CHANGE OF BASIS EXAMPLE

1. A Small Example

In this section we will deal with the vector space

$$V = \left\{ \begin{bmatrix} \alpha & \beta \\ 0 & \delta \end{bmatrix} \middle| \alpha, \beta \text{ and } \delta \text{ are real} \right\}.$$

In other words, V is the vector space of all upper-triangular two-by-two matrices.

Here is (without proof) an ordered basis $\mathcal{B} = \{B_1, B_2, B_2\}$ of V :

$$B_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}.$$

Here is another basis, $\mathcal{C} = \{C_1, C_2, C_3\}$ of V :

$$C_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Let's find A, the transition matrix from $\mathcal{B}$ coordinates to $\mathcal{C}$ coordinates.

We need to solve three equations, since V is three-dimensional. They all of are the form

$$r\text{new}_1 + s\text{new}_2 + t\text{new}_3 = \text{old}_*.$$

Task 1: Find the one solution to

$$r \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + s \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + t \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Task 2: Find the one solution to

$$r \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + s \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + t \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}.$$

Task 3: Find the one solution to

$$r \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + s \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + t \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}.$$

So here's the work:

Doing Task 1: From

$$r \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + s \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + t \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

we get

$$\begin{bmatrix} r & r \\ 0 & r \end{bmatrix} + \begin{bmatrix} s & s \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} t & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and so

$$\begin{bmatrix} r+s+t & r+s \\ 0 & r \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{array}{ccccccc} r & + & s & + & t & = & 1 \\ r & + & s & & & = & 1 \\ r & & & & & = & 1 \\ & & & & 0 & = & 0 \end{array}$$

We can solve this in our heads (the last equation is not helpful):

$$r = 1,\ s = 0,\ t = 0.$$

These numbers form the **1st** column of A :

$$A = \begin{bmatrix} 1 & ? & ? \\ 0 & ? & ? \\ 0 & ? & ? \end{bmatrix}.$$

Doing Task 2: From

$$r \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + s \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + t \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$$

we get

$$\begin{bmatrix} r & r \\ 0 & r \end{bmatrix} + \begin{bmatrix} s & s \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} t & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$$

and so

$$\begin{bmatrix} r+s+t & r+s \\ 0 & r \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$$

$$\begin{array}{ccccccc} r & + & s & + & t & = & 0 \\ r & + & s & & & = & 2 \\ r & & & & & = & 2 \end{array}$$

Let's use the array stuff to solve:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

so

$$\begin{aligned} r &= 2, \\ s &= 0 \\ t &= -2 \end{aligned}$$

These numbers tell us the **2**nd column of A :

$$A = \begin{bmatrix} 1 & 2 & ? \\ 0 & 0 & ? \\ 0 & -2 & ? \end{bmatrix}.$$

Doing Task 3: From

$$r\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + s\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + t\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$$

we get

$$\begin{bmatrix} r & r \\ 0 & r \end{bmatrix} + \begin{bmatrix} s & s \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} t & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$$

and so

$$\begin{bmatrix} r+s+t & r+s \\ 0 & r \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\begin{array}{rcl} r + s + t &=& 0 \\ r + s &=& 0 \\ r &=& 3 \end{array}$$

Let's do this in our heads:

$$\begin{array}{rcl} r &=& 3, \\ s &=& -3 \\ t &=& 0 \end{array}$$

These numbers tell us the **3**rd column of A :

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -3 \\ 0 & -2 & 0 \end{bmatrix}.$$

Done!

Clearly this had a lot of repetitive stuff, and there are much shorter was to do this.

2. A bigger Example

In this section we will deal with the vector space

$$V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} \;\middle|\; \begin{array}{c} \text{the } x_j \text{ are real} \\ x_5 = x_1 \\ x_6 = x_2 \\ x_7 = x_3 \\ x_8 = x_4 \end{array} \right\}.$$

In other words, V is the vector space of all vectors in 8-space whose last four elements equal the first four.

Here is (without proof) an ordered basis $\mathcal{B} = \{b_1, b_2, b_2, b_3\}$ of V :

$$b_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \ b_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \ b_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \ b_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Here is another basis, $\mathcal{C} = \{c_1, c_2, c_3, c_4\}$ of V :

$$c_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ c_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ c_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ c_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Let's find A, the transition matrix from $\mathcal{B}$ coordinates to $\mathcal{C}$ coordinates.

We need to solve four equations of are the form

$$r_1\text{new}_1 + r_2\text{new}_2 + r_3\text{new}_3 + r_4\text{new}_4 = \text{old}_*.$$

Equation(s) 1:

$$r_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r_3 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + r_4 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

so

$$\begin{array}{ccccccccc} r_1 & & & + & r_3 & + & r_4 & = & 1 \\ & & & & r_3 & + & r_4 & = & -1 \\ & & r_2 & & & + & r_4 & = & 1 \\ & & & & & & r_4 & = & -1 \end{array}.$$

In our heads, we solve:

$$\begin{aligned} r_1 &= 2 \\ r_2 &= 2 \\ r_3 &= 0 \\ r_4 &= -1 \end{aligned}$$

So far, we have figured out:

$$A = \begin{bmatrix} 2 & ? & ? & ? \\ 2 & ? & ? & ? \\ 0 & ? & ? & ? \\ -1 & ? & ? & ? \end{bmatrix}.$$

Equation(s) 2:

$$r_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r_3 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + r_4 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

so

$$\begin{array}{ccccccccc} r_1 & & & + & r_3 & + & r_4 & = & 1 \\ & & & & r_3 & + & r_4 & = & 0 \\ & & r_2 & & & + & r_4 & = & -1 \\ & & & & & & r_4 & = & 0 \end{array}.$$

In our heads, we solve:

$$\begin{aligned} r_1 &= 1 \\ r_2 &= -1 \\ r_3 &= 0 \\ r_4 &= 0 \end{aligned}$$

So far, we have figured out:

$$A = \begin{bmatrix} 2 & 1 & ? & ? \\ 2 & -1 & ? & ? \\ 0 & 0 & ? & ? \\ -1 & 0 & ? & ? \end{bmatrix}.$$

Equation(s) 3:

$$r_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r_3 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + r_4 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

so

$$\begin{array}{cccccccc} r_1 & & + & r_3 & + & r_4 & = & 0 \\ & & & r_3 & + & r_4 & = & 1 \\ & r_2 & & & + & r_4 & = & 0 \\ & & & & & r_4 & = & -1 \end{array}.$$

In our heads, we solve:

$$\begin{aligned} r_1 &= -1 \\ r_2 &= 1 \\ r_3 &= 2 \\ r_4 &= -1 \end{aligned}$$

So far, we have figured out:

$$A = \begin{bmatrix} 2 & 1 & -1 & ? \\ 2 & -1 & 1 & ? \\ 0 & 0 & 2 & ? \\ -1 & 0 & -1 & ? \end{bmatrix}.$$

Equation(s) 4:

$$r_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r_3 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + r_4 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

so

$$\begin{array}{cccccccc} r_1 & & + & r_3 & + & r_4 & = & 1 \\ & & & r_3 & + & r_4 & = & 1 \\ & r_2 & & & + & r_4 & = & 1 \\ & & & & & r_4 & = & 1 \end{array}.$$

In our heads, we solve:

$$\begin{aligned} r_1 &= 0 \\ r_2 &= 0 \\ r_3 &= 0 \\ r_4 &= 1 \end{aligned}$$

The answer is that the transition matrix is

$$A = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 2 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{bmatrix}.$$