Lecture 22: Derivatives of analytic functions

1. Cauchy-integral Formula (review)

Reminder: if f is analytic in a region containing a simple closed curve C, containing z_0, with counterclockwise orientation,then

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz \tag{1}$$

Example 1: Evaluate $\displaystyle\int_0^\infty \frac{e^{-z}}{(z - i\pi/2)z}\, dz$

2. Derivatives of analytic functions

Alternatively, (1) can be written as

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(s)}{s - z}\, ds \tag{2a}$$

We can now differentiate this equation with respect to z on both sides. Since the integrand on the right is sufficiently nice, the derivative can be taken inside the integral:

$$\frac{d}{dz}\Big[\oint_C \frac{f(s)}{s-z}\, ds\Big] = \oint_C \frac{d}{dz}\Big[\frac{f(s)}{s-z}\Big]\, ds$$

and we get

$$f'(z) = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s-z)^2}\, ds \tag{2b}$$

$$f''(z) = \frac{2}{2\pi i} \oint_C \frac{f(s)}{(s-z)^3}\, ds \tag{2c}$$

$$f'''(z) = \frac{3\cdot 2}{2\pi i} \oint_C \frac{f(s)}{(s-z)^4}\, ds \tag{2d}$$

$$\vdots$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(s)}{(s-z)^{n+1}}\, ds \tag{3}$$

This has two important consequences

Theorem: If f is analytic in a region containing z, then it has derivatives of all orders at z!

This is a very powerful property of analytic functions $f(z)$ that functions of a real variable, $f(x)$, dont have. We gave two examples in class of functions $f(x)$ that have one derivative at one point, but not two.

Exercise: Can you give two such examples (or reproduce the ones we wrote down) ??

The theorem says that if a function is integrable, then it is differentiable, and that is all we need to get the "if and only if" in the following theorem, which supersedes the one we saw earlier in class:

Theorem: (**Morera's theorem**) $\int_C f(z)\, dz = 0$ for each closed curve inside a region $D \in \Re^2 \iff f(z)$ is analytic in D

Example 1: Let $C : |z| = r > 0$, with counterclockwise orientation. Find $\displaystyle\oint_C \frac{ds}{s^2}$