TOPICS COVERED

Functions of a complex variable
limits and continuity

1. Remember that $\log z$ is defined as
$$\log(z) = \log(re^{i\theta+2n\pi}) = \mathrm{Log}(r) + i(\theta + 2n\pi)$$
where $\mathrm{Log}(r) = \ln r$ is the usual natural logarithm of a positive real number. We use Log instead of ln nonetheless since it agrees with the principal logarithm value of r. The principal logarithm is defined as
$$\mathrm{Log}(z) = \mathrm{Log}(r) + i\theta\ , \quad -\pi < \theta < \pi\ .$$
Using the definition and the properties of the natural logarithm for positive real numbers, show that $\log(z)$ satisfies
$$\log(z_1z_2) = \log(z_1) + \log(z_2)\ , \quad \log(z_1/z_2) = \log(z_1) - \log(z_2)\ , \quad \log(z^p) = p\log(z)\ .$$

2. Remember that $z^c = e^{c\log(z)}$. The principal branch of z^c is $z^c = e^{c\mathrm{Log}(z)}$. Using the principal branch of z^i, find its real and imaginary parts $u(r,\theta)$ and $v(r,\theta)$, using polar coordinates,
$$z^i = u(r,\theta) + iv(r,\theta)\ .$$

3. (a) Show that the level curves of $f(z) = z^2$ are orthogonal to each other.
(b) Repeat for $f(z) = \log z$.

4. Use the definition of limits to prove the following statements.
(a) $\displaystyle\lim_{z\to 2+i}(y^2 + i(2x - y)) = 1 + 3i$, (b) $\displaystyle\lim_{z\to\infty}\frac{1}{z^2} = 0$, (c) $\displaystyle\lim_{z\to 0}\frac{1}{z^2} = \infty$.

5. Using the definition of limits, show that
(a) $\displaystyle\lim_{z\to\infty} f(z) = w_0 \iff \lim_{z\to 0} f\Big(\frac{1}{z}\Big) = w_0$, (b) $\displaystyle\lim_{z\to z_0} f(z) = \infty \iff \lim_{z\to z_0}\frac{1}{f(z)} = 0$.

6. Using the results of the previous exercise, show that
(a) $\displaystyle\lim_{z\to\infty}\frac{4z^2}{(z-1)^2} = 4$, (b) $\displaystyle\lim_{z\to 1}\frac{1}{(z-1)^3} = \infty$.

7. Using the definition of the limit, show that the function $f(z) = \overline{z}$ is nowhere differentiable.

8. (a) Prove that the quadratic formula solves the quadratic equation
$$az^2 + bz + c = 0$$
when the coefficients a, b, c are complex numbers. Specificallly, by completing the square on the left-hand side, prove that the roots of the equation are
$$z = \frac{-b + (b^2 - 4ac)^{1/2}}{2a}\ ,$$
where the two square roots are to be considered when $b^2 - 4ac \neq 0$.
(b) Use the result in part (a) to find the roots of the equation
$$z^2 + 2z + (1 - i) = 0\ .$$