**Lecture 1: Complex numbers - algebra and geometry**

## 1. Definitions

A *complex number* is any of the form

$$z = a + ib\ , \quad \text{where} \quad a, b \in \Re \quad \text{and} \quad i^2 = -1$$

The *real part of* $z$ is $Re(z) = a$.
The *imaginary part of* $z$ is $Im(z) = a$.
The *complex conjugate of* $z$ is $\overline{z} = a - ib$.
The *modulus of* $z$ is $|z| = \sqrt{a^2 + b^2}$.

*Example 1:* If $z = 2 - 3i$, then $\overline{z} = 2 + 3i$ and $Im(z) = -3$.

## 2. Algebra

Suppose $z_1 = a + ib$ and $z_2 = c + id$. Then define

*Addition:* $z_1 + z_2 = (a + c) + i(b + d)$

*Subtraction:* $z_1 - z_2 = (a - c) + i(b - d)$

*Multiplication:* $z_1 z_2 = (a + ib)(c + id) = (ac - bd) + i(bc + ad)$

*Division:* $\dfrac{z_1}{z_2} = \dfrac{a + ib}{c + id} = \dfrac{(a + ib)(c - id)}{(c + id)(c - id)}$ and continue simplifying from there (multiplication by conjugate makes the denominator real).

Always write your answer in form so that real and imaginary part are clear.

Note: Addition and multiplication satisfies the usual commutative, associative and distributive laws of the real numbers.

## 3. Geometry

We can identify $z = a + ib$ with the point $P(a, b)$ in the complex plane, or more precisely, with the vector $\overline{OP}$. Then $\overline{z}$ is symmetric about x-axis, and $|z|$ is the length of the vector $\overline{OP}$. We interpreted *addition, scalar multiplication, subtraction, multiplication by* $i$ geometrically.

*Example 2:* If $A(1, -2)$, $B(-3, 4)$, $C(2, 2)$ and $D$ is the midpoint of $\overline{AB}$, find $|\overline{CD}|$.

*Example 3:* Show that the diagonals of a parallelogram bisect each other (HW).

## 4. Some properties of modulus and complex conjugate

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \quad \text{(prove it - in class)}$$
$$\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2} \quad \text{(prove it - HW)}$$
$$|z_1 z_2| = |z_1||z_2| \quad \text{(prove it - HW)}$$

Nice and useful is the following:

$$z\overline{z} = |z|^2$$

*Triangle Inequality:*

$$|z_1 + z_2| \leq |z_1| + |z_2| \quad \text{(prove it+interpret geometrically - in class)}$$

It follows that:

$$||z_1| - |z_2|| \leq |z_1 + z_2| \quad \text{(prove it using triangle ineq - HW)}$$