MATH 313 Exam 2— Review Topics and Problems

1. Evaluating Integrals

Evaluate integrals using any of the below (including improper integrals):

(1) $\displaystyle\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$

(2) $\displaystyle\int_C f(z)\,dz = \int_a^b f'(z(t))z'(t)\,dt$ where the curve C is parametrized by $z(t)$, $t \in [a,b]$.

(3) If f analytic in region D containing a smooth curve C then

- If C simple, closed: $\displaystyle\oint_C f(z)\,dz = 0$ (this generalizes to regions with holes.)
- If C goes from A to B: $\displaystyle\int_C f(z)\,dz = \int_A^B f(z)\,dz$ is path independent.

(4) If $f(z) = F'(z)$ in a region D containing a smooth curve C from A to B, then

- If C goes from A to B: $\displaystyle\int_C f(z)\,dz = \int_A^B f(z)\,dz = F(B) - F(A)$

(Note: F has to be analytic in D. Why?)

(5) Cauchy integral theorem and its derivatives

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(s)}{s-z}\,ds\,, \qquad f^{(n)}(z) = \frac{n!}{2\pi i}\oint_C \frac{f(s)}{(s-z)^{n+1}}\,ds$$

Examples: Homework #5: 6,7.
Homework #6: 4,5,6,7,
Homework #7: 1,4,5,6.
Homework #7: 8. Similar examples done in class.

Example: Let f be analytic in a region D. Let C be a generic closed simple smooth curve in D enclosing a point z_0. Let $C_\epsilon : |z - z_0| = \epsilon$ be a circle completely contained in C. Show that

$$\oint_C \frac{f(z)}{z-z_0}dz = \oint_{C_\epsilon} \frac{f(z)}{z-z_0}dz$$

Example: Evaluate $\displaystyle\oint_{|z-i|=4} \frac{z^4 - 3z^2}{(z-1)(z+2i)^2}dz$

Example: Show that $\displaystyle\oint_{|z+i|=1.2} \frac{f(z)}{(z-1)(z+2i)^2}dz = -2\pi i\Big[\frac{f'(-2i)}{1+2i} + \frac{f(-2i)}{(1+2i)^2}\Big]$

2. Bounding Integrals

Bound integrals and use the bound to determine certain limits.
Examples: HW 6: 3, HW 7: 3

3. Analytic functions

(1) Find real and imaginary parts, find singular points, define branch cuts.
Examples: HW 6: 1, HW 7: 1,7.

(2) Show $f(z_0)$ is average of values on circle around it. Conclude.
Examples: Homework #7: 2.

(3) List all properties of analytic functions you can think of.

(4) Give an example of a function that is not analytic at a point. Explain.
Give an example of a function that is not analytic anywhere. Explain.