## Lecture 19: More on Cauchy-Goursat. Antiderivatives and path-independence

## 1. Consequences of Cauchy-Goursat

First we derive and use three consequences of the Cauchy-Goursat Theorem. Again, assume $f=u_{i} v$ is analytic in a region $D$. Then
(1) If $C$ is a simple closed oriented curve contained in $D$ then $\oint_{C} f(z) d z=0$. (This is just a restatement of the theorem)
(2) If $C$ is a simple curve contained in $D$ going from point $A$ to point $B$, then the line integral


$$
\int_{C} f(z) d z
$$

is path independent! Why? Suppose you have two paths $C_{1}$ and $C_{2}$ going from $A$ to $B$ (that dont intersect), then $C_{1} \cup\left(-C_{2}\right)$ is a closed simple curve and by Cauchy-Goursat

$$
\begin{gathered}
\int_{C} f(z) d z=\int_{C_{1}} f(z) d z+\int_{-C_{2}} f(z) d z \\
=\int_{C_{1}} f(z) d z-\int_{C_{2}} f(z) d z=0
\end{gathered}
$$

so $\int_{C_{1}} f(z) d z=\int_{C_{2}} f(z) d z$
(3) If $C$ is a closed oriented curve that intersects itself, then the result of Cauchy-Goursat still holds:

$$
\oint_{C} f(z) d z=0 .
$$

Why? Because we can replace the line integral over $C$ by the sum of two line integrals over closed simple curves,

$$
\oint_{C} f(z) d z=\oint_{C_{1}} f(z) d z-\oint_{C_{2}} f(z) d z
$$

where both $C_{1}$ and $C_{2}$ have counterclockwise orientation. See
 picture. Then, by (1), $\oint_{C_{1}}=\oint_{C_{2}}=0$, so $\oint_{C}=0$.
(4) Now suppose $f$ is analytic in a region $D$ that has a hole in it. Then we can apply the theorem to a properly oriented curve $C$ that bounds a region $R$ contained in $D$ within which $f$ is analytic. Now suppose $R$ has a hole excluding the point(s) where $f$ is not analytic. Then the curve $C$ consists of 2 pieces, both shown in black in the figure (next page). The result

$$
\oint_{C} f(z) d z=0
$$

holds provided the two pieces have the proper orientation. What is the poper orientation? It is such that as you are travelling on each piece so that the enclosed region (shown as dashed region in figure) lies to your left (see picture). Why? Because in that case we can replace the line integral over $C$ by the sum of two line integrals over two closed simple curves (shown in blue),

$$
\oint_{C} f(z) d z=\oint_{C_{1}} f(z) d z+\oint_{C_{2}} f(z) d z
$$


where both $C_{1}$ and $C_{2}$ have counterclockwise orientation. See picture. This works because the two little straight pieces shown in the picture cancel each other with the given orientations. Then, by (1), $\oint_{C_{1}}=\oint_{C_{2}}=0$, so $\oint_{C}=0$.
Example 1: Evaluate $\oint_{C} z^{2} d z$ where $C$ is the boundary of the annular region $r=1$ and $r=2$, with proper orientation.
Example 2: Evaluate $\oint_{C} \frac{d z}{z^{2}\left(z^{2}+9\right)}$ where $C$ is as in Example 1.
Example 3: Suppose $C$ is as in Example 1. Let $C_{1}, C_{2}$ be the outer and inner boundaries of the annulus, both with counterclockwise orientation, so that $C=C_{1} \cup\left(-C_{2}\right)$ According to the above it follows that

$$
\oint_{C} \frac{1}{z} d z=\oint_{C_{1}} \frac{1}{z} d z-\oint_{C_{2}} \frac{1}{z} d z=0
$$

since $f(z)=1 / z$ is analytic for $z \neq 0$. Note that the orientation of all parts matter. Suppose you consider the curve with the wrong orientation $C_{1} \cup C_{2}$. Then we already know that

$$
\oint_{C_{1} \cup C_{2}} \frac{1}{z} d z=\oint_{C_{1}} \frac{1}{z} d z+\oint_{C_{2}} \frac{1}{z} d z=4 \pi i
$$


which is not equal to zero!

## 2. Antiderivatives and independence of path

Suppose $f(z)$ has an antiderivative. That is, there exists a function $F(z)$ such that $F^{\prime}(z)=f(z)$. Suppose $C$ is a curve parametrized by $z(t), t \in[a, b]$ going from $A=z(a)$ to $B=z(b)$. Then

$$
\int_{C} f(z) d z=\int_{a}^{b} f(z(t)) z^{\prime}(t) d t=[F(z(t))]_{a}^{b}=F(B)-F(A)
$$

Why? Because $\frac{d}{d t}[F(z(t))]=F^{\prime}(z(t)) z^{\prime}(t)=f(z(t)) z^{\prime}(t)$. In particular, it is path-indendent! So we also denote this integral by $\int_{A}^{B} f(z) d z$.
Example 4: Evaluate $\int_{0}^{1+3 i} z^{2} d z$
Example 5: Evaluate $\int_{-i}^{i} \frac{1}{z} d z$
Example 6: Evaluate $\int_{-1}^{1} \frac{1}{z} d z$

