LINEAR APPROXIMATIONS

Notes:
• The linear approximation of a function $f(x)$ about a base point $a$ is

$$ L(x) = f(a) + f'(a)(x - a) $$

The graph of this linear function is the line tangent to $f$ at $x = a$.
A useful application is to approximate $f$ by its linearization near a basepoint, $f(x) \approx L(x)$ if $x \approx a$. For example

$$ \sin x \approx x \quad \text{if} \quad x \approx 0 $$
$$ \sqrt{1 + x} \approx 1 + x/2 \quad \text{if} \quad x \approx 0 $$

• The change in the function $f$ between $x = a$ and $x = a + \Delta x$ can be approximated by the change in the linear approximation $L(x)$,

$$ f(a + \Delta x) - f(a) \approx L(a + \Delta x) - L(a) \quad (*) $$

This is useful since the right hand side is particularly simple to evaluate. Namely, if we denote the change in $f$ by $\Delta y$, equation (*) reduces to

$$ \Delta y \approx f'(a)\Delta x \quad (***) $$

Draw a picture showing $\Delta y$ and $f'(a)\Delta x$.

• One possible application of this: if you know an upper bound for a possible change $\Delta x$ in $x$, $|\Delta x| \leq \epsilon$, then you can approximately bound the change $\Delta y$ in the function by

$$ |\Delta y| \approx |f'(a)||\Delta x| \leq |f'(a)|\epsilon $$