Lecture 17: Line Integrals of functions of a complex variable

1. Curves

Curves in the complex plane are parametrized by

$$z(t) = x(t) + iy(t) \ , \quad t \in [a, b] \ .$$

The derivative $z'(t) = x'(t) + iy'(t)$ is a vector tangent to the curve. The **unit tangent vector** is

$$\frac{z'(t)}{|z'(t)|}$$

Example 1: Parametrize circle of radius r centered at z_0, clockwise orientation

Example 2: Graph of $z(t) = (5 - t) + i(2 + 2t)$, $t \in [-2, 3]$

Definition: A **simple** curve is a curve that does not intersect itself ($z(t_1) \neq z(t_2)$ if $t_1 \neq t_2$).

Definition: A **smooth** curve is given by differentiable functions $x(t)$, $y(t)$ such that $|z'(t)| \neq 0$.

Definition: A **closed** curve satisfies $z(a) = z(b)$.

2. Line Integrals I, independent of orientation

Example 1: Arclength $\int_C ds = \int_a^b |z'(t)|\, dt$

3. Line Integrals II, $\int_C f(z)dz$, dependent on orientation

Definition: Let $f(z) = u(x, y) + iv(x, y)$ be function of a complex variable $z = x + iy$, and C be a curve bien by $z(t) = x(t) + iy(t)$, $\quad t \in [a, b]$. Define

$$\int_C f(z)\, dz = \int_a^b f(z(t))z'(t)\, dt$$

Note 1: $\int_C f(z)\, dz = \int_C u\, dx - v\, dy + i \int_C v\, dx + u\, dy$

Note 2: Suppose that $|f(z)| \leq M$ along C. Then

$$|\int_C f(z)\, dz| \leq \int_a^b |f(z)z'(t)| dt \leq M \int_a^b |z'(t)|\, dt = ML$$

Example 1: Find $\int_C z^2\, dz$ where C is the line from origin to $2 + i$

Example 2: Find $\int_C z^2\, dz$ where C is segment of two lines from origin to $2 + i$

Example 3: Find $\int_C \overline{z}\, dz$ where C is half-circle from 1 to -1.

Example 4: Find $\int_C \overline{z}\, dz$ where C is line from 1 to -1.

Example 5: Find $\int_C \frac{1}{z - z_0}\, dz$ where C is closed circle around z_0.