Lectures 28-29: Laurent Series

Now suppose $f(z)$ is analytic only inside an annulus around the origin, $D : \epsilon \le |z| < R$. That is, it is not analytic inside $C_\epsilon : |z| = \epsilon$. Let $z \in D$ with $|z| = r$. Let $C : |z| = R_1$ where $\epsilon < r < R_1 < R$. Since f is analytic between C_ϵ and C, we can show, using the Cauchy Integral Formula, that

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(s)}{s-z}ds - \frac{1}{2\pi i}\oint_{C_\epsilon} \frac{f(s)}{s-z}ds$$

Using the formula (1) you proved in the homework

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \ldots + z^N + \frac{z^{N+1}}{1-z}$$

it now follows that

$$\begin{aligned} f(z) = & \frac{1}{2\pi i}\oint_C \frac{f(s)}{s}\frac{1}{1-(z/s)}ds + \frac{1}{2\pi i}\oint_{C_\epsilon} \frac{f(s)}{z}\frac{1}{1-(s/z)}ds \\ = & \frac{1}{2\pi i}\oint_C \frac{f(s)}{s}\Big(1 + \frac{z}{s} + \frac{z^2}{s^2} + \frac{z^3}{s^3} + \ldots \frac{z^N}{s^N}\Big)ds + \rho_N^1(z) \\ & + \frac{1}{2\pi i}\oint_{C_\epsilon} \frac{f(s)}{z}\Big(1 + \frac{s}{z} + \frac{s^2}{z^2} + \frac{s^3}{z^3} + \ldots \frac{s^N}{z^N}\Big)ds + \rho_N^2(z) \end{aligned}$$

where

$$\rho_N^1(z) = \frac{1}{2\pi i}\oint_C \frac{f(s)}{s}\frac{(z/s)^{N+1}}{1-(z/s)}\ , \qquad \rho_N^2(z) = \frac{1}{2\pi i}\oint_{C_\epsilon} \frac{f(s)}{z}\frac{(s/z)^{N+1}}{1-(s/z)}\ .$$

As we showed last time, $|\rho_N^1| \to 0$ as $N \to \infty$ (because $|z/s| < 1$ on C). Also $|\rho_N^2| \to 0$ as $N \to \infty$ (because $|s/z| < 1$ on C_ϵ). It therefore follows that the series converges and

$$\begin{aligned} f(z) = & \frac{1}{2\pi i}\oint_C \frac{f(s)}{s}ds + \frac{z}{2\pi i}\oint_C \frac{f(s)}{s^2}ds + \frac{z^2}{2\pi i}\oint_C \frac{f(s)}{s^3}ds + \ldots \\ & + \frac{1}{2\pi i z}\oint_{C_\epsilon} f(s)ds + \frac{1}{2\pi i z^2}\oint_{C_\epsilon} f(s)s\,ds + \frac{1}{2\pi i z^3}\oint_{C_\epsilon} f(s)s^2ds + \ldots \\ = & \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{b_n}{z^n} \\ = & a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots + \frac{b_{-1}}{z} + \frac{b_{-2}}{z^2} + \frac{b_{-3}}{z^3} + \ldots \end{aligned}$$

where

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(s)}{s^{n+1}}\,ds\ , \quad b_n = \frac{1}{2\pi i}\int_{C_\epsilon} f(s)s^{n-1}\,ds\ .$$

This proves that every function that is analytic in an annulus has a Laurent series that consists of sums of **positive and negative powers of** z. The result is easily generalized to:

Theorem: If f is analytic inside an annulus centered at z_0 , $\epsilon < |z - z_0| < R$, where $\epsilon > 0$, then it has a Laurent series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^{n+1}}\, ds \ , \qquad b_n = \frac{1}{2\pi i} \oint_{C_\epsilon} f(s)(s - z_0)^{n-1}\, ds$$

Note, in particular, that

$$b_1 = \frac{1}{2\pi i} \oint_{C_\epsilon} f(s)\, ds \ .$$