MATH 313 — HOMEWORK # 4 Due : Friday, Feb 19, 2016.

TOPICS COVERED

Functions of a complex variable

Differentiability
Cauchy-Riemann Equations
Analytic functions
Harmonic functions

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1. Assume a function $f(z) = u(x,y) + iv(x,y)$, where $z = x + iy$, is differentiable. Show that u, v must satisfy the Cauchy-Riemann Equations.

2. Use your result in problem 1 to show that $f'(z)$ does not exist at any point for
(a) $f(z) = \overline{z}$ (b) $f(z) = 2x + ixy^2$ (c) $f(z) = z - \overline{z}$ (d) $f(z) = e^x e^{-iy}$

3. Determine where $f'(z)$ exists and find its value when
(a) $f(z) = 1/z$ (b) $f(z) = x^2 + iy^2$ (c) $f(z) = zIm(z)$

4. Prove that each of these functions is entire
(a) $f(z) = 3x + y + i(3y - x)$
(b) $f(z) = \sin x \cosh y + i \cos x \sinh y$
(c) $f(z) = (z^2 - 2)e^{-x}e^{-iy}$

5. Assume a function is given in polar coordinates, $f(z) = u(r,\theta) + iv(r,\theta)$, where $u(r,\theta) = u(x(r,\theta), y(r,\theta))$ and $v(r,\theta) = v(x(r,\theta), y(r,\theta))$. Derive the Cauchy-Riemann equations for u, v in terms of derivatives with respect to r, θ, as follows:
Use $x = r\cos\theta$, $y = r\sin\theta$ and the chainrule to write $u_r, u_\theta, v_r, v_\theta$ in terms of u_x, u_y, v_x, v_y.
Write your result as a linear system
$$\begin{pmatrix} u_r & v_r \\ u_\theta & v_\theta \end{pmatrix} = A \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix}$$
for some matrix A.
Invert A to write u_x, u_y, v_x, v_y in terms of $u_r, u_\theta, v_r, v_\theta$.
Rewrite the Cauchy-Riemann equations in terms of polar derivatives.

6. Show that $u(x,y)$ is harmonic in some domain and find a harmonic conjugate $v(x,y)$ for
(a) $u(x,y) = 2x(1-y)$ (b) $u(x,y) = 2x - x^3 + 3xy^2$.