Lectures 24-25: Sequences, Series, Taylor series

## 1. Sequences

Definition: A sequence of complex numbers $\left\{z_{n}\right\}_{n=1}^{\infty}$ converges to $z$ if for all $\epsilon>0$ there exists an integer $N>0$ such that

$$
n>N \Rightarrow\left|z_{n}-z\right|<\epsilon .
$$

In that case we write

$$
\lim _{n \rightarrow \infty} z_{n}=z \text { or } z_{n} \rightarrow z \text { as } n \rightarrow \infty
$$

Example 1: $\lim _{n \rightarrow \infty}|z|^{n}=0$ for all $z$ with $|z|<1$.

## 2. Series

Definition: A series $\sum_{n=1}^{\infty} z_{n}$ converges to $S$ if the partial sums

$$
s_{N}=\sum_{n=1}^{N} z_{n} \rightarrow S \text { as } n \rightarrow \infty .
$$

Theorem: For a series to converge the summands $z_{n}$ must approach 0 as $n \rightarrow \infty$ sufficiently fast. That is, $\lim _{n \rightarrow \infty} z_{n}=0$ is necessary for convergence, but not sufficient.
Example 2: The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the Integral Test, even though $1 / n \rightarrow 0$ as $n \rightarrow \infty$.
Example 3: Consider the geometric series $\sum_{n=0}^{\infty} z^{n}=0$. In the homework you showed that if $z \neq 1$, the partial sums equal

$$
\begin{equation*}
s_{N}=1+z+z^{2}+z^{3}+\ldots+z^{N}=\frac{1-z^{N+1}}{1-z} \tag{1}
\end{equation*}
$$

It now follows that if $|z|<1$, then

$$
\sum_{n=0}^{\infty} z^{n}=\frac{1}{1-z}
$$

It is also easy to check that if $|z| \geq 1$ then

$$
\sum_{n=0}^{\infty} z^{n} \text { diverges }
$$

since the $n$th term in the series does not $\rightarrow 0$ as $n \rightarrow \infty$.

## 3. Taylor series

Suppose $f(z)$ is analytic inside a circle of radius $R$ centered on the origin, $D:|z|<R$. Let $z \in D$ with $|z|=r$. Let $C:|z|=R_{1}$ where $r<R_{1}<R$. By the Cauchy Integral Formula, since $z$ is inside $C, C$ is closed and simple, and $f$ analytic,

$$
f(z)=\frac{1}{2 \pi i} \oint_{C} \frac{f(s)}{s-z} d s
$$

Using the formula (1) you proved in the homework

$$
\frac{1}{1-z}=1+z+z^{2}+z^{3}+\ldots+z^{N}+\frac{z^{N+1}}{1-z}
$$

it follows that

$$
\begin{aligned}
f(z) & =\frac{1}{2 \pi i} \oint_{C} \frac{f(s)}{s} \frac{1}{1-(z / s)} d s \\
& =\frac{1}{2 \pi i} \oint_{C} \frac{f(s)}{s}\left(1+\frac{z}{s}+\frac{z^{2}}{s^{2}}+\frac{z^{3}}{s^{3}}+\ldots \frac{z^{N}}{s^{N}}\right) d s+\rho_{N}(z) \\
& =\frac{1}{2 \pi i} \oint_{C} \frac{f(s)}{s} d s+\frac{z}{2 \pi i} \oint_{C} \frac{f(s)}{s^{2}} d s+\frac{z^{2}}{2 \pi i} \oint_{C} \frac{f(s)}{s^{3}} d s+\ldots+\frac{z^{N}}{2 \pi i} \oint_{C} \frac{f(s)}{s^{N+1}} d s+\rho_{N}(z) \\
& =f(0)+f^{\prime}(0) z+\frac{f^{\prime \prime}(0)}{2} z^{2}+\frac{f^{\prime \prime \prime}(0)}{3!} z^{3}+\ldots+\frac{\left.f^{(N)}\right)(0)}{N!} z^{N}+\rho_{N}(z)
\end{aligned}
$$

where

$$
\rho_{N}=\frac{1}{2 \pi i} \oint_{C} \frac{f(s)}{s} \frac{(z / s)^{N+1}}{1-z / s} d s=\frac{1}{2 \pi i} \oint_{C} f(s) \frac{z^{N+1}}{(s-z) s^{N+1}} d s
$$

Since $s \in C$ it follows that $|s|=R_{1}$. Also $|z|=r<R_{1}$. Also

$$
|s-z| \geq||s|-|z||=R_{1}-r \quad(\text { as you showed in HW })
$$

and $f$ is bounded on $C$, since it is analytic (and therefore continous). That is, $|f| \leq M$ on $C$ for some $M$. So

$$
\left|f(s) \frac{z^{N+1}}{(s-z) s^{N+1}}\right| \leq \frac{M}{\left(R_{1}-r\right)}\left(\frac{r}{R_{1}}\right)^{N+1}=B_{N}
$$

and thus

$$
\left|\rho_{N}\right| \leq \frac{1}{2 \pi} 2 \pi R_{1} B_{N}=R_{1} B_{N}
$$

Since $B_{N} \rightarrow 0$ as $N \rightarrow \infty$ (since $r / R_{1}<1$ ), it follows that the upper bound $R_{1} B_{N} \rightarrow 0$ and thus

$$
\rho_{N} \rightarrow 0 \text { as } N \rightarrow \infty
$$

This proves that the infinite Taylor series converges. We have thus used the Cauchy Integral formula to prove, in a quite simple and direct way, a rather amazing result:

Theorem: If $f$ is analytic inside a circle centered at the origin, $|z|<R$, then the Taylor series

$$
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^{n}=f(0)+f^{\prime}(0) z+\frac{f^{\prime \prime}(0)}{2!} z^{2}+\frac{f^{\prime \prime \prime}(0)}{3!} z^{3}+\ldots
$$

converges to $f(z)$, that is, it is $=f(z)$ for all $z \in D!!$

The result can be easily gereneralized to a circle centered at $z_{0}$ :
Theorem: If $f$ is analytic inside the circle $D:\left|z-z_{0}\right|<R$, then the Taylor series about $z_{0}$ converges to $f$,

$$
f(z)=\sum_{n=0}^{\infty} \frac{f^{(n)}\left(z_{0}\right)}{n!}\left(z-z_{0}\right)^{n}
$$

for all $z \in D$.
Example 4: $f(z)=e^{z}$. Write down Taylor series. State region of convergence.
Example 5: $f(z)=\sin z$. Write down Taylor series. State region of convergence.
Example 6: $f(z)=\cos z$. Write down Taylor series. State region of convergence.
Theorem: Uniqueness of Taylor series. If $f(z)=\sum_{n=0}^{\infty} a_{n}\left(z-z_{0}\right)^{n}$, then $a_{n}=\frac{f^{(n)}\left(z_{0}\right)}{n!}$ and the series is the Taylor series about $z_{0}$.

Theorem: Region of convergence. If a Taylor series converges at a point $z_{1}$ then it converges at all points $z$ with $\left|z-z_{0}\right|<\left|z_{1}-z_{0}\right|$.

Draw a picture. This implies that the region of convergence of a Taylor series about $z_{0}$ is a disk centered at $z_{0}$, plus possibly some points on the boundary. This in turn implies that the region of convergence is a circle about $z_{0}$ that reaches up to the nearest singularity!

Example 7: $f(z)=\frac{1}{1-z}$. Write down Taylor series about $z=0$. State region of convergence.
Example 8: $f(z)=\frac{1}{1+z^{2}}$. Write down Taylor series about $z=0$. State region of convergence.
Example 9: $f(z)=\frac{1}{z}$. Write down Taylor series about $z=1$. State region of convergence. About $z=2$.

Example 10: $f(z)=\frac{1+2 z}{z^{2}+z^{3}}$. Expand into series involving powers of $z$. State region of convergence.
Example 11: $f(z)=\frac{e^{z}}{z^{2}}$. Expand into series involving powers of $z$. State region of convergence.
One further interesting property of analytic functions follows from the Taylor series representation that they all have. Suppose $f$ is analytic at $z_{0}$ and $f\left(z_{0}\right)=0$. Then its Taylor series about $z_{0}$ converges in a disk around $z_{0}$. This Taylor series either is identically zero, or there is one first nonzero term, so that

$$
\begin{aligned}
f(z) & =a_{m}\left(z-z_{0}\right)^{m}+a_{m+1}\left(z-z_{0}\right)^{m+1}+a_{m+1}\left(z-z_{0}\right)^{m+2}+\ldots \\
& =\left(z-z_{0}\right)^{m}\left[a_{m}+a_{m+1}\left(z-z_{0}\right)+a_{m+2}\left(z-z_{0}\right)^{2}+\ldots\right] \\
& =\left(z-z_{0}\right)^{m} g(z)
\end{aligned}
$$

where $g$ is some analytic function around $z_{0}$ with $g\left(z_{0}\right)=a_{m} \neq 0$. Since $g$ is continous at $z_{0}$, and $\left(z-z_{0}\right)^{m} \neq 0$ for $z \neq z_{0}$, it follows that $f(z) \neq 0$ near $z_{0}$ !! This proves that

Theorem: Zeros of analytic functions. The zeros of an analytic function are isolated, unless $f$ is identically zero.

Inside their domain of convergence, Taylor series can be differentiated and integrated term by term. Two Taylor series can be multiplied term by term within their joint domain of convergence.

Example 12: $f(z)=\frac{e^{z}}{1+z}$. Find Taylor series about 0 .
Example 13: $f(z)=\frac{z}{(1+z)^{2}}$. Find Taylor series about 0 .
Example 14: $f(z)=\log (1+z)$. Find Taylor series about 0 .

