**Lecture 19: More on Cauchy-Goursat. Antiderivatives and path-independence**

## 1. Main results on integrals so far

We have the following results that we can use to evaluate integrals.

(1) $\displaystyle\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$

*Example 1:* Evaluate $\int_0^\infty e^{-zt}\,dt$, where $Re(z) > 0$.

(2) $\displaystyle\int_C f(z)\,dz = \int_a^b f'(z(t))z'(t)\,dt$ where the curve $C$ is parametrized by $z(t)$, $t \in [a,b]$.

(3) If $f$ analytic in region $D$ containing $C$, where $C$ closed, then $\displaystyle\oint_C f(z)\,dz = 0$ (this follows from Green's Theorem). This can be generalized to regions $D$ with holes, where $C$ surrounds region $R$ with a hole, as long as all parts of curve $C$ bounding $R$ have the proper orientation. Similarly if $C$ is any curve in the region $D$ from $A$ to $B$, then the line integral is path independent, depends only on $A$ and $B$, so can write as $\displaystyle\int_C f(z)\,dz = \int_A^B f(z)\,dz$

(4) If $f(z) = F'(z)$ in a region (in which $F$ necessarily is analytic) containing $C$, then $\displaystyle\int_C f(z)\,dz = F(B) - F(A)$ (if C from A to B) and $\displaystyle\oint_C f(z)\,dz = 0$ (if C closed)

*Example 2:* Evaluate $\int_{-i}^{i} \frac{1}{z}dz$

*Example 3:* Evaluate $\int_{-1}^{1} \frac{1}{z}dz$

*Example 4:* Evaluate $\oint_C \frac{1}{z}dz$, where $C$ is any simple closed curve surrounding origin. Use two different approaches, one using (1) above plus the theorem below, the other using (4) above, where we write integral as a limit.)

*Theorem:* If $f$ is analytic in a region $R$ between two curves $C_1$ and $C_2$, then

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

*Proof:* In class. We used this result in example 4 above.

## 2. Cauchy integral theorem

*Theorem:* If $f$ is analytic in a region $R$ containing a simple closed curve $C$ surrounding a point $z_0$, with counterclockwise orientation, then

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz\ ,$$

or alternatively

$$2\pi i f(z_0) = \oint_C \frac{f(z)}{z - z_0}\,dz\ .$$

*Proof:* Outline of proof given in class.

*Example 5:* Evaluate $\oint_C \frac{1}{z}\,dz$