Math 316 Overview

0. **General.**
   - Integration Rules (Int by parts, some trig identities, partial fraction)
   - Differentiation (eg, finding maximum values)
   - Determining limiting/dominant behaviour in certain limits
   - Linear operators and superposition principle.
1. **First Order Differential Equations- Chapter 1.**
   - Linear vs. nonlinear DEs
   - Infinitely many solutions to differential equation, vs. Unique solution to an initial value problem
   - Direction fields, integral curves
   - Separable Equations
   - Linear Equations
   - Autonomous equations, phase line, equilibria, stability
   - Domain of existence/uniqueness. Linear vs Nonlinear.
2. **Mathematical Models and Numerical Solutions - Chapter 2.**
   - Numerical Solution (Euler's Method and Improved Euler) **(not on final exam)**
   - Applications (population models, acceleration and damping, simple physical principles)
3. **Second Order Linear Differential Equations - Chapter 3.**
   - Homogeneous case (characteristic equation, 3 cases depending on discriminant)
   - Nonhomogeneous case:
     - method of undetermined coefficients
     - method of variation of parameters
   - Springs: undamped, over- or underdamped, forced, resonance, beats
4. **Laplace Transforms - Chapter 7** (our table of Laplace Transforms will be provided)
   - Definition, convergence of improper integral
   - Solving IVP with Laplace Transform
   - Shifts. Step Functions. Impulse function.
5. **2x2 Linear Systems $x' = Ax$ - Chapter 5.**
   - Rewriting 2nd order linear equations as 2x2 linear system
   - The eigenvalue problem
   - Find the general solution (3 cases, depending on eigenvalues)
   - Solve IVP
   - Phase Portraits in all cases. Determine behaviour of components.
   - Classify equilibria and their stability.
6. **Nonlinear systems**
   - Equilibria. Linearization. Classify equilibria and their stability.
   - Completing the phase portrait
   - Applications (Pendulum, Competing species)

All exam and homework problems are also good review problems.

Remember: Credit will be given for work shown to obtain the answer. Work must be shown completely and clearly.

Sample Problems

1. Solve the following differential equations or IVPs: For the IVPs, state the interval of validity

(a) $y' = xy^3$, $y(0) = 1$ (b) $y' = xy^3$, $y(0) = 0$
(c) $y' = xy^3(1+x^2)^{-1/2}$ (d) $dy/dx = k - ry$, $y(0) = k/(2r)$ (where k, r are const
(e) $y' = 2xy + x$, $y(0) = 0$ (f) $\frac{dy}{dt} = .01(1 - y/3)y$, $y(0) = 10$.

Answers: (a) $y = 1/\sqrt{1-x^2}, |x| < 1$ *(b)* $y = 0$ *(c)* $y = \pm 1/\sqrt{C - 2\sqrt{1+x^2}}$
(d) $y = \frac{k}{r}(1 - \frac{1}{2}e^{-rt})$ *(e)* $y = \frac{1}{2}(e^{-x^2} - 1)$ *(f)* $y(t) = \frac{30}{10 - 7e^{-.01t}}$

2. Solve the following differential equations and determine the behavior of the solutions as $t \to \infty$. Determine the value of a that separates one type of solution behavior from another type.

   (a) $ty' + (1+t)y = 2te^{-t}$, with the initial condition $y(1) = a$, $t > 0$ *(Answer:* $y = te^{-t} + (ea-1)e^{-t}/t$. $y \to 0$ *as* $t \to \infty$ *for all a.)*

   (b) $y' - y = 1 + 3\sin t$, with the initial condition $y(0) = a$. *(Answer:* $y = -1 - \frac{3}{2}(\cos t + \sin t) + (a + \frac{5}{2})e^t$. *If* $a > -\frac{5}{2}$ *then* $\lim_{t\to\infty} y = \infty$. *If* $a < -\frac{5}{2}$ *then* $\lim_{t\to\infty} y = -\infty$. *The value of a that separates the types of behaviour is* $a = -5/2$.*)*

3. (a) State conditions for which local existence and uniqueness of a first order initial value problem $y' = f(x, y)$, $y(t_0) = y_0$ is guaranteed. *(Answer: Theorem 2.4.2)*

   (b) State conditions for which the interval of existence of a unique solution to an initial value problem is known. *(Answer: Theorem 2.4.1 for first order IVPs, Theorem 3.2.1. In particular, such results are only known for linear DEs.)*

   (c) Without solving the differential equation, determine the interval of validity of the solution.

$$y' + \frac{1}{t^2 - 9}y = \frac{\ln|20 - 4t|}{t^2 - 9}, \quad y(4) = -3.$$

   (Answer: Valid on $(3, 5)$.*)*

4. For the following autonomous equations, (i) sketch the phase line and some solutions in the ty-plane, (ii) state all equilibria and classify as asymptotically stable or unstable, (iii) compare with your particular solutions in problem 1 where applicable.
(a) $\frac{dy}{dt} = .01(1 - y/3)y$
(b) $\frac{dy}{dt} = k - ry$
(c) $y' = -k(y-1)^2$, where $k > 0$
(d) $y' = y^2(y^2-1)^2$, where $k > 0$
(e) $y' = y(1-y^2)^2$, where $k > 0$
Partial answers: (a) $y = 0$ *unstable,* $y = 3$ *stable.*

5. Determine the value(s) of α such that the solution to the following differential equations goes to zero as $t \to \infty$, and also those value(s) of α such that the solution is unbounded as $t \to \infty$.

(a) $y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0$. *[Answer: since $y = c_1 e^{\alpha t} + c_2 e^{(\alpha-1)t}$, we see that if $\alpha < 0$, then $\lim_{t\to\infty} y = 0$, and if $\alpha > 0$, then y is unbounded. If $\alpha = 0$, then $\lim_{t\to\infty} y = c_1$.]*

(b) $y'' - (3 - \alpha)y' - 2(\alpha - 1)y = 0$. *[Answer: since $y = c_1 e^{2t} + c_2 e^{(1-\alpha)t}$, we see that y is unbounded as $t \to \infty$ for all α.]*

6. Application: Population models.

(a) State the exponential growth model for a population $P(t)$ and the solution if $P(0) = P_0$. Explain the interpretation of all parameters

(b) State the logistic growth model for a population $P(t)$. Explain the interpretation of all parameters. Draw the phase line. Draw a set of solutions in the $P - t$ plane. Describe the behaviour of the solutions as $t \to \infty$ and how it depends on the initial condition. You may assume $P \geq 0$.

(c) A pond in a fish hatchery starts out with 1000 fish, which is also the capacity κ of the pond. The fish are known to grow at the rate of $r = 0.01$/day. They are harvested at the rate of h fish/day.

i. A simple exponential growth model for the fish population is given by

$$\frac{dP}{dt} = rP - h$$

With this model, how many fish h_{max} can be caught at most per day without causing the population to go extinct in the long run? What will the fish population $P(t)$ be in this case? Explain your answer. (Hint: draw a phase line showing the equilibrium solution P_{equil} of the ODE, and explain what happens if $P(0) > P_{equil}$, $P(0) = P_{equil}$, $P(0) < P_{equil}$.)

ii. Another model for the fish hatchery is given the logistic growth law

$$\frac{dP}{dt} = rP(1 - \frac{P}{\kappa}) - h$$

With this model, how many fish h_{max} can be caught at most per day without causing the population to go extinct in the long run? What will the fish population $P(t)$ be in this case? Explain your answer. (Hint: sketch P' vs P for various values of h, as well as the corresponding phase line, then find the critical value h_{max} that still gives you a nonzero equilibrium population.)

iii. Which model do you think is more realistic?

(d) Give an example of a 2nd order nonlinear system modeling the population of two competing species and explain the contribution of each term in the model

(e) The interaction of two species one a predator, the other the prey, whose populations are given by $x(t)$ and $y(t)$ is modeled by

$$\frac{dx}{dt} = x(1 - 0.5y)\ , \frac{dy}{dt} = y(-0.75 + 0.25x)\ ,$$

where x, y are non-negative. Explain the contribution of each term in the model. Determine all equilibria, analyze their stability, draw a phase portrait and describe the interaction between the species. You may assume that the nonlinear perturbation of the linear center has closed trajectories about the equilibrium (that is, stable or unstable spirals are not possible in this case). Remember, this is not a guarantee of all linearized systems! Some linearizations will have stable centers while the actual system will be unstable. Here this means that the life cycle of both species are periodic, and exhibit a cyclic variation.
(Answer: The equilibria are $(0, 0)$ and $(3, 2)$. $(0, 0)$ is unstable and $(3, 2)$ is a stable center (in the linearization).

7. Application: pendulum

   (a) State the equation modeling the motion of a nonlinear pendulum and describe the meaning of all variables, all parameters, and all terms in the equation.

   (b) Find all equilibria and classify them based on the value of the damping parameter. (You may assume that in the undamped case, linear centers are nonlinear centers. Beware that this is not always the case! It is only so in this case since the energy of the system is conserved)

   (c) Sketch a phase portrait in each case. Be able to describe the motion of the pendulum in each case.

8. *Method of Undetermined Coefficients.* Determine the general solution to the following equations. In (d), solve the IVP. In each case be able to describe the behaviour as $t \to \infty$.

   (a) $y'' + 4y = 5t^2e^t$ (Answer: $y(t) = c_1 \cos(2t) + c_2 \sin(2t) + (t^2 - 4/5t - 2/25)e^t$)

   (b) $y'' + 2y' - 3y = 5\sin(3t)$ (Answer: $y(t) = c_1 et + c_2 e^{-3t} - 1/6\cos(3t) - 1/3\sin(3t)$)

   (c) $y'' - y' - 2y = 3e^{-t}$ (Answer: $y(t) = c_1 e^{2t} + c_2 e^{-t} - te^{-t}$)

   (d) $y'' - 2y' + y = e^{-t}$, $y(0) = 1$, $y'(0) = 3$

9. State the form of the general solution without solving for the unknown coefficients.

   (a) $y'' + 4y' = e^{2t} + t^2$ *(Answer:* $y = c_1 \cos(2t) + c_2 \sin(2t) + Ae^{2t} + (Bt^2 + Ct + d)$*)*

   (b) $y'' - 4y' + 4y = te^t + e^{2t} + 4$ *(Answer:* $y = c_1 e^{2t} + c_2 te^{2t} + At^2e^{2t} + (Bt + C)e^t + D$*)*

   (c) $y'' + 2y' + 2y = 3e^{-t} + 4e^{-t}\sin t + t^2 \cos t$ *Answer:* $y = e^{-t}(c_1 \cos t + c_2 \sin t) + Ae^{-t} + te^{-t}(B\sin t + C\cos t) + (Dt^2 + Et + F)\sin t + (Gt^2 + Ht + I)\cos t$

   (d) $y'' + 4y = t(1 + \sin 2t)$ *Answer:* $y = c_1 \cos 2t + c_2 \sin 2t + At + B + t(C\sin t + D\cos t)$

10. Solve the differential equation for the velocity of a falling body,

$$mv' + \gamma v + mg = 0$$

where $m, \gamma, g > 0$, using four different methods: Integrating factor, Separation of variables, Undetermined coefficients, Laplace Transform (assume initial velocity v_0) (Answer: HW 7 solutions)

11. *Damped springs, unforced.* The position of a certain spring-mass system satisfies the initial value problem

$$2y'' + \gamma y' + y = 0\ , \quad y(0) = 2, \quad y'(0) = 0$$

For which values of the coefficient γ is the system undamped? underdamped? critically damped? overdamped? (Answers: $\gamma = 0$, $0 < \gamma < 2\sqrt{2}$, $\gamma = 2\sqrt{2}$, $\gamma > 2\sqrt{2}$ respectively)

12. Undamped spring, forced. A mass weighing 4lb stretches a spring 1.5 in. The mass is given a positive displacement of 2 in from this equilibrium position and released with no initial velocity. Assume that there is no damping and that the mass is acted on by an external force of $2\cos 3t$ lb.

   (a) Formulate the initial value poblem describing the motion of the mass. (Answer: $x''/8 + 32x = 2\cos(3t)\ ,\ x(0) = 1/6\ ,\ x'(0) = 0$)

   (b) Solve using Laplace Transorms (Answer: hw 7 solution)

   (c) If the external force is replaced by a force $4\sin \omega t$ of frequency ω, find the value of ω for which resonance occurs. State the form of the forced solution and sketch a graph of it. (Answer: $\omega = 16$, $x_f = t(A\cos 16t + B\sin 16t)$)

13. Show that the Laplace transform is a linear operator

14. Use the Laplace transform to solve the following differential equations:

    (a) $y'' + 3y' + 2y = \delta(t-5) + u_{10}(t)\ , y(0) = 0\ , \quad y'(0) = \frac{1}{2}\ .$

    (b) $y'' + 4y = \sin(3t)$, $y(0) = y'(0) = 0$. (Answer: $y(t) = 3/10\sin(2t) - 1/5\sin(3t)$)

15. Find the general solution to

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} = \mathbf{x}$$

Answer:

$$\mathbf{x}(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\left(te^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{2t}\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$$

16. The system

$$\mathbf{x}' = \begin{pmatrix} \alpha & 2 \\ -2 & -0 \end{pmatrix}\mathbf{x}$$

contains a parameter α. Describe how the solutions depend qualitatively on α; in particular, find the critical values of α at which the qualitative behavior of the trajectories in the phase plane change markedly.
(Answer: $\alpha \in (-\infty, -4)$:stable node, $\alpha \in (-4, 0)$:stable spiral, $\alpha \in (0, 4)$:unstable spiral, $\alpha \in (4, \infty)$:unstable node. Stability changes at the critical value of $\alpha = 0$.)

17. Nonlinear phase portraits. For the following nonlinear autonomous systems $\mathbf{x}' = F(\mathbf{x})$ of differential equations: Draw the nullclines, find all equilibria, linearize about each equilibrium, classify he equilibria, plot a possible phaseportrait consistent with the information you found.

    (a) $x' = x - y$, $y' = x^2 - 4$ (Answer: saddle and unstable spiral)

    (b) $x' = x - y^3$, $y' = x^3 - y$ (Answer: 2 saddles and 1 unstable node)

    (c) $x' = -y - x^3$, $y' = x$ (Answer: 1 center or unstable spiral or stable spiral)

    (d) $\frac{dx}{dt} = 3x - x^2 - xy$, $\quad \frac{dy}{dt} = y + y^2 - 3xy$. (Answer: $(0,0)$ unstable, $(1,2)$ unstable, $(3,0)$ asymptotically stable, $(0,-1)$ unstable)