MATH 313 — HOMEWORK # 2 Due : Friday, Feb 5, 2016.

TOPICS COVERED

Complex numbers

Sets of complex numbers

Functions of a complex variable

complex and real valued
single- and multi-valued functions
examples: polynomials, exponentials, $\sin(z)$, $log(z)$, $Log(z)$

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1. Prove that $\sqrt{2}|z| \geq |Re(z)| + |Im(z)|$
2. Prove that z is real if and only if $\overline{z} = z$
3. (a) Show that $|z - z_0| = R$ is the equation of a circle centered at z_0 of radius R.
   (b) Show that the hyperbola $x^2 - y^2 = 1$ can be written as $z^2 + \overline{z}^2 = 2$
4. (a) Show that a set is open if and only if it does not contain any of its boundary points.
   (b) Explain why a set S and its complement S^c have the same boundary.
   (c) Explain why a set is open if and only if its complement is closed.
5. Sketch the following sets and determine whether they are open or closed, or neither.
   (a) $|z - 2 + i| \leq 1$ (b) $|2z + 3| > 4$ (c) $Im(z) > 1$ (d) $Im(z) = 1$
   (e) $0 \leq arg z \leq \pi/4$, $z \neq 0$ (f) $|z - 4| \geq |z|$
   (g) $0 < |z - z_0| < \delta$ where z_0 is a fixed point and $\delta > 0$ (a positive real number).
6. Sketch each of the following sets and their closure
   (a) $-\pi < arg z < \pi, z \neq 0$ (b) $|Re z| < |z|$ (c) $|Re(1/z)| < 1/2$ (d) $Re(z^2) > 0$
7. (a) Is the set $\{1/n, n = 1, 2, 3, \ldots\}$ open, closed or neither? Explain. Is it bounded? Is it connected?
   (b) Is the set $\{1/n, n = 1, 2, 3, \ldots\} \cup \{0\}$ open, closed or neither? Explain.
   (c) Give an example of a bounded, connected set.
   (d) Give an example of a open and bounded set that is not connected.
8. Establish the identity

$$1 + z + z^2 + \ldots + z^n = \frac{1 - z^{n+1}}{1 - z} \ , \quad z \neq 1$$

and then use it to derive the identity:

$$1 + \cos\theta + \cos 2\theta + \ldots + \cos(n\theta) = \frac{1}{2} + \frac{\sin[(n + \frac{1}{2})\theta]}{2\sin(\theta/2)} \ , \quad 0 < \theta < 2\pi$$

Suggestion: As for the first identity, write $S = 1 + z + z^2 + \ldots + z^n$ and consider the difference $S - zS$. To derive the second identity, write $z = e^{i\theta}$ and then use that

$$1 - z = e^{i\theta/2}e^{-i\theta/2} - e^{i\theta/2}e^{i\theta/2}$$

on the bottom, and that

$$1 - z^n = e^{i\theta/2}e^{-i\theta/2} - e^{i\theta/2}e^{i(n+1/2)\theta}$$

on the top, cancel common factors, use Euler's identity and match real parts.

9. Find the Cartesian representation of (a) $e^{2\pm 3\pi i}$, (b) $e^{(2+\pi i)/4}$, (c) $\log e$, (d) $\log i$, (e) $\log(1+\sqrt{3}i)$, (f) $\mathrm{Log}(-ei)$, (g) $\mathrm{Log}(1-i)$.

10. Show that $|e^{z^2}| \leq e^{|z|^2}$

11. Show that
    (a) $e^{\overline{z}} = \overline{e^z}$
    (b) $e^{i\overline{z}} = \overline{e^{iz}}$ if and only if $z = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$

12. Schaums: 2.48, 2.49.

13. Find the principal value of (a) i^i , (b) $\frac{e}{2}(-1-\sqrt{3}i)^{3\pi i}$, (c) $(1-i)^{4i}$.

14. Find the preimage in the z plane of the horizontal and vertical lines $u = const$ and $v = const$ in the w plane if

$$w = u + iv = \log(z) \ .$$

Sketch the corresponding curves in the z-plane and the image in the w-plane.