Lecture 31: Dividing series. Zeros. Singular points.

1. Finding Laurent series by division

Trick: Use the series for $\frac{1}{1-z}$.

Example 1: Consider $f(z) = \dfrac{1}{z^2 \sinh(z)}$. Find singular points. Find Laurent series in $0 < |z| < \pi$.

2. Zeros of analytic functions

Suppose $f(z)$ is analytic and has a zero at z_0, $f(z_0) = 0$. Then

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

where $a_0 = 0$. If f is not identically equal to zero, then there is at least one $a_k \neq 0$. Let m be the smallest index so that $a_m \neq 0$. Then

$$f(z) = \sum_{k=m}^{\infty} a_k (z - z_0)^k = (z - z_0)^m [a_m + a_{m+1}(z - z_0) + a_{m+2}(z - z_0)^2 + \ldots] = (z - z_0)^m g(z)$$

where $g(z_0) \neq 0$ and g is analytic and therefore continuous at z_0. Then there is a neighbourhood of z_0 within which $g(z) \neq 0$. Also, $(z - z_0)^m \neq 0$ if $z \neq 0$. We have thus shown that $f(z) \neq 0$ in a neighbourhood of z_0, with $z \neq z_0$, that is,

the zeros of an analytic function are isolated points!

3. Singular points

Definition: A point z_0 is a singular point of f if f is not analytic at z_0 but is analytic at some point in every neighbourhood of z_0

Definition: A point z_0 is an isolated singular point of f if it is a singular point of f but f is analytic at every point in some neighbourhood of z_0.

Example 2: State singular points of $\sinh z$. Are they isolated or not?

Example 3: $z = 0$ is a singular point of $\text{Log} z$. Is it isolated or not?

Example 4: If $f(z)$ is analytic, the singular points of $1/f(z)$ are isolated, in view of above.

Note: If f has an isolated singularity at z_0 then it is analytic in an annular region $D : 0 < |z - z_0| < R_1$ for some R_1, and, in this region D, it has a Laurent series representation

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \frac{b_3}{(z - z_0)^3} + \ldots$$

Definition: We defined the **residue of f at** z_0, the **principal part** of f, as well as **essential singularities** and **pole singularities**