MATH 313 — Final Summary and review topics

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1. Complex numbers
- ∘ Add, multiply, divide complex numbers. Write in exponential form. Find logarithms and exponentials. Plot in the complex plane. Represent complex numbers, their sums, products,
- ∘ Find roots and powers.
- ∘ Derive simple trig identities using rules

2. Complex functions
- ∘ definition of the derivative
- ∘ Cauchy-Riemann equations (can you derive them?)
- ∘ Determine where functions are differentiable. Justify.
- ∘ Give an example of a function that is not analytic at a point. Explain.
- ∘ harmonic functions (what are they? show given function is harmonic)
- ∘ harmonic conjugate (find them)

3. Mappings
- ∘ Find where points, curves and regions are mapped to by a given function $f(z)$.
- ∘ Properties of conformal maps.
- ∘ Given a map $w = f(z)$ and a harmonic function in the w-plane, find a harmonic function in the z-plane.
- ∘ What are linear fractional transformations? What special properties do they have?

4. Cauchy integral theorem

If f is analytic in a region containing simple close curve C, then

$$\oint_C f(z)\,dz = 0\ .$$

Furthermore,

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(s)}{s-z}\,ds\ , \qquad f^{(n)}(z) = \frac{n!}{2\pi i}\oint_C \frac{f(s)}{(s-z)^{n+1}}\,ds$$

Conclusions:
- If f has n singularities at z_k, $k = 1, \ldots, n$, contained in a simple closed curve C, then

$$\oint_C f(z)\,ds = \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz$$

  where C_k is a simple closed curve containing only the singularity at z_k. (Why? Show this for $n = 2$)
- $f(z_0)$ is average of values on circle around it (show this)
- Extrema of f occur on boundary of domain, not in interior (explain why)

5. Taylor and Laurent series, singular points, residues

Every function has a Laurent series representation of the form

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \frac{b_2}{(z-z_0)^3} + \ldots$$

in an annular region centered at z_0 in which f is analytic. If z_0 is an isolated singularity of f and C is a closed curve containing z_0 and no other singularities of f, it follows from the Cauchy Integral Theorem that

$$\oint_C f(z)\,dz = 2\pi i b_1$$

The number b_1 is called the **residue of f at z_0.**

- Find Taylor series for analytic functions, using definition or manipulation of known series. Why are zeros of (nontrivial) analytic functions isolated?
- Find Laurent series using manipulation of known Taylor series.
- What is the domain of convergence of Taylor and Laurent series?
- Find and classify singular points (poles of finite order, essential singularities, removable singularities).
- List all properties of analytic functions you can think of.
- Find residues at z_0 using any of the methods summarized below.
  (1) Find the Laurent Series about z_0, at least a few terms thereof.
  (2) Using the Cauchy Integral formula, as follows: Assume f has an isolated singularity at z_0. If the singularity is a simple pole, $f = \phi(z)/(z - z_0)$ where ϕ analytic at z_0, then

$$\oint_C f(z)\,dz = \oint_C \frac{\phi(z)}{z - z_0}\,dz = 2\pi i \phi(z_0)\ ,$$

  so the residue of f at z_0 is $b_1 = \phi(z_0)$. If the singularity is a multiple pole, $f = \phi(z)/(z - z_0)^n$ where ϕ analytic at z_0, then

$$\oint_C f(z)\,dz = \oint_C \frac{\phi(z)}{(z - z_0)^{n+1}}\,dz = \frac{2\pi i}{n!}\phi^{(n)}(z_0)$$

  so the residue of f at z_0 is $b_1 = \phi^{(n)}(z_0)/n!$.
  (3) If $f(z) = p(z)/q(z)$ has a simple pole at z_0, where p, q are analytic at z_0, then

$$\oint_C f(z)\,dz = \oint_C \frac{p(z)}{q(z)}\,dz = 2\pi i \frac{p(z_0)}{q'(z_0)}$$

  so the residue at z_0 is $b_1 = p(z_0)/q'(z_0)$.

6. Evaluating Line Integrals

Evaluate integrals using any of the below (including improper integrals):

(1) $\displaystyle\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$

(2) $\displaystyle\int_C f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt$ where the curve C is parametrized by $z(t)$, $t \in [a, b]$. Be able to parametrize any curve (circles, lines, parabolas)!

(3) If f is analytic, with $f(z) = F'(z)$, in a region D containing a smooth curve C from A to B, then

$$\int_C f(z)\,dz = \int_A^B f(z)\,dz = F(B) - F(A)$$

(4) If f analytic in region D containing a simple closed curve C, except for n isolated singularities at $z_k, k = 1, \ldots, n$, then

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} B_k$$

where B_k are the residues at z_k.

7. Evaluating real integrals using residues

- Bound integrals and use the bound to determine certain limits.
- Integrals of the form $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$.
- Integrals of the form $\displaystyle\int_{-\infty}^{\infty} f(x)\cos x\,dx$ and $\displaystyle\int_{-\infty}^{\infty} f(x)\sin x\,dx$.
- Jordan's inequality. When do you need it?

Review Problems: Choose problems from HW and Exams on the topics listed above. You can skip the following problems, HW 1: 5, 8. HW 2: 1, 3, 4, 7, 8. HW 3: 1, 2, 3(read only), 4, 5, 6, 8. HW 4: 5. HW 5: 1, 2(read only), 3, 4, 8(read only). HW 6: 2. HW 7: 1. HW 8: 2, 3, 5, 6, 7. HW 10: 1a. HW 11: 3.