Lecture 15: Integrals $\int_a^b f(t)\,dt$, where $f(t) = u(t) + iv(t)$

1. Review of multi-variable calculus

How to represent curves C in the plane?

How to compute arclength? Derive $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

2. Integrals $\int_a^b f(t)\,dt$, $f(t) = u(t) + iv(t)$

Let $f(t) = u(t) + iv(t)$, $t \in [a, b]$, u, v piecewise continuous, with finite left and right-hand limits. This corresponds to sampling the function $f(z) = u(x, y) + iv(x, y)$ on the real line where $z = t$. Define

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i \int_a^b v(t)\,dt$$

Note: Supposed U and V are antiderivatives of u, v, so that $W = U + iV$ is antiderivative of $w = f(t)$. Then

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i \int_a^b v(t)\,dt = U + iV\Big|_a^b = W\Big|_a^b$$

Example 1: Evaluate $\int_0^{\pi/4} e^{it}\,dt$ in two ways: (1) integrate real and imaginary parts separately; (2) find antiderivative W directly.

Example 2: Evaluate $\int_0^{2\pi} \sin(3x)\cos(5x)\,dx$. What methods can you think of? (we have seen two methods so far for this one) Now use complex variables.

3. Inequalities of integrals.

Example 1: Assume $f(x)$ is real-valued. We showed $|\int_a^b f(x)\,dx| \le \int_a^b |f(x)|\,dx$ (also HW).

Theorem: Let $f(z) = u(t) + iv(t)$, $t \in [a, b]$. Then

$$|\int_a^b f(t)\,dt| \le \int_a^b |f(t)|\,dt$$

Proof: next class