Lectures 7-8: Limits of Functions of a complex variable

1. Definition of the limit

Review: Let $f(x)$ be a real-valued function of a real variable. What does

$$\lim_{x\to c} f(x) = L$$

mean, where L is a finite number?

Review: Let $f(x, y)$ be a real-valued function of two real variable. What does

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

mean, where L is a finite number?

Example 1: Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ does not exist.

Example 2: Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$.

Definition: Let $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$. Then

$$\lim_{z\to z_0} f(z) = w_0$$

if for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|z - z_0| < \delta \Rightarrow |f - w_0| < \epsilon$$

What does this mean geometrically? Note: δ typically depends on ϵ.

Example 3: Show that $\displaystyle\lim_{z\to 1} \frac{iz}{2} = \frac{i}{2}$.

Example 4: Show that $\displaystyle\lim_{z\to 2i} (2x + iy^2) = 4i$.

Definition: Let $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$. Then

$$\lim_{z\to z_0} f(z) = \infty$$

if for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|z - z_0| < \delta \Rightarrow |f| > \frac{1}{\epsilon}$$

Definition: Let $f(z) = u(x,y) + iv(x,y)$, where $z = x + iy$. Then

$$\lim_{z \to \infty} f(z) = w_0$$

if for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|z| > \frac{1}{\delta} \Rightarrow |f - w_0| < \epsilon$$

Example 5: Show that $\lim_{z \to 0} \frac{1}{z^2} = \infty$.

Example 6: Show that $\lim_{z \to \infty} \frac{1}{z^2} = 0$.

2. Limit rules

Theorem: Suppose $f(z) = u(x,y) + iv(x,y)$, $z = x + iy$, $z_0 = x_0 + iy_0$. Then

$$\lim_{z \to z_0} f(z) = \lim_{(x,y) \to (x_0,y_0)} u(x,y) + i \lim_{(x,y) \to (x_0,y_0)} v(x,y)$$

Theorem: Suppose $\lim_{z \to z_0} f(z)$ and $\lim_{z \to z_0} g(z)$ exist. Then

$$\lim_{z \to z_0} \big[f(z) + g(z)\big] = \big[\lim_{z \to z_0} f(z)\big] + \big[\lim_{z \to z_0} g(z)\big]$$

$$\lim_{z \to z_0} \big[f(z)g(z)\big] = \big[\lim_{z \to z_0} f(z)\big]\big[\lim_{z \to z_0} f(z)\big]$$

$$\lim_{z \to z_0} \Big[\frac{f(z)}{g(z)}\Big] = \frac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} g(z)} \quad \text{provided} \quad \lim_{\mathrm{z} \to \mathrm{z}_0} \mathrm{g(z)} \neq 0$$

Example 7: If $p(z)$ is a polynomial, it follows that $\lim_{z \to z_0} p(z) = p(z_0)$.