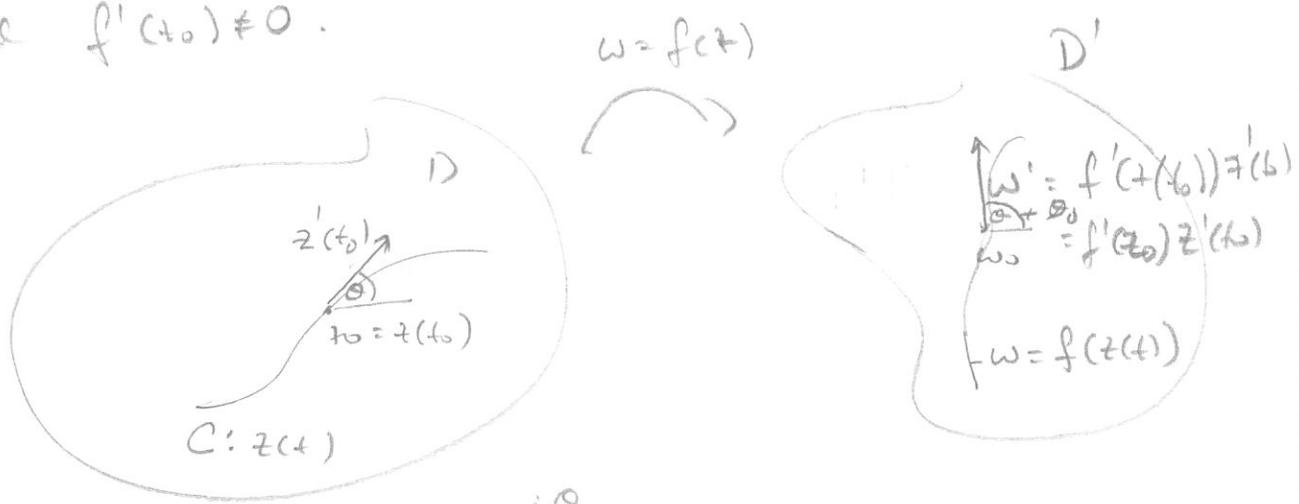


Lect 4.1 Conformal Maps, Harmonic Functions

Conformal Maps

Suppose f is analytic in D
and $f'(z_0) \neq 0$.



Note: If $z'(t_0) = r e^{i\theta}$
 the tangent vector w' to curve $f(z(t))$
 (image of C) is

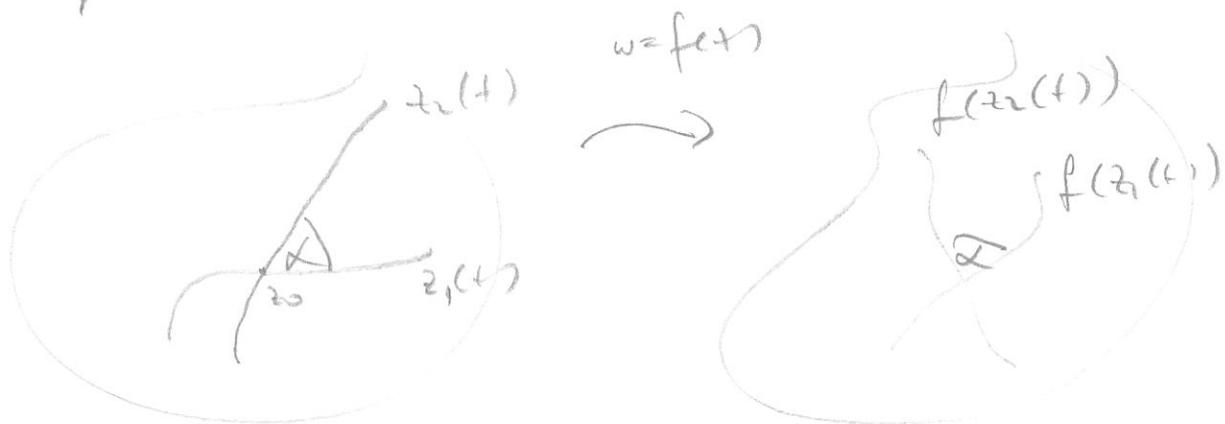
$$f'(z_0)z'(t_0) = r_0 e^{i\theta_0} r e^{i\theta} = r_0 r e^{i(\theta_0 + \theta)}$$

which is a rotation by

$$\theta_0 = \arg(f'(z_0))$$

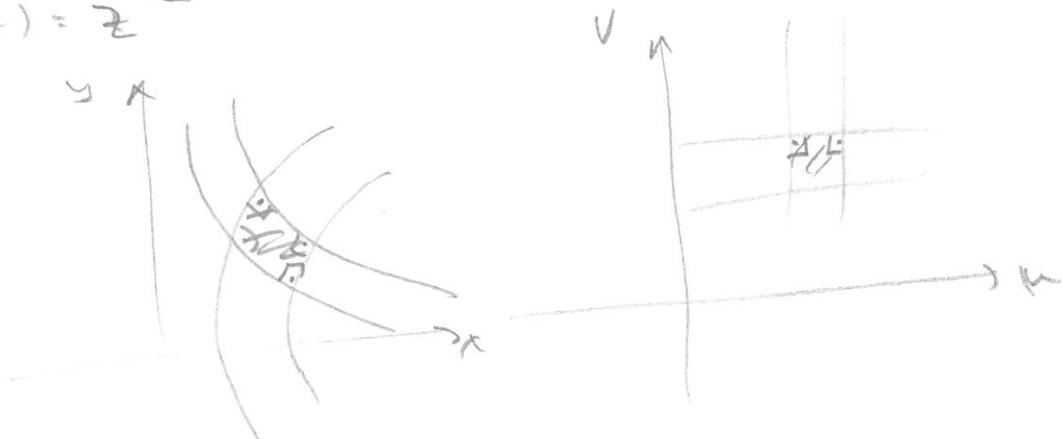
The angle of rotation is
 independent of curve $z(t)$!

Therefore

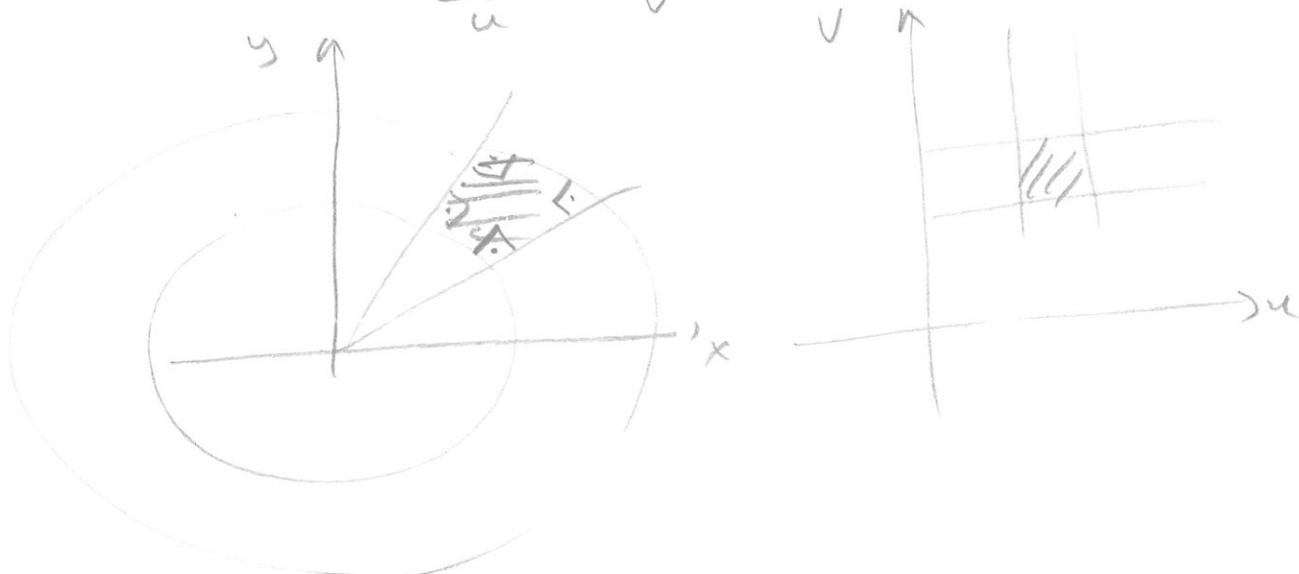


The angle between curves is preserved under the map!

Ex: $f(z) = z^2$



Ex: $f(z) = \text{Log } z = \underbrace{\text{log } r}_u + i \underbrace{\theta}_v$



Harmonic Functions

Def: $h(x,y)$ harmonic if $h_{xx} + h_{yy} = 0$

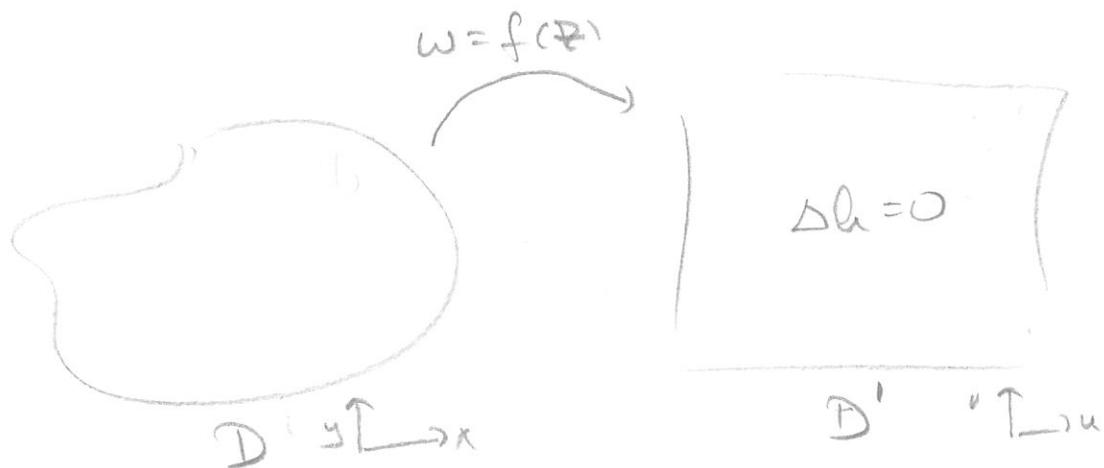
- Note:
- if $f(z) = u(x,y) + i v(x,y)$ is analytic then u, v are harmonic (v is the harmonic conjugate of u)
 - if $h(x,y)$ is harmonic then there exist a $g(x,y)$ s.t. $f(z) = h(x,y) + i g(x,y)$ is analytic.

Goal: to solve $\Delta H = 0$ in a domain D by mapping $D \rightarrow D'$ conformally where D' is simpler and finding a solution $\Delta h = 0$ is simpler domain.

So:

Suppose f analytic, mapping $D \rightarrow D'$

(4)



and we have a solution $h(u, v)$ s.t.

$$\Delta h = h_{uu} + h_{vv} = 0 \quad \text{in } D'$$

then h is real part of analytic fun

$$\phi(w) = h(u, v) + i g(u, v), \quad \text{some } g$$

Now consider the composition-

$$\begin{aligned} \phi(f(z)) &= h(u(x, y), v(x, y)) + i g(u(x, y), v(x, y)) \\ &= H(x, y) + i G(x, y) \end{aligned}$$

Since composition of analytic functions are analytic we know

$$\Delta H = H_{xx} + H_{yy} = 0 \quad \text{in } D.$$

So: a solution to Laplace equation in D' plus mapping f gives solution in D .

Boundary Value Problems

The correct boundary conditions for the partial differential equation

$$\Delta h = h_{xx} + h_{yy} = 0 \text{ in } D$$

that give a unique solution are either

$$h(x,y) = \text{specified function on } \partial D$$

(Dirichlet Problem) ↖
boundary of D

or

$$\frac{\partial h}{\partial n} = \nabla h \cdot \underline{n} = \text{specified, on } \partial D$$

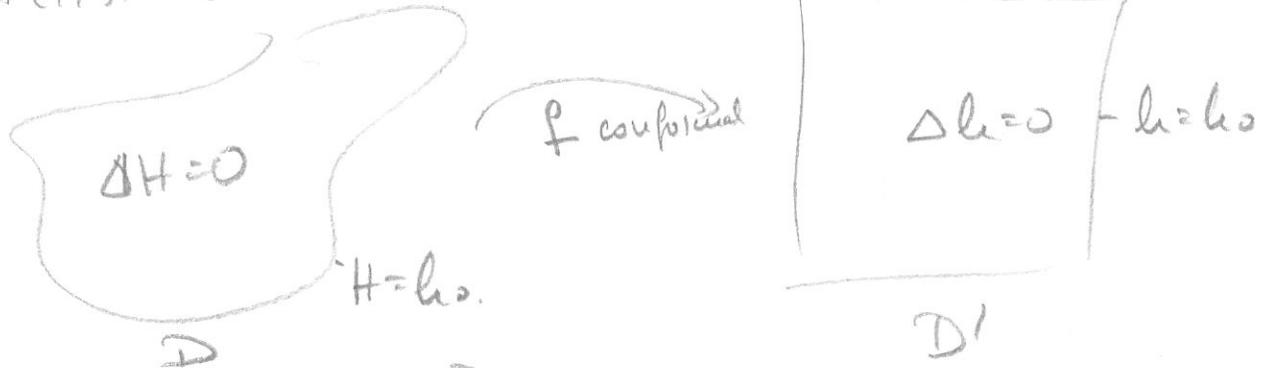
where \underline{n} is the outward normal.

(Neumann problem)

Here we consider the simpler problem (6) of

$$(1) \begin{cases} \Delta h = 0 & \text{in } D \\ h = h_0 & \text{(constant!!)} \\ & \text{on } \partial D \end{cases} \quad \text{or} \quad (2) \begin{cases} \Delta h = 0 & \text{in } D \\ \frac{\partial h}{\partial n} = 0 & \text{on } \partial D. \end{cases}$$

Note that if h satisfies (1) in D , then $H(x, y) = h(\varphi(x, y), \psi(x, y))$

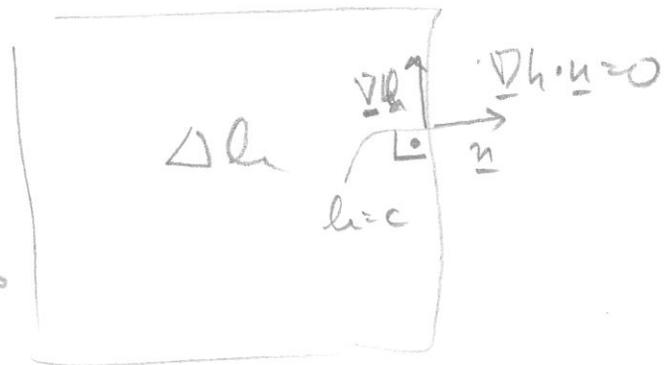


satisfies (1) in D

since

$$H \text{ on } \partial D = h \text{ on } \partial D' = h_0$$

Also, if h satisfies (2) in D' , the ∇h is normal to \underline{n} , so ∇h is parallel to $\partial D'$, so level curves of h are normal to $\partial D'$.



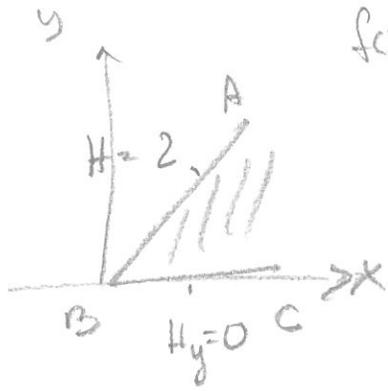
By conformality of mapping it follows the level curves of H are normal to ∂D !

$$\text{so } \frac{\partial H}{\partial n} = 0 \text{ on } \partial D$$

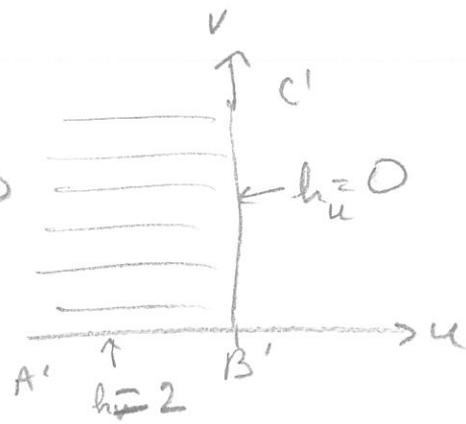


So H satisfies (2) in D !

Ex:
(hw)

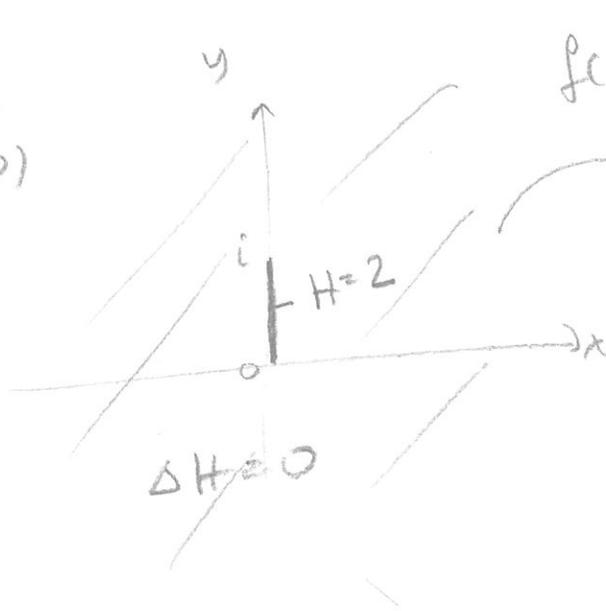


$$f(z) = iz^2$$

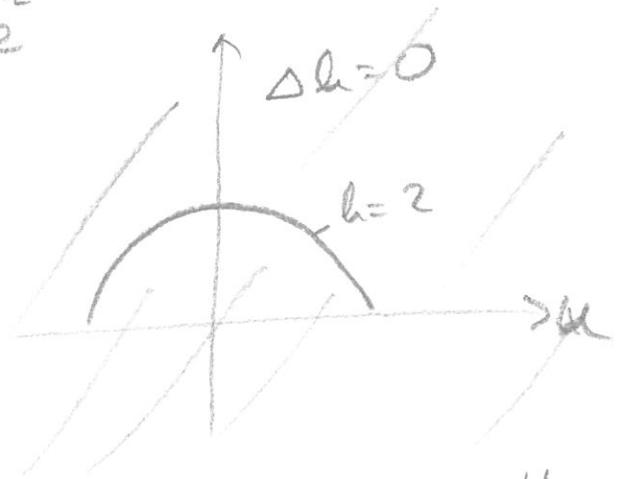


$$h(u, v) = v + 2$$

Ex:
(hw)



$$f(z) = e^z$$



$$h(u, v) = 2 - u + \frac{u}{u^2 + v^2}$$