\text{span}(B_m) = \text{span}(B)$$

so that B_m is a basis for span(B).
Method:

(1) Find the first nonzero vector $b_{k'}$ in B. Let $B_{k'} = \{b_{k'}\}$ and let $k = k' + 1$. (Cast out leading zero vectors).
(2)
  (a) If b_k is in span(B_{k-1}) then let $B_k = B_{k-1}$. (Cast out).
  (b) If b_k is not in span(B_{k-1}) then let $B_k = B_{k-1} \cup \{\mathbf{b}_k\}$. (Keep)
(3) Let $k = k + 1$.
(4) If $k \leq m$ repeat 2 and 3.

Example 1. Suppose the vector space is that of all 2-by-2 matrices. Find a subset of $\{\mathbf{b_1}, \ldots, \mathbf{b}_5\}$ that is a basis for the span of $\{\mathbf{b_1}, \ldots, \mathbf{b}_5\}$, where

$$\mathbf{b}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ \mathbf{b}_2 = \begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 1 & 5 \\ 3 & 1 \end{bmatrix}, \ \mathbf{b}_4 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \mathbf{b}_5 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Using the casting our method, we throw out $\mathbf{b}_1$ and start with $\{\mathbf{b}_2\}$:

$$B_2 = \left\{ \begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix} \right\}.$$

The $k = 3$ **Step:** Is $\mathbf{b}_3$ in the span of B_2? I.e., can we solve

$$r_1 \begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 3 & 1 \end{bmatrix}?$$

Well, certainly not. We get $r_1 = 1$ and $4r_1 = 5$ and can't do both. The new vector (matrix) is not in the span of the current set, so add it:

$$B_3 = \left\{ \begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 5 \\ 3 & 1 \end{bmatrix} \right\}.$$

The $k = 4$ **Step:** Is $\mathbf{b}_4$ in the span of B_3? I.e., can we solve

$$r_1 \begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix} + r_2 \begin{bmatrix} 1 & 5 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}?$$

This is the same as the equations

$$\begin{array}{rcrcl} r_1 & + & r_2 & = & 1 \\ 4r_1 & + & 5r_2 & = & 0 \\ 3r_1 & + & 3r_2 & = & 3 \\ r_1 & + & r_2 & = & 1 \end{array}$$

which we try to solve

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 5 & 0 \\ 3 & 3 & 3 \\ 1 & 1 & 1 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The system has a solution. So we cast out $\mathbf{b}_4$:

$$B_4 = B_3 = \left\{ \begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 5 \\ 3 & 1 \end{bmatrix} \right\}.$$

The $k = 5$ **Step:** Is $\mathbf{b}_5$ in the span of B_4? I.e., can we solve

$$r_1 \begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix} + r_2 \begin{bmatrix} 1 & 5 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}?$$

This is the same as the equations

$$\begin{array}{rcrcl} r_1 & + & r_2 & = & 1 \\ 4r_1 & + & 5r_2 & = & 1 \\ 3r_1 & + & 3r_2 & = & 1 \\ r_1 & + & r_2 & = & 1 \end{array}$$

which we try to solve

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 5 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

The system does not have a solution. So we keep $\mathbf{b}_5$:

$$B_5 = \left\{ \begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 5 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}.$$

Our answer is B_5.

Now another example, in $\mathbb{R}^5$.

Example 2. Find a subset of $\{\mathbf{b_1}, \ldots, \mathbf{b}_5\}$ that is a basis for the span of

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \ \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \ \mathbf{b}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{b}_5 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 3 \end{bmatrix}.$$

Using the casting our method, we start with

$$B_1 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

The $k = 2$ **Step:** Is $\mathbf{b}_2$ in the span of B_1? Can we solve

$$r_1 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}?$$

No. So $\mathbf{b}_2$ is a keeper:

$$B_2 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

The $k = 3$ **Step:** Is $\mathbf{b}_3$ in the span of B_2? Can we solve

$$r_1 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} + r_2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}?$$

Jumping to the augemented matrix of the resulting equations, we have

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -4 \\ 0 & 0 & 0 \\ 0 & -1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and so this has a solution. We cast-out $\mathbf{b}_3$:

$$B_3 = B_2 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

The $k = 4$ **Step:** Is $\mathbf{b}_4$ in the span of B_3? Can we solve

$$r_1 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} + r_2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} ?$$

Jumping to the augemented matrix of the resulting equations, we have

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and so there is no a solution. We keep $\mathbf{b}_4$:

$$B_4 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

The $k = 5$ **Step:** Is $\mathbf{b}_5$ in the span of B_5? Can we solve

$$r_1 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} + r_2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + r_3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 3 \end{bmatrix}?$$

Jumping to the augemented matrix of the resulting equations, we have

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 2 \\ 1 & 1 & 0 & 3 \\ 2 & 1 & 0 & 2 \\ 1 & 1 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and so there is a solution. We don't use $\mathbf{b}_5$:

$$B_5 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

This is our answer.

Well *that* took forever.

2. Fast, but only in $\mathbb{R}^n$

If you want to do casting out in the standard Euclidian spaces, there is a much faster way.

(1) Put all the $\mathbf{b}_k$ in a matrix A :

$$A = [\mathbf{b}_1\ \mathbf{b}_2\ \cdots\ \mathbf{b}_m].$$

(2) Row reduce until you can see the pivot positions:

$$A \to \cdots \to U.$$

(3) Keep b_j if there is a a pivot position in column j of U.

Notice that the pivots in U tell you which columns of A to use.

Let's repeat the last example.

Example 3. Find a subset of $\{\mathbf{b_1}, \ldots, \mathbf{b}_5\}$ that is a basis for the span of

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \mathbf{b}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{b}_5 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 3 \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 2 & 1 & 0 & 0 & 2 \\ 1 & 1 & 2 & 0 & 3 \\ 2 & 1 & 0 & 0 & 2 \\ 1 & 1 & 2 & 0 & 3 \end{bmatrix}$$

and do the row ops needed:

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 2 & 1 & 0 & 0 & 2 \\ 1 & 1 & 2 & 0 & 3 \\ 2 & 1 & 0 & 0 & 2 \\ 1 & 1 & 2 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 0 & -1 & -4 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & -1 & -4 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 0 & -1 & -4 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$

There are pivots positions in columns $1, 2$ and 4 and so we keep $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_4$: Answer

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

So why did that work? I'll not give a proof, but notice the following:

In our first solution, we tossed $\mathbf{b}_3$ because

$$-2\begin{bmatrix}1\\2\\1\\2\\1\end{bmatrix}+4\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}=\begin{bmatrix}2\\0\\2\\0\\2\end{bmatrix}$$

If we had applied the slow algorithm to the columns of U we would have found

$$-2\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}+4\begin{bmatrix}1\\-1\\0\\0\\0\end{bmatrix}=\begin{bmatrix}2\\-4\\0\\0\\0\end{bmatrix}$$

and so would have tossed the third column for exactly the same "reason."

3. As a test for Linear Independence

- Suppose you have m vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ in V. If the casting out algorithm tells you to *keep all* these vectors, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ was *linearly independent* to begin with. Otherwise, it wasn't.
- Suppose you know that V has dimension n and you have n vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. If the casting out algorithm tells you to *keep all* the vectors, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ was *a basis* for V to begin with. Otherwise, it wasn't.

Casting out is such a pain in general that these statements are often little help. But, in $\mathbb{R}^n$, casting out is fast, so these really help. In terms of rank:

- Suppose you have m vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ in $\mathbb{R}^n$. If

$$\operatorname{rank}\begin{bmatrix}| & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_m \\ | & | & & |\end{bmatrix}=m$$

  then $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are linearly independent. If the rank is less, they are dependent.
- Suppose you have n vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in $\mathbb{R}^n$. If

$$\operatorname{rank}\begin{bmatrix}| & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & |\end{bmatrix}=n$$

  then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are a basis for $\mathbb{R}^n$. If the rank is less, they are not.

Recall: If you have more or less than n vectors in $\mathbb{R}^n$, then you don't have a basis.

In terms of row ops, if $U = \operatorname{rref}(A)$:

- The columns of A are linearly independent $\Leftrightarrow$ there's a pivot in every column of U.
- The columns of A are a basis of $\mathbb{R}^n$ $\Leftrightarrow$ A is n-by-n and there's a pivot in every column of U.

Putting together a bunch of stuff:

Theorem 1. *The columns of an n-by-n matrix A are a basis for $\mathbb{R}^n$*
$\Leftrightarrow$ *there's a pivot in every column of $U = rref(A)$*
$\Leftrightarrow$ *A is invertible*
$\Leftrightarrow$ *$rank(A) = n$*
$\Leftrightarrow$ *$det(A) \neq 0$.*