Lecture 32: The Residue Theorem.

1. Finding residues

Example 1: Find the residue of $f(z) = \dfrac{e^{-z}}{(z-1)^2}$ at $z = 1$, the hard way. The easy way. Classify the singularity.

Example 2: Let $f(z) = \dfrac{1}{(z-1)(z-2)}$. Find residues at $z = 1$, $z = 2$. Classify the singularity.

Example 3: Let $f(z) = \dfrac{\sin z}{z}$. Find residues at $z = 0$. Classify the singularity.

2. The Residue Theorem

Let C be a simple closed curve. Suppose $f(z)$ is analytic inside C, except for isolated singularities at z_k, $k = 1, \ldots, n$. Let B_k be the residue at z_k. Then

$$\oint_C f(z)\, dz = 2\pi i \sum_{k=0}^{n} B_k$$

Example 4: Show that under conditions of theorem, $\oint_C f(z)\, dz = \oint_{C_1} f(z)\, dz + \oint_{C_2} f(z)\, dz + \ldots + \oint_{C_n} f(z)\, dz$ where each C_k encloses only one singularity at z_k. We showed it for $n = 2$.

Example 5: Find $\displaystyle\oint_{|z-1|=2} \frac{1}{(z-1)(z-2)}\, dz$.

Example 6: Find $\displaystyle\oint_{|z-1|=2} \frac{\sin z}{z}\, dz$.

Example 7: Classify the singularity $z = 0$ of $f(z) = \frac{1}{z(1+z^2)}$. Hint: need to use the Laurent series about $z = 0$ that is valid in $0 < |z| < 1$, not the one that holds for $1 < |z| < \infty$. Result: the singularity is a pole singularity of order 1.

3. Special Cases

If $f(z) = \frac{\phi(z)}{z-z_0}$ where ϕ is analytic at z_0, and C is a curve enclosing only the singularity at z_0, then

$$\oint_C f(z) dz = \oint_C \frac{\phi(z)}{z - z_0}\, dz = 2\pi i \phi(z_0)$$

so the residue of f at z_0 is $b_1 = \phi(z_0)$.

If $f(z) = \frac{\phi(z)}{(z-z_0)^{n+1}}$ where ϕ is analytic at z_0, and C is a curve enclosing only the singularity at z_0, then

$$\oint_C f(z) dz = \oint_C \frac{\phi(z)}{(z - z_0)^{n+1}}\, dz = \frac{2\pi i}{n!} \phi^{(n)}(z_0)$$

so the residue of f at z_0 is $b_1 = \phi^{(n)}(z_0)/n!$.

Example 8: Use the above to find $\oint_{|z|=2} \frac{1}{z(1+z^2)}\, dz$.