MATH 313 — HOMEWORK # 5 Due : Monday, Feb 29, 2016.

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1. Assume a function $f(z) = u(r,\theta) + iv(r,\theta)$, where $z = x + iy$, and $x = r\cos\theta$, $y = r\sin\theta$, is differentiable at a nonzero point $z_0 = re^{i\theta} \neq 0$. Show that

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$

You may use any of the results you derived in your previous homework.

2. Suppose $f(z) = u(x,y) + iv(x,y)$ is analytic in a region R. Show that the level curves of u and v in the z-plane, $u = const$ and $v = const$ are orthogonal to each other.

3. (a) The inverse hyperbolic sine function is defined as $f(z) = \sinh^{-1} z$

$$w = \sinh^{-1} z \iff \sinh(w) = z$$

where $\sinh(w) = (e^w - e^{-w})/2$. Solve the equation on the right for w to show that

$$\sinh^{-1} z = \log[z + (z^2+1)^{1/2}]$$

(Hint: first multiply both sides by e^w and solve a quadratic equation for e^w.)

(b) Use (a) to show that

$$\sin^{-1} z = -i\log[iz + (1-z^2)^{1/2}]$$

(Hint: $w = \sin^{-1} z \iff z = \sin w = \sinh(iw)/i$. Use (a) to solve this equation for w.)

4. Show that $\mathrm{Log}[(1+i)^2] = 2\mathrm{Log}(1+i)$ but $\mathrm{Log}[(-1+i)^2] \neq 2\mathrm{Log}(-1+i)$.

5. Suppose that v is the harmonic conjugate of u. Explain why u is not the harmonic conjugate of v, unless both u and v are constant.

6. Evaluate (a) $\displaystyle\int_0^1 (1+it^2)\,dt$, (b) $\displaystyle\int_0^{\pi/3} e^{it}\,dt$, (c) $\displaystyle\int_0^{\infty} e^{-zt}\,dt$, if $Re(z) > 0$.

7. (a) Evaluate $\displaystyle\int_0^{2\pi} e^{ikx}\,dx$ is $k \neq 0$ and if $k = 0$.

(b) Use your result in (a) to evaluate $\displaystyle\int_0^{2\pi} \sin(mx)\sin(nx)\,dx$, if $n \neq m$ and if $n = m$, where both $n, m > 0$.

8. Suppose $f(x)$ is a real-valued continuous function. Show that

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$