Lecture 9: Continuity and differentiability

1. Continuity

Definition: $f(z)$ is continuous at $z = z_0$ if

$$(i) \quad \lim_{z \to z_0} f(z) \quad \text{exists}$$

$$(ii) \quad f(z_0) \quad \text{exists}$$

$$(iii) \quad \lim_{z \to z_0} f(z) = f(z_0)$$

The following statement is equivalent: for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon$$

How does this statement differ from the one giving existence of a limit?

Theorem: $f(z) = u(x, y) + iv(x, y)$ is continuous at $z_0 = x_0 + iy_0 \iff u$ and v are continuous at (x_0, y_0)

It follows that sums, products and compositions of continous functions are continuous.

2. Differentiability

Definition: $f(z)$ is differentiable at $z = z_0$ if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \to 0} \frac{f(z0 + \Delta z) - f(z_0)}{\Delta z}$$

exists. In that case, the limit is denoted by $f'(z_0)$.

Example 1: Use the definition to find the derivative of $f(z) = z^2$.

Example 2: Use the definition to show that $f(z) = |z|^2$ is not differentiable anywhere except at $z = 0$.

Example 2 above illustrates that a continuity does not imply differentiability. However:

Theorem: If $f(z)$ is differentiable at z_0, then it is continuous there.

Proof: In class. Also HW

Theorem: Suppose that $f(z)$ and $g(z)$ are differentiable at z. Then

$$(1) \quad (f+g)'(z) = f'(z) + g'(z)$$

$$(2) \quad (fg)'(z) = f'(z)g(z) + f(z)g'(z)$$

$$(3) \quad (f/g)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g^2(z)} \text{ provided } g(z) \neq 0$$

$$(4) \quad (f \circ g)'(z) = f'(g(z))g'(z) \text{ provided } f \text{ differentiable at } g(z)$$

Proof: We proved (3) in class.

Example 3: Find $f'(z)$ if $f(z) = (2z^2 + i)^5$.