Lecture 3: Complex numbers - exponential form

1. Rules of complex exponential

The complex exponential satisfies basic rules for exponents

$$(1)\ e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}\ , \quad (2)\ (e^{i\theta})^n = e^{in\theta}\ , \quad (3)\ e^{-i\theta} = \frac{1}{e^{i\theta}}$$

Proof: We showed that (1) follows from the summation formula from trigonometry $\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$, $\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$ and (2) and (3) follow from (1).

Given these rules you can derive all other trigonometric identities. These are therefore much easier thought of in terms of complex exponentials.

2. Albegra in exponential form

Let $z_1 = 1 + i = \sqrt{2}e^{i\pi/4}$, $z_2 = -1 + \sqrt{3}i = 2e^{2\pi i/3}$.

2.1. Multiplication

Example 1: $z_1 z_2 = 2\sqrt{2}e^{11\pi i/12}$

2.2. Division

Example 2: $z_1/z_2 = (\sqrt{2}/2)e^{-5\pi i/12}$

2.3. Powers

Example 3: $z_1^{20} = -1024$

2.4 Roots

nth Roots of unity satisfy $z^n = 1$. Write $1 = e^{2n\pi i}$, $z = e^{i\theta}$ to find all n roots z_0^k, $k = 0, \ldots, n-1$, $z_0 = e^{2\pi i/n}$. Plot them in the complex plane. Using the same approach can solve any equation $z^n = a + ib$.

3. Fundamental Theorem of Albegra

Theorem: A polynomial $p(z) = z^n + a_{n-1}z^{n-1} + z_{n-2}z^{n-2} + \ldots a_1 z + a_0$ has precisely n complex roots $z_k, k = 1, \ldots, n$, and can be written as

$$p(z) = (z - z_1)(z - z_2)\ldots(z - z_n)$$

where z_k may be repeated.