Lecture 23: Computing integrals. Maximum principle.

1. Evaluating integrals

Example 1: Find $\displaystyle\oint_{|z|=4} \frac{\sin z}{z^2}\,dz$

Example 2: Find $\displaystyle\oint_{|z|=4} \frac{\cos z}{z^3(z^2+8)}\,dz$

Example 3: Show that $\displaystyle\oint_C \frac{f(z)}{(z-z_1)(z-z_2)^2(z-z_3)^3}\,dz = \oint_{C_1} + \oint_{C_2} + \oint_{C_3}$ where C_k enclose only z_k and no other singularity

Example 4: Find $\displaystyle\oint_C \frac{s^3+2s}{(s-z)^2}\,ds$

2. Maximum principle

Note: Let $C : |z-z_0| = r$ and f be analytic in a region containing C. The curve C can be parametrized by $z(t) = z_0 + re^{i\theta}$, $\theta \in [0, 2\pi]$. Then

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z(\theta))}{re^{i\theta}} z'(\theta)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta = f_{av}$$

That is, $f(z_0)$ is the average of function values around the surrounding curve C! This implies that if $f > f(z_0)$ somewhere on C, then f must be $< f(z_0)$ somewhere on C, and vice versa.

Theorem: (**Maximum principle**.) If f is analytic in a domain D, then f cannot attain a maximum value inside D, unless it is the constant function.

Similarly: if f is entire, then f is either unbounded, or identically equal to a constant.

Again, these are very strong properties that differentiable functions of a real variable do not have. And all follows from the Cauchy-Riemann equations! (how?)