Lecture 2: Complex numbers - Euler's formula

Some remarks on wednesday's material: if $z_1 = a + ib$, $z_2 = c + id$,

$$|z_1 - z_2| = \text{ distance between (a,b) and (c,d)}$$

$$\bar{\bar{z}} = z$$

iz appears to be counterclockwise rotation by 90^o

We also worked through example 2 on wednesday's notes.

1. Polar representation of z

To every point $z = a + ib$ we can associate polar coordinates r, θ.

Example 1: $z = 1 + i$ has values $r > 1, \theta = \pi/4 + 2n\pi$ for any positive or negative integer $n = \ldots, -2, -1, 0, 1, 2, \ldots$

Note: Any pair of radius $r > 0$ and argument θ uniquely defines a point in the plane. However, the argument θ that a point z has is **not unique**. Relations between a, b and r, θ

$$r = |z| = \sqrt{a^2 + b^2}, \quad \theta = arg(z) = \tan^{-1}(b/a)$$

$$a = r\cos\theta \ , \quad b = r\sin\theta$$

We will often be using the **Principal argument** $Arg(z) = \theta$, where $-\pi < \theta \leq \pi$.

2. Euler's formula

We define e^z in terms of known Taylor series for real arguments, so that it agrees with e^z if $z \in \Re$:

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^3}{4!} + \frac{z^3}{5!} + \ldots$$

It follows that

$$e^{i\theta} = \cos\theta + i\sin\theta \ , \quad \theta \in \Re$$

In turn, it follows that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \ , \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \ .$$

Thus we can write

$$z = a + ib = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

What makes this so convenient is that the complex exponential follows the same rules as the real exponential function, in particular

$$e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)} \ , \quad (e^{i\theta})^n = e^{in\theta} \ , \quad e^{-i\theta} = \frac{1}{e^{i\theta}}$$

These are also referred to as DeMoivre's formulas. They will be particularly useful to compute powers, roots, products and quotients of complex numbers.