Lectures 5-6: Functions of a complex variable

A function $w = f(z)$ of a complex variable $z = x + iy$ is of the form

$$f(z) = u(x, y) + iv(x, y)$$

where u and v are real-valued functions, the real and imaginary part of f. It *maps* a point z in the x-y plane into a point w in the u-w plane, is therefore also called a **mapping** or **transformation**.

Example 1: $f(z) = z^2$. Find u and v. Find the image in the w-plane of a sample point in the z-plane. Find the image in the w-plane of a wedge in z-plane, for example the wedge $\{z : |z| \leq 2, arg(z) \in [0, 3\pi/4]$. Find the pre-image in the z-plane of horizontal and vertical lines in the w-plane, $u = const$, $v = const$. HW: show that these pre-image curves, namely the *level curves* of u and v, are orthogonal with respect to each other.

Example 2: Any two real-valued $u(x, y)$, $v(x, y)$ define a function of a complex variable $f = u + iv$, for example $f(z) = y \displaystyle\int_0^\infty e^{-xt}\, dt + i \sum_{n=0}^\infty y^n$, for $x > 0$, $|y| < 1$. Graph the domain of this function.

Example 3: $p(z) = a_0 + a_1 z + a_2 z^2 + \ldots a_n z^n$. We know how to add and multiply complex numbers, so we can evaluate any polynomial $p(z)$.

Example 4: $f(z) = |z|$. This is a *real-valued* function. Note that for this function the level curves of u and v are not normal to each other.

Example 5: $f(z) = \overline{z}$.

Example 6: $f(z) = e^z = e^{x+iy} = e^x(\cos y + i \sin y)$. This function satisfies the usual rules for exponentials:

$$e^{z_1} e^{z_2} = e^{z_1 + z^2} \ , \quad e^{z_1}/e^{z_2} = e^{z_1 - z_2} \ , \quad (e^z)^n = e^{nz} \ .$$

Example 7: $f(z) = \sin z = \dfrac{e^{iz} - e^{-iz}}{2i} \ , \quad f(z) = \cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

Example 8: $f(z) = \sinh z = \dfrac{e^{z} - e^{-z}}{2} \ , \quad f(z) = \cosh z = \dfrac{e^{z} + e^{-z}}{2}$

Example 9: $f(z) = z + 1/z$. This is called a *linear fractional transformation.* We showed that it maps the upper half plane outside the unit circle 1-1 and onto the upper half plane.

Example 10: $f(z) = \log(z) = \log(re^{i\theta+2n\pi)}) = \text{Log}(r) + i(\theta + 2n\pi)$. This is our first example of a **multi-valued** function. Here $\text{Log}(r) = \ln(r)$, our usual single valued logarithm of a function of one variable. The multivalued complex log has several **branches**, corresponding to the several arguments of z. We can restrict the domain to make it single-valued. The **principal logarithm** (denoted by $\text{Log}(z)$) is

$$\text{Log}(z) = \text{Log}(r) + i\theta \ , \quad \text{where } \theta \in (-\pi, \pi] \ .$$

The **branch-cut** corresponding to this branch is the line across which the principal log is discontinuous, in this case $\theta = -\pi$. Note that with this definition $\text{Log}(z)$ agrees with $\ln(r)$ when $z = r$ is real, positive.

We could have used any other value of θ as a branch-cut to make the log single-valued. All these lines must include the origin, which is called a **branch-point**.

Note:

$$e^{\log z} = z \quad \text{but} \quad \log(e^z) \neq z$$

because of the multi-valuedness. $\log(z)$ satisfies the usual log properties (HW)

$$\log(z_1 z_2) = \log(z_1) + \log(z_2) \ , \quad \log(z_1/z_2) = \log(z_1) - \log(z_2) \ , \quad \log(z^p) = p\log(z) \ .$$

HW: Find the level curves of u and v for $\log(z) = u + iv$.

Example 11: $f(z) = z^{1/2}$ also multivalued: it has two values, that is, two branches.

Example 12: $f(z) = z^{1/3}$ has three branches.

Example 13: You can compute $f(z) = z^c$ for any complex c using the multi-valued or the principal logarithm using

$$z^c = e^{c\log(z)} \ , \quad \text{Principal branch:} \quad z^c = e^{c\ \text{Log}(z)} \ .$$

Example 14: HW: Using the principal branch of z^i, find its real and imaginary parts $u(r,\theta)$ and $v(r,\theta)$, using polar coordinates,

$$z^i = u(r,\theta) + iv(r,\theta) \ .$$