Lecture 15: Line Integrals of real-valued functions

1. Curves

Curves are parametrized by

$$\mathbf{x}(t) = \langle x(t), y(t) \rangle, t \in [a, b]$$

The derivative $\mathbf{x}'(t) = \langle x'(t), y'(t) \rangle$ is a vector tangent to the curve. The **unit tangent vector** is

$$\mathbf{T}(t) = \frac{\mathbf{x}'(t)}{|\mathbf{x}'(t)|}$$

Example 1: Parametrize 3/4 circle, clockwise orientation

Example 2: Parametrize line

Example 3: Parametrize section of parabola

Example 4: Graph of $x = t^3$, $y = t^2$. What happens at $t = 0$? Why there is no tangent vector at that point.

2. Line Integrals I, $\int_C f(x, y)\, ds$

Definition: $\int_C f(x, y)\, ds = \lim_{n \to \infty} \sum_{j=1}^{n} f(x_k, y_k) \Delta s$

Example 1: Arclength $\int_C ds$

This type of integral does not depend on the orientation of the curve C.

To compute them use that $\Delta s \approx \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\, \Delta t$ to get the formula

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Then write

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b f(x(t), y(t))\, |\mathbf{x}'(t)|\, dt$$

3. Line Integrals II, $\int_C \mathbf{F}(x, y) \cdot \mathbf{T}\, ds = \int_C \mathbf{F}(x, y) \cdot d\mathbf{s}$

Let $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ be a vector field, $C : \langle x(t), y(t) \rangle$ $t \in [a, b]$ be a given curve and $\mathbf{T}(t)$ be the unit tangent vector at t. The **oriented** line integral

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

integrates the component of F tangent to the curve C. To compute it use

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b (P,Q)\cdot\frac{\mathbf{x}'(t)}{|\mathbf{x}'(t)|}|\mathbf{x}'(t)|\,dt = \int_a^b (P,Q)\cdot\mathbf{x}'(t)\,dt = \int_a^b \big(P\frac{dx}{dt} + Q\frac{dy}{dt}\big)\,dt$$

which is often also written as

$$= \int_C Pdx + Qdy$$

Theorem: If $\mathbf{F} = \nabla f$ then the line integral is path independent, with

$$\int_C \nabla f\cdot\mathbf{T}\,ds = f(B) - f(A)$$

where $A = \mathbf{x}(a)$ and $B = \mathbf{x}(b)$

Theorem: If C is closed, then

$$\int_C Pdx + Qdy = \int\int_A \frac{dQ}{dx} - \frac{dP}{dy}\,dA$$