The Torus and Noncommutative Topology

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Abstract

Let A_θ denote the rotation algebra, generated by unitaries U and V which satisfy $UV = e^{2\pi i\theta}VU$, $0 \leq \theta < 1$. For irrational values of θ , Pimsner and Voiculescu have shown that there is an embedding of A_θ into an AF algebra that induces an isomorphism of the K_0 groups. It remains unsettled whether such an embedding can be found for rational θ . In this paper we partially solve the problem by constructing a unital embedding of $A_0 = C(T^2)$ into an AF algebra for which the induced map on K_0 is injective. Using strong Morita equivalence, an analogous embedding can be found of the other rational rotation algebras.

The existence of these embeddings shows that cohomology theory cannot be extended to C*-algebras in a way that satisfies the axioms of stability and continuity while still admitting a well behaved Chern character. It also follows that A_θ is not semiprojective [EK1] for any value of θ . We show, however, that the quotient $C(S^1 \vee S^1)$ is semiprojective. These examples suggest that semiprojectivity may be a good characterization of one-dimensionality for C*-algebras.

The embedding of $C(T^2)$ is determined by two commuting unitaries in an AF algebra. By approximating these by unitaries that lie in finite dimensional subalgebras, one obtains two sequences of unitary matrices that commute asymptotically but which cannot be approximated by sequences of commuting unitaries. The first example of such a pair of sequences was given by Voiculescu. The calculations that are needed for calculating the K-theory of the AF embedding of $C(T^2)$ provide a new proof of Voiculescu's example, that illustrates the exact role played by the second-cohomology of the torus.

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Of course, no one can begin a thesis without first struggling through the terrible transition from student to researcher. For me, this process was helped immensely by the confidence that several people had in me, notable my advisor, my family, and my wife. And to Larry Carter, thanks for teaching me early on what it is that mathematicians really do. To wit, "find patterns, make conjectures, prove things."

Introduction

In the study of C*-algebras, it is common to look to the commutative case for inspiration. Since a commutative C*-algebra is isomorphic to $C_0(X)$ for a locally compact Hausdorff space, it is very tempting to try to extend the methods of algebraic topology to work for noncommutative C*-algebras. This extension process has worked well enough to deserve a name, and is often called noncommutative topology. Among the triumphs in this field are K-theory for C*-algebras and shape theory for C*-algebras. The K-groups $K_0(A)$ and $K_1(A)$ have a very natural definition [B4, T1] for any C*-algebra, and they agree exactly with topological K-theory in that $K_i(C_0(X)) \cong K^i(X)$ for $i = 1, 2$. Shape theory is not nearly so well developed, but some interesting results have been obtained [B1, EK1, EK2].

There must, however, be some limits on how far the methods of algebraic topology can be generalized to work for C*-algebras. There have not been any successful definitions of a cohomology for C*-algebras, extending, say, singular cohomology. (Connes' cyclic cohomology works well only for suitable proper dense subalgebras of C*-algebras.) It may be that no definitions are possible unless we limit our expectations on what properties of topological cohomology carry over to C*-algebras.

K-theory is a nonstandard cohomology that contains much less information than the standard cohomology theories. Let H^{ev} and H^{odd} denote the direct sums of the even and odd cohomology groups respectively. The Chern character provides a natural homomorphism $ch : K^0(X) \to H^{ev}(X, \mathbf{Z})$ which is an isomorphism up to torsion. A good cohomology theory for C*-algebras should include a Chern character, but as Blackadar pointed out to me, this may be too much to hope for, given the many maps that can exist [B3] from commutative C*-algebras into approximately finite-dimensional C*algebras (AF algebras).

AF algebras are generally thought of as being zero-dimensional. By definition [B2], an AF algebra A has an increasing sequence A_n of finite-dimensional subalgebras so that $A = \overline{\bigcup A_n}$. This is generally denoted $A = \lim A_n$. A commutative AF algebra is isomorphic to $C(\Sigma)$ where Σ is a zero-dimensional space. As a consequence, $H_n(A) = 0$ for $n > 0$ for a commutative AF algebra. As we shall now see, under reasonable assumptions, $H_n(A)$, $n>0$, will be zero for any AF algebra.

In the next propositions, the term cohomology theory will be used loosely to mean any abelian group valued functor defined on C*-algebras. If H_n, is such a collection of functors, then as usual, H_* and H_{ev} shall denote $\bigoplus_{n\geq 0} H_n$ and $\bigoplus_{n\geq 0} H_{2n}$ respectively.

0.1 Proposition: Assume that H_* is a cohomology theory for C*-algebras that satisfies the following axioms:

1) Stability: $H_*(A\otimes M_n) \cong H_*(A)$ where M_n denotes the algebra of n by n matrices
2) Continuity: $H_*(\lim A_n) \cong \lim H_*(A_n)$
3) $H_*(A\oplus B) \cong H_*(A)\oplus H_*(B)$
4) $H_*(\mathbf{C}) = H^*(pt) = \mathbf{Z}$ in dimension zero, 0 in higher dimensions

Then for any AF algebra A, $H_n(A) = 0$ for $n > 0$.

Proof: Since any finite-dimensional C*-algebra is a direct sum of matrix algebras, axioms 1, 3 and 4 imply that, for n greater than zero, H_n is zero on any finite-dimensional algebra. By axiom 2 this is also true for any AF algebra.

Q.E.D.

In chapter **II** we show that there exists an AF algebra A and a homomorphism $\varphi : C(T^2) \to A$ such that the induced map $\varphi_* : K_0(C(T^2)) \to K_0(A)$ is an injection. The following proposition, mentioned to me by Blackadar, shows that, given this example, the four axioms above are incompatible with a well-behaved Chern character.

0.2 Proposition: Suppose H_* is a cohomology theory for C*-algebras, and suppose that there is a natural transformation $ch : K_0 \to H_{ev}$ which is an isomorphism up to torsion. If $\varphi : C(T^2) \to A$ is a homomorphism that induces an injection on K_0, then $H_2(A)$ contains an infinite cylic element.

Proof: The groups $K^0(T^2) \cong \mathbf{Z}^2$, $H^0(T^2) \cong \mathbf{Z}$ and $H^2(T^2) \cong \mathbf{Z}$ are all torsion-free so $ch : K^0(T^2) \to H^0(T^2) \oplus H^2(T^2)$ is an isomorphism. Since ch is natural, the following diagram is commutative.

$$\begin{array}{ccc} K^0(T^2) & \xrightarrow{\varphi_*} & K_0(A) \\ ch^{-1} \uparrow & & \downarrow ch \\ H^0(T^2) \oplus H^2(T^2) & \xrightarrow{\varphi_* \oplus \varphi_*} & H_0(A) \oplus H_2(A) \oplus \ldots \end{array}$$

On the top row, φ_* maps injectively into the torsion-free part of $K_0(A)$. Since ch is injective on torsion-free elements, the composition $ch\ \varphi_*\ ch^{-1} = \varphi_* \oplus \varphi_*$ is injective. In particular, $\varphi_* : H^2(T^2) \cong \mathbf{Z} \to H^2(A)$ is an injection which proves the theorem. Q.E.D.

What does this mean for cohomology theory for C*-algebras? It may be that there are cohomologies which satisfy axioms 1 to 4, but for which there is either no Chern character or for which the associated Chern character is only an isomorphism (up to torsion) in the commutative case. Notice that proposition 0.2 says nothing about the Chern character of Connes [C1, C2], which takes its values in the cyclic cohomology. Blackadar has suggested that K_0 may still break up into the various even-dimensional parts, but in a more subtle way than a direct sum, such as a filtration. It may be the case that we can describe what it means for $a \in K_0(A)$ to be a two-dimensional element, but $f_*(a)$ may be zero-dimensional for some homomorphism f. Another interpretation of these propositions is that one of the axioms 1 through 4 should be discarded so that some AF algebras can have nonzero second-cohomology. Stability is certainly suspect since it is motivated only by the stability of the K-theory of C*-algebras, there being no counterpart for stability in topological cohomology. See [S3] for a study of axiomatic cohomology for C*-algebras in which stability is not assumed. In any case, it is fair to say we need to rethink the notion that AF algebras should be thought of as zero-dimensional.

The best place to find motivation for the existence of an embedding φ from $C(T^2)$ into an AF algebra which is injective on K_0 is in the work of Pimsner and Voiculescu, since they have done the most work on embeddings into AF algebras. By looking at the Fourier transform, it is clear that $C(T^2)$ is generated by two commuting unitaries, $e^{2\pi ix}$ and $e^{2\pi iy}$. In fact, $C(T^2)$ is the universal C*-algebra generated by two commuting unitaries. This is a special case of the rotation algebras A_θ. For any real number θ, A_θ is defined as the universal C*-algebra generated by two unitaries U and V subject to the commutation relation $UV = e^{2\pi i\theta}VU$. We call A_θ a rational or an irrational rotation algebra depending on whether or not θ is rational. The first important AF embeddings were Pimsner and Voiculescu's embeddings of the irrational rotation algebras into an AF algebra. These embeddings induced injections on K_0. This was important since, combined with the work of Rieffel [R1], it provided the first calculation of the image of the trace on $K_0(A_\theta)$. Since $C(T^2)$ is the zero-rotation algebra, it seems likely that a similar construction will work in this case.

Pimsner [P1] went on to find a method for embedding a wide class of transformation group C*-algebras $C^*(X, \mathbf{Z}, \alpha)$ into AF algebras. Here α is an action of $\mathbf{Z}$ on a topological space X. When the space is taken to be S^1 and α is rotation though an angle of $2\pi\theta$, $C^*(X, \mathbf{Z}, \alpha)$ is isomorphic to the rotation algebra A_θ. For θ irrational, Pimsner's embedding is more or less the same as the original embedding of Pimsner and Voiculescu. Corollary **II**.5.4, which depends on theorem **II**.5.2, shows that, for $\theta = 0$, Pimsner's embedding also will induce an injection on the K_0 groups.

Any map from $C(T^2)$ to an AF algebra A is defined by a pair of commuting unitaries U and V in A. Any unitary in an AF algebra will be approximated by unitaries in the finite subalgebras. Commuting unitaries in an AF algebra are therefore defined by convergent sequences U_n and V_n of unitary matrices (of increasing dimension) that commute asymptotically in the sense that $\| U_nV_n - V_nU_n \| \to 0$. In [V1] Voiculescu has exhibited two (nonconvergent) sequences of unitaries which commute asymptotically, but which cannot be approximated by commuting unitaries. He mentions that his example depends on the nonzero second-cohomology of the torus. This will be made precise in chapter **I** where a new proof of this result is given using K-theory.

The fact that there exist homomorphisms from $C(T^2)$ to AF algebras which are injective on K_0 is of interest independent of its implications for cohomology of C*-algebras. For instance Spielberg [S2, section 4.5] discusses such maps in relation to the AF embeddings of suspended solenoids. It also has implications regarding the shape theory of C*-algebras. Shape theory for compact spaces, as developed in [MS1], is the study of the ways that a space can be written as a limit of ANR's. The reason ANR's are choosen as the basic building block is that they have the useful homotopy property that any map to an ANR from a limit of spaces is homotopic to a map from one of those spaces [MS1, lemma 3]. By abstracting this homotopy property, Effros and Kaminker have defined what it means for a C*-algebra to be a "noncommutative ANR."

0.3 Definition: A C*-algebra A is semiprojective if for every system of C*-algebras

$$B_1 \xrightarrow{\varphi_1} B_2 \xrightarrow{\varphi_2} B_3 \xrightarrow{\varphi_3} \cdots \qquad \varphi_n \text{ injective, unital}$$

every unital homomorphism $\psi : A \to \lim B_n$ is homotopic to a unital homomorphism into some B_n. (It is actually a theorem that this agrees with the usual definition of semiprojectivy. See section III.1.)

A shape system for a C*-algebra B is a system of C*-algebras $B_n \to B_{n+1}$ such that $B \cong \lim B_n$. Since finite-dimensional algebras are semiprojective [EK1], a Bratteli diagram [B2] is an example of a shape system.

A natural question to ask is whether or not C(X) is semiprojective for X an ANR. While C(X) will have the required homotopy property for limits of commutative C*-algebras, it is too much to hope that it will have the required property for general B_n. It is possible that C(X) will be semiprojective only for spaces that have the homotopy type of a one-dimensional space. We shall see that the K-theory calculations in chapter II imply that $C(T^2)$, $C(S^2)$ and the rotation algebras are not semiprojective. In chapter III it is proved that $C(S^1 \vee S^1)$ is semiprojective.

The first chapter investigates the K-theory of the torus from a C*-algebra point of view. An explicit formula for a nontrivial projection in $M_2(C(T^2))$ is obtained, leading to the definition of a projection e(U,V) which is defined for any pair of unitaries U and V which commute. This formula is extended to pairs of unitaries with small, but nonzero, commutator for which it defines matrices that are approximately projections. This extended formula is then used to give a K-theoretic proof of Voiculescu's example.

The second chapter, which relies heavily on the results in the first, defines the embedding φ of $C(T^2)$ into an AF algebra and calculates its K-theory. The construction is basically just an application of the results in [P1], but since this example illuminates Pimsner's techniques while avoiding some technicalities, the embedding is constructed directly in proposition II.4.1. It should be possible to read the proof of proposition II.4.1 without prior knowlege of Pimsner's work.

The main results in the third chapter are that $C(S^1 \vee S^1)$ is semiprojective, while the rotation algebras are not. The proof that $C(S^1 \vee S^1)$ is semiprojective is logically independent of the previous chapters, while the proof that the rotation algebras are semiprojective depends on theorem II.5.2.

Chapter I: A K-theoretic obstruction to commuting approximants for asymptotically commuting unitary matrices

§1 Introduction: Voiculescu [V1] has shown by example that the following question can sometimes have a negative answer: Given unitaries $U_n, V_n \in \mathcal{U}(m_n)$ such that $\lim_{n\to\infty} \| [U_n, V_n] \| = 0$, do there exist unitaries $U_n', V_n' \in \mathcal{U}(m_n)$ so that $[U_n', V_n'] = 0$ and $\lim_{n\to\infty} (\| U_n - U_n' \| + \| V_n - V_n' \|) = 0$? This question can be translated into a lifting problem for a certain homomorphism from $C(T^2)$. Voiculescu remarks that his counterexample seems to depend on the nonzero second-cohomology of the space T^2, and so is unlikely to have any bearing on the (unsolved) corresponding question for bounded sequences of selfadjoint matrices. This chapter should make the role of the second-cohomology in Voiculescu's example explicit.

The matrices in Voiculescu's example are S_n and Ω_n where

$$S_n = \begin{pmatrix} 0 & & & & & 1 \\ 1 & 0 & & & & \\ & 1 & 0 & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & 1 & 0 \end{pmatrix} \quad \Omega_n = \begin{pmatrix} \omega & & & & & & \\ & \omega^2 & & & & & \\ & & \omega^3 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \omega^n \end{pmatrix} \quad \omega = n^{th} \text{ root of unity.}$$

Voiculescu gives a proof of the following result that is based on the non-quasidiagonality of the unilateral shift.

1.1 Theorem: Let S_n and Ω_n be the matrices above. Then $\lim \| [S_n, \Omega_n] \| = 0$, but there do not exist unitaries U_n and V_n such that $U_nV_n = V_nU_n$ and

$$\lim \| S_n - U_n \| = \lim \| \Omega_n - V_n \| = 0 .$$

The proof of theorem 1.1 given in section 4 follows a different course. While the second-cohomology of the torus is not explicitly mentioned in the proof, it is always in the background as explained below.

Let $m(n)$ be any sequence of integers. Voiculescu considers the C*-algebra

$$\mathcal{A} = \{(T_n)_1^\infty \mid T_n \in M_{m(n)} , \sup_n \| T_n\| < \infty \}$$

and the ideal $\mathcal{J}$ which consists of sequences (T_n) such that $\lim \| T_n \| = 0$. Asymptotically commuting unitaries define commuting unitaries in the quotient $\mathcal{A}/\mathcal{J}$, and so define a $*$-homomorphism of $C(T^2)$ into $\mathcal{A}/\mathcal{J}$. The approximation question is equivalent to asking whether every $*$-homomorphism $C(T^2) \to \mathcal{A}/\mathcal{J}$ can be lifted to $\mathcal{A}$. Lemma 4.1 below will show that any $\psi : C(T^2) \to \mathcal{A}$ induces a map $\psi_* : K_0(C(T^2)) \to K_0(\mathcal{A})$ whose kernel contains the second-cohomology of T^2, where $K_0(C(T^2))$ is identified with the even cohomology of the torus via the Chern character. Thus one obstruction to lifting φ is that it must also contain the second-cohomology in its kernel. Stated more concretely, φ cannot be lifted unless $\varphi_*(1)$ and $\varphi_*(e)$ are equivalent projections, where e is the projection to be defined in section two.

Now let $\varphi : C(T^2) \to \mathcal{A}/\mathcal{J}$ denote the $*$-homomorphism corresponding to S_n and Ω_n. Using the six-term exact sequence for K-theory, it quickly follows that $K_0(\mathcal{A}/\mathcal{J})$ is isomorphic to the group of all sequences of integers where two sequences are identified if they agree except on a finite portion. The content of theorem 4.2 is that $\varphi_*(e)$ corresponds to the equivalence class of the sequence $(n - 1)$ while clearly $\varphi_*(1)$ corresponds to the equivalence class of the sequence (n). These are not equivalent, and so φ cannot be lifted.

§2 The K-theory of the torus. The complex vector bundles over the torus T^2 are classified up to isomorphism by their images in $K_0(T^2)$, which is isomorphic to $\mathbf{Z}^2$. The first integer corresponds to the dimension of the fibers and the second integer is the first Chern class. This section will describe how to find a projection in $M_2(C(T^2))$ of a particulary simple form which corresponds to the bundle of dimension one with first Chern class equal to one.

Following along the lines of [R1], we take a guess that the desired projection can be taken to be of the form

$$e = \begin{pmatrix} f & g + hU \\ hU^* + g & 1 - f \end{pmatrix}$$

where $U = e^{2\pi iy}$ and $f(x)$, $g(x)$ and $h(x)$ are nonnegative functions on $S^1 = \mathbf{R}/\mathbf{Z}$ (i.e., are functions on the real line of period one). Setting e^2 equal to e imposes the condition

$$f = f^2 + ghU + g^2 + h^2 + ghU^*,$$

or equivalently,

$$gh = 0$$
$$\text{and} \quad g^2 + h^2 = f - f^2$$

One way to satisfy these equations is to choose any f for which

1) $0 \leq f \leq 1$

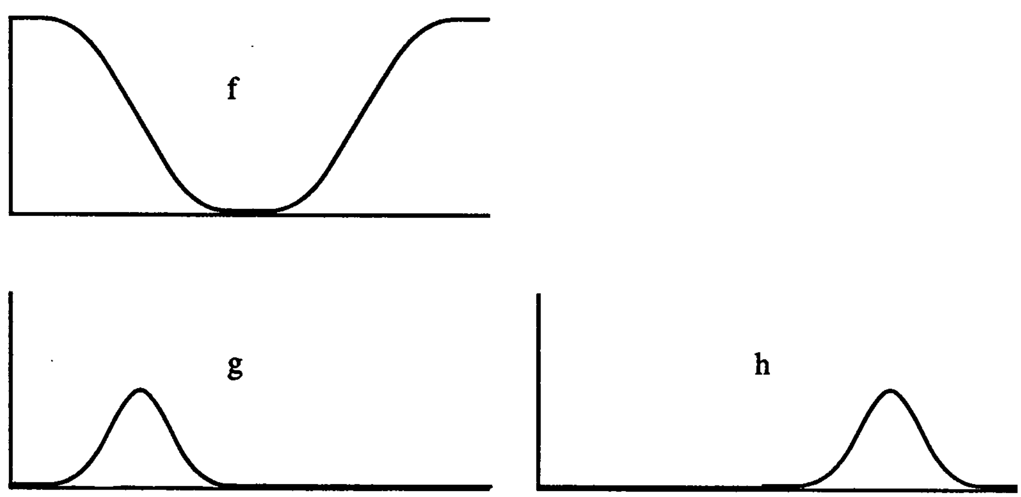
and 2) $f(0) = 1$, $f(1/2) = 0$ and $f(1) = 1$

and then define g and h by

3) $g = \chi_{[0,1/2]}(f - f^2)^{1/2}$

4) $h = \chi_{[1/2,1]}(f - f^2)^{1/2}$

where χ_X denotes the characteristic function on the set X . Furthermore, we assume that f , g and h are smooth functions.



As defined above, e will have trace one and so represents a bundle with one-dimensional fibers. To calculate the first chern class $c_1(e)$ we use the formula [see C1]

$$c_1(e) = \tau(e(\delta_1(e)\delta_2(e) - \delta_2(e)\delta_1(e))) / 2\pi i$$

where δ_1 and δ_2 are the component-wise extensions to $M_2(C^\infty(T^2))$ of differentiation by x and y respectively. The symbol τ denotes the trace on $C(T^2)$ corresponding to Lebesgue measure, extended to matrices in the usual way.

2.1 Proposition: For any choice of smooth functions f , g and h satisfying conditions 1) to 4), the first chern class $c_1(e)$ is equal to 1 .

Proof:

$$\delta_1(e) = \begin{pmatrix} f' & g' + h'U \\ h'U^* + g' & -f' \end{pmatrix} \qquad \delta_2(e)/2\pi i = h \begin{pmatrix} 0 & U \\ -U^* & 0 \end{pmatrix}$$

Since g and h have disjoint supports, $gh = g'h = gh' = 0$ and so

$$\delta_1(e)\delta_2(e)/2\pi i = h \begin{pmatrix} -h' - g'U^* & f'U \\ f'U^* & h' + g'U \end{pmatrix} = h \begin{pmatrix} -h' & f'U \\ f'U^* & h' \end{pmatrix}$$

$$\delta_2(e)\delta_1(e)/2\pi i = h \begin{pmatrix} h' + g'U & -f'U \\ -f'U^* & -h' - g'U^* \end{pmatrix} = h \begin{pmatrix} h' & -f'U \\ -f'U^* & -h' \end{pmatrix}$$

Therefore, $e(\delta_1(e)\delta_2(e) - \delta_2(e)\delta_1(e)) / 2\pi i =$

$$2h \begin{pmatrix} f & g + hU \\ hU^* + g & 1-f \end{pmatrix} \begin{pmatrix} -h' & f'U \\ f'U^* & h' \end{pmatrix} = 2 \begin{pmatrix} fh & h^2U \\ h^2U^* & h - fh \end{pmatrix} \begin{pmatrix} -h' & f'U \\ f'U^* & h' \end{pmatrix}$$

$$= 2 \begin{pmatrix} -fhh' + h^2f' & ff'hU + h^2h'U \\ -h^2h'U^* + (h - fh)f'U & -fhh' + h^2f' + hh' \end{pmatrix}$$

Applying the trace, we get

$$c_1(e) = 2\tau(-fhh' + h^2f') + 2\tau(-fhh' + h^2f' + hh')$$
$$= \int 2hh' - 4fhh' + 4h^2f'$$

which, by the next lemma, is equal to 1. Q.E.D.

It is interesting to note that these integrals will appear again in section four when computing the dimension of certain finite-rank projections.

2.2 Lemma: For functions f, g and h which satisfy conditions 1) to 4):

(i) $\int hh' = 0$

(ii) $\int fhh' = -1/12$

(iii) $\int h^2f' = 1/6$.

Proof: Since h is periodic, $2\int hh' = \int(h^2)' = 0$. By integration by parts, we find $2\int fhh' = \int f(h^2)' = -\int f'(h^2)$ so (ii) follows from (iii). On the interval $[0, 1/2]$, h is equal to zero, while on the interval $[1/2, 1]$ we have $h^2 = f - f^2$. Therefore,

$$\int_0^1 h^2f' = \int_{1/2}^1 h^2f' = \int_{1/2}^1 (f - f^2)f' = \int_0^1 (\lambda - \lambda^2)d\lambda = 1/6 .$$

Q.E.D.

§3 Approximate projections and almost commuting unitaries. When U and V are unitaries in a C*-algebra A that commute, we can define a projection $e(U, V)$ in $M_2(A)$ as the image of e under the map of $C(T^2)$ to A defined by sending $e^{2\pi ix}$ and $e^{2\pi iy}$ to U and V. More generally, when U and V are close to commuting, we can define $e(U, V)$ to be a two by two matrix over A that is almost a projection.

3.1 Definition : Let U and V be unitaries in a C*-algebra A. We now consider f, g and h to be functions defined on $\{ z \in \mathbf{C} : |z| = 1 \}$ so we can use the functional calculus to define $f(V)$, $g(V)$ and $h(V)$. Define $e(U, V)$ to be the matrix over A

$$e(U,V) = \begin{pmatrix} f(V) & g(V) + h(V)U \\ U^*h(V) + g(V) & 1 - f(V) \end{pmatrix}$$

For any unitaries, $e(U, V)$ is selfadjoint, and if U and V commute, then $e(U, V)$ is a projection. In order to explore the continutity properties of the function e, we need two lemmas.

3.2 Lemma: For any $f \in C(S^1)$, the function $U \longmapsto f(U)$ is uniformly continuous on the set of unitary operators.

Proof: Given any $\varepsilon > 0$, there is a polynomial p which appoximates f to within $\varepsilon/3$ on the unit circle. Any polynomial is uniformly continuous on a bounded set of operators, hence there is a number $\delta > 0$ such that $\|U - V\| < \delta$ implies that $\| p(U) - p(V) \| < \varepsilon/3$ for any U and $V \in \mathcal{U}(\mathcal{H})$. Whenever $\|U - V\| < \delta$,

$$\| f(U) - f(V) \| \leq \| f(U) - p(U) \| + \| p(U) - p(V) \| + \| p(V) - f(V) \| < \varepsilon$$

Q.E.D.

3.3 Lemma: Let $f \in C(S^1)$. For unitaries U and V, $\|f(V)U - Uf(V)\|$ tends toward zero uniformly as $\|UV - VU\|$ tends toward zero.

Proof: Let ε be given. By lemma 3.2 there exists δ such that $\| U^*VU - V\| < \delta$ implies that $\|f(U^*VU) - f(V)\| < \varepsilon$. Since

$$\|f(V)U - Uf(V)\| = \|U^*f(V)U - f(V)\| = \|f(U^*VU) - f(V)\|$$

we are done. Q.E.D.

These lemmas prove:

3.4 Proposition: The function $e : \mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H}) \rightarrow M_2(\mathcal{B}(\mathcal{H}))$ is uniformly norm-continuous.

3.5 Proposition: There exists a constant M such that if $\|UV - VU\| < M$, then 1/2 is not in the spectrum of $e(U,V)$. More generally, $\| e(U, V)^2 - e(U, V) \|$ tends toward zero uniformly as the commutation error $\| UV - VU \|$ tends to zero.

Proof: Notice that this says nothing about U and V converging. In fact, if U_n and V_n are unitaries in varying C*-algebras A_n, the lemma says that if

$\| U_nV_n - V_nU_n \| \to 0$ then $\| e(U, V)^2 - e(U, V) \| \to 0$.

We begin by calculating the square of $e(U, V)$. Identifying f, g and h with $f(V)$, $g(V)$ and $h(V)$,

$$\begin{pmatrix} f & g + hU \\ U^*h + g & 1 - f \end{pmatrix} \begin{pmatrix} f & g + hU \\ U^*h + g & 1 - f \end{pmatrix}$$

$$= \begin{pmatrix} gU^*h + f^2 + g^2 + h^2 + hUg & fg + (1 - f)g + fhU + hU(1 - f) \\ U^*fh + (1 - f)U^*h + fg + (1 - f)g & U^*hg + (1 - f)^2 + g^2 + U^*h^2U + ghU \end{pmatrix}$$

$$= \begin{pmatrix} gU^*h + f + hUg & g + fhU - hUf + hU \\ U^*h + U^*fh - fU^*h + g & (1 - f) + U^*h^2U - h^2 \end{pmatrix}$$

$$= \begin{pmatrix} f & g + hU \\ U^*h + g & 1 - f \end{pmatrix} + \begin{pmatrix} gU^*h + hUg & fhU - hUf \\ U^*fh - fU^*h & U^*h^2U - h^2 \end{pmatrix}$$

Working out the norms in each of the entries in the error term we get:

$$\| gU^*h + hUg \| = 2 \| hUg \| = 2 \| hUg - hgU \| = 2 \| h(Ug - gU) \|$$
$$\le 2 \|h\| \| Ug - gU \| = 2 \| Ug - gU \|$$

$$\| fhU - hUf \| = \| h(fU - Uf) \| \le \|h\| \| fU - Uf \| = \| fU - Uf \|$$

$$\| U^*fh - fU^*h \| = \| fhU - hUf \| \le \| fU - Uf \|$$

$$\| U^*h^2U - h^2 \| = \| h^2U - Uh^2 \| \le \| h^2U - hUh \| + \| hUh - Uh^2 \|$$

$$\leq \| hU - Uh \| + \| Uh - hU \| = 2 \| Uh - hU \|$$

Thus by lemma 3.3 we are done. Q.E.D.

Since we will need these estimates again later, we record them as a corollary.

3.6 Corollary: For any unitaries U and V in a C*-algebra A, $\| e(U, V)^2 - e(U, V) \|$ is less than or equal to

$$2(\| Ug(V)U^* - g(V) \| + \| Uf(V)U^* - f(V) \| + \| Uh(V)U^* - h(V) \|) .$$

§4 Voiculescu's example: This section provides a proof of theorem 1.1. The main idea in the proof is to compare the spectral projections of $e(S_n, \Omega_n)$ to the projections $e(U_n, V_n)$ obtained from commuting unitary matrices.

4.1 Lemma: If U and V are commuting unitaries in $M_n(\mathbf{C})$, then the projection $e(U, V) \in M_{2n}(\mathbf{C})$ has dimension n.

Proof: Let τ denote the trace on $M_n(\mathbf{C})$ normalized so that $\tau(I) = n$. Then $\tau(e(U, V)) = \tau(f) + \tau(1-f) = n$. Q.E.D.

For the rest of this section, let χ denote the characteristic function of the interval $[1/2, 2]$. As long as $\| UV - VU \|$ is less than the constant M of proposition 3.5, the spectrum of $e(U, V)$ will have a gap around 1/2, and the projection $\chi(e(U, V))$ will be in the C*-algebra generated by U and V.

4.2 Theorem: For large values of n, $\chi(e(S_n, \Omega_n))$ is a projection of dimension $n - 1$.

Before proving theorem 4.2, we shall see how theorem 1.1 follows from it. Actually, we prove a slightly stronger result:

4.3 Theorem: There do not exist paths of unitaries $U_n^{(t)}$ and $V_n^{(t)}$ from S_n, Ω_n to the identity such that

$$\lim_n \sup_t \| U_n^{(t)} V_n^{(t)} - V_n^{(t)} U_n^{(t)} \| = 0 .$$

Proof: Let $e_n^{(t)} = e(U_n^{(t)}, V_n^{(t)})$. This is a continuous path of matrices from $e(S_n, \Omega_n)$ to $e(I, I)$. If $\lim_n \sup_t \| U_n^{(t)} V_n^{(t)} - V_n^{(t)} U_n^{(t)} \|$ were equal to zero then, for large n, $\chi(e_n^{(t)})$ would be a continuous path of projections from $\chi(e(S_n, \Omega_n))$ to $e(I, I) = e_{11}$. This would imply that $\dim \chi(e(S_n, \Omega_n)) = n$ for large n, contradicting theorem 4.2. Q.E.D.

We now proceed with the proof of theorem 4.2. It seems odd that $e(S_n, \Omega_n)$ has trace n and yet only has $n - 1$ eigenvalues near one. The reason that this can happen is that, as the next lemma shows, the "spectral errors"

$$\max_{\lambda \in \sigma(e(S_n, \Omega_n))} \min_{s=0,1} |\lambda - s| = \| e(S_n, \Omega_n) - \chi(e(S_n, \Omega_n)) \|$$

are tending toward zero only on the order of $1/n$. Therefore the trace does not give an accurate reading of the dimension of $\chi(e(S_n, \Omega_n))$.

For notation, we now write e_n for $e(S_n, \Omega_n)$. Also, S_n implements the shift automorphism on $C^*(\Omega_n)$ which we denote by α. Any element of $C^*(\Omega_n)$ is a diagonal matrix, and so a multiplication operator. If k is any function on **R/Z** then we let k_n (or simply by k again when this is clear from the context) denote the n by n matrix

$$\begin{pmatrix} k(\Delta) & & & & & \\ & k(2\Delta) & & & & \\ & & k(3\Delta) & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \\ & & & & & & k(1) \end{pmatrix}$$

where Δ (or Δ_n) equals $1/n$. Sometimes we shall write S instead of S_n.

4.4 Lemma: As $n \to \infty$, $\| e_n - \chi(e_n) \| \to 0$ at least on the order of $1/n$.

Proof: What we wish to show is that $n\| e_n - \chi(e_n) \|$ is bounded. Since

$$| x - \chi(x) | \leq 2| x^2 - x |$$

for all real numbers x,

$$\| e_n - \chi(e_n) \| \leq 2 \| e_n{}^2 - e_n \|$$

so it suffices to show that $n\| e_n{}^2 - e_n \|$ is bounded.

Let k denote any smooth function on the circle. Then

$$\| S_n k - k S_n \| = \| S_n k S_n^* - k \| = \| \alpha(k) - k \|$$
$$= \sup | k(\rho\Delta) - k((\rho+1)\Delta) |$$

which is bounded by $\| k' \|_\infty \Delta = \| k' \|_\infty / n$. By corollary 3.6,

$$n \| e_n^2 - e_n \| \leq 2(\| g' \|_\infty + \| f' \|_\infty + \| h' \|_\infty)$$

Q.E.D.

The trace can be used to give an accurate count of the eigenvalues near one if we replace the e_n by matrices whose spectral errors vanish on the order of $1/n^2$. We cannot calculate $\chi(e_n)$ without knowing a priori the spectral decomposition of e_n, but we can apply a polynomial approximation. The polynomial $3x^2 - 2x^3$ is a second degree approximation to χ at zero and one, and so will turn the $1/n$ convergence into $1/n^2$ convergence.

4.5 Lemma: $\lim (\dim \chi(e_n) - \tau(3e_n^2 - 2e_n^3)) = 0$.

Proof: Let $p(x) = 3x^2 - 2x^3$. Since $p(0) = 0$ and $p'(0) = 0$, if $\lambda_n \to 0$ on the order of $1/n$ then $p(\lambda_n) \to 0$ on the order of $1/n^2$. Similarly, $p(1) = 1$ and $p'(1) = 0$ so if $\lambda_n \to 1$ on the order of $1/n$ then $p(\lambda_n) \to 1$ on the order of $1/n^2$.

By the spectral mapping theorem and lemma 4.4, $\| p(e_n) - \chi(e_n) \| \to 0$ on the order of $1/n^2$. Therefore the n eigenvalues of $\chi(e_n)$ and those for $p(e_n)$ differ by at most a constant times $1/n^2$, so $\lim (\tau(\chi(e_n)) - \tau(p(e_n))) = 0$. Q.E.D.

4.6 Lemma: $\tau(e_n^2) = n$.

Proof: In lemma 3.5 we saw that e_n^2 equaled e_n plus

$$M_n = \begin{pmatrix} gS^*h + hSg & hf(S - Sf) \\ (S^*f - fS^*)h & S^*h^2S - h^2 \end{pmatrix}$$

Thus it suffices to show that $\tau(M_n) = 0$. Since gS^*h and hSg are zero on the diagonal, they have trace zero. Therefore, $\tau(M_n) = \tau(gS^*h + hSg) + \tau(S^*h^2S - h^2) = \tau(h^2SS^*) - \tau(h^2) = 0$. Q.E.D.

4.7 Lemma: $\lim (\tau(e^3) - n) = 1/2$.

Proof: Using the last lemma, we find that $\tau(e_n^3) = \tau(e_n(e_n + M_n)) = \tau(e_n^2) + \tau(e_nM_n)) = n + \tau(e_nM_n)$. It suffices therefore to prove that $\lim \tau(e_nM_n) = 1/2$.

The coefficient of S^0 in the top left-hand corner of the matrix e_nM_n is

$$hS(S^*f - fS^*)h = h(f - \alpha(f))h .$$

The coefficient of S^0 in the lower right-hand corner is

$$S^*h^2(fS - Sf) + (1 - f)(S^*h^2S - h^2) .$$

Using the fact that $\tau(xy) = \tau(yx)$, we see that

$$\tau(e_nM_n) = \tau((f - \alpha(f))h^2) + \tau(h^2(f - \alpha(f))) + \tau((1 - f)(\alpha^{-1}(h^2) - h^2))$$

$$= 2\tau((f - \alpha(f))h^2) + \tau((1 - f)(\alpha^{-1}(h^2) - h^2))$$

For any two smooth functions r and s on **R/Z**

$$\begin{aligned}\tau(r(s - \alpha(s)) &= \Sigma\, r(\rho\Delta)[s(\rho\Delta) - s((\rho - 1)\Delta)] \\ &\to \int r(t)\, ds(t) \\ &= \int rs'\end{aligned}$$

and similarly , $\tau(r(s - \alpha^{-1}(s)) \to -\int rs'$. Therefore

$$\begin{aligned}\tau(e_nM_n) &\to 2\int h^2f' + \int((1 - f)(h^2)' \\ &= 2\int h^2f' + 2\int hh' - 2\int fhh' \\ &= 2(1/6) + 0 - 2(-1/12) \\ &= 1/2\end{aligned}$$

by lemma 2.2. Q.E.D.

This finishes the proof of theorem 4.2 since

$$\begin{aligned}\lim(\dim \chi(e_n) - n) &= \lim(\tau(3e_n{}^2 - 2e_n{}^3) - n) = \lim(-\tau(2e_n{}^3) + 2n) \\ &= -2\lim(\tau(e_n{}^3) - n) = -1 .\end{aligned}$$

Chapter II: An AF embedding of $C(T^2)$ which is faithful on K_0

§1 Introduction: The map $\psi : C(T^2) \to \mathcal{A}/\mathcal{J}$ defined in the first section of chapter one induces an injection on K_0. While $\mathcal{A}/\mathcal{J}$ itself is not an AF algebra, it is constructed out of finite-dimensional C*-algebras, and the map ψ demonstrates that sequences of matrices are able to capture all of $K_0(C(T^2))$. As we shall see, it is in fact possible to construct an AF embedding φ of $C(T^2)$ which is closely related to ψ. Therefore, we will be able to use the results from chapter **I** can be used to calculate the K-theory of φ.

One possible AF embedding comes from looking at $C(T^2)$ as the transformation group C*-algebra $C(T) \rtimes \mathbf{Z}$, with trivial action, and applying the construction in [P1]. This will turn out to be injective on K_0 (corollary 5.4 to theorem 5.2), but the K-theory of the AF algebra is hard to describe. By modifying Pimsner's techniques a little, one obtains an embedding into a simpler AF algebra. Our main result, theorem 5.2, is that there is a map φ from $C(T^2)$ to an AF algebra A such that the induced map on K_0 is the natural embedding of $\mathbf{Z}^2$ into $\mathbf{Q}(2^\infty)\oplus\mathbf{Z}$. Here $\mathbf{Q}(2^\infty)$ denotes the dyadic rationals. Proposition 2.1 then tells us that φ is itself an embedding of C*-algebras.

§2 Some negative results: This section collects together a few results concerning mappings from $C(T^2)$ to AF algebras which cannot be faithful on K_0.

2.1 Proposition: Let $\varphi : C(T^2) \to A$ be a homomorphism into any C*-algebra A. If $\varphi_*: K_0(C(T^2)) \to K_0(A)$ is injective then φ is injective.

Proof: Suppose that $\varphi : C(T^2) \to A$ has nonzero kernel J. Then $J = C_0(O)$ for some open subset O of T^2. Let D be an open disk in O that does not intersect the one skeleton $S^1 \vee S^1$ of T^2. Since $\varphi(C_0(D)) = 0$, φ factors through $C(T^2 \setminus D)$. This latter C*-algebra is homotopic to $C(S^1 \vee S^1)$. Thus φ_* factors through $K_0(C(S^1 \vee S^1)) \cong \mathbf{Z}$ so φ_* is not injective. Q.E.D.

2.2 Lemma: For any unitary U in a C*-algebra A, the projections $e(U, 1)$ and $e(1, U)$ are homotopic to the trivial projection e_{11}.

Proof: It is immediate from definition I.3.1 that $e(U, 1)$ equals e_{11}. By definition, $e(1, U)$ is equal to

$$\begin{pmatrix} f(U) & g(U) + h(U) \\ h(U) + g(U) & 1 - f(U) \end{pmatrix}$$

Let f_t be a homotopy of f to the constant function one such that $0 \le f_t \le 1$, and let $k_t = (f_t - f_t^2)^{1/2}$. Then since $g + h = (f - f^2)^{1/2}$, the matrices

$$\begin{pmatrix} f_t(U) & k_t(U) \\ k_t(U) & 1 - f_t(U) \end{pmatrix}$$

form a path of projections from $e(1, U)$ to e_{11}. Q.E.D.

2.3 Proposition: There exists a constant $C > 0$ such that the following holds: Suppose U and V are commuting unitaries in a C*-algebra A. If there exist commuting unitaries U' and V' such that $\| U - U' \| \le C$ and $\| V - V' \| \le C$, and if either U' or V' generates a finite-dimensional subalgebra, then $e(U,V)$ is homotopic to the trivial projection e_{11}.

Proof: Because the functon e is uniformly continuous, there exists a constant $C > 0$ such that if $\| U - U' \| \leq C$ and $\| V - V' \| \leq C$ then $\| e(U,V) - e(U',V') \|$ is less than one half, and so these projections are homotopic. By assumption, the spectrum of U' or V' is discrete; assume this holds for U'. Since the spectrum of U' is discrete, the functional calculus gives a path U_t from U' to the identity which, at every point, consists of unitaries in A which commute with V'. Then $e(U_t, V')$ is a path of projections from $e(U,V)$ to $e(I,V')$ which is homotopic to the trivial projection. Since $e(U', I)$ equals the trivial projection, the same proof works if it is V' that lies in a finite-dimensional subalgebra. Q.E.D.

2.4 Corollary: Suppose A is an AF algebra. If $\varphi : C(T^2) \to A$ maps into the center of A, then $\varphi_*(K_0(T^2))$ is cyclic.

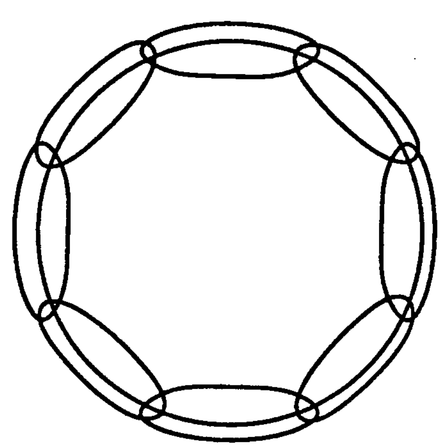
2.5 Proposition: Suppose $\{\tau_\lambda\}$ is a family of finite traces on a C*-algebra A for which the corresponding homomorphisms $\tau_\lambda : K_0(A) \to \mathbf{R}$ are separating. Then for any map $\varphi : C(T^2) \to A$, $\varphi_* : K_0(C(T^2)) \to K_0(A)$ is not injective.

Proof: A trace τ is extended to matrices by defining $\tau(\Sigma a_{ij} \otimes e_{ij}) = \Sigma \tau(a_{ii})$. With this definition it follows that $\tau(e(U,V)) = \tau(e_{11})$. If there is a separating family of traces then $\varphi_*([e(e^{2\pi i x}, e^{2\pi i y})]) = e(U,V) = \varphi_*([e_{11}])$. Q.E.D.

§3 The Bratteli diagram: According to proposition 2.1 and corollary 2.4, to find a mapping of $C(T^2)$ to an AF algebra which is injective on K_0 we must consider only embeddings into noncommutative AF algebras. It would also be nice to find an embedding that allows us to take advantage of the K-theory calculations from chapter I. These results cannot be used directly; the matrices S_n and Ω_n must be modified since there seems to be no way to construct an AF algebra in which both the S_n and the Ω_n converge. This type of problem comes up in Pimsner's theorem on embedding cross-product algebras into AF algebras. The connection here is that U_n and V_n represent the natural generators of $C(\mathbf{Z}_n) \rtimes \mathbf{Z}_n$. (The action being rotation of the n points.)

For the present purposes, it is best to apply the Fourier transform on only one of the variables on the torus and identify $C(T^2)$ with $C(T) \rtimes \mathbf{Z}$ where the action is the trivial action. Pimsner's theorem then provides an embedding into a highly noncommutative algebra. Indirectly we will prove that this embedding is injective on the level of K_0. It is not necessary to know Pimsner's proof [P1] to follow the proof of theorem 4.1, but it does provide the motivation for definition 3.2 and figure 3.1.

To construct the n^{th} level of Pimsner's AF algebra, we need to pick a covering of the circle by open sets. The natural way to do this is to use 2^n intervals, each of which has intersection only with its two neighbors.



We also need to pick integers m(n) that determine the power of the homeomorphism that we use. Although in this case the homeomorphism T is the identity map, so $T^{m(n)} = T$, the conversion of pseudo-orbits for one power of T to a lower power still depends on the choices of the m(n). The easiest choice is $m(n) = 2^n$.

Any point on the circle will be in at least three pseudo-orbits; the one that consists of just that point, and the orbits that run around the circle in one or the other direction. Deviating from Pimsner's construction a little, we only consider the trivial pseudo-orbits and the one that runs around the circle in the positive direction. The algebra that is associated to this collection of orbits is

$$M_{2^n} \oplus \underbrace{M_1 \oplus M_1 \oplus \ldots \oplus M_1}_{(2^n \text{ copies})}.$$

It is easier to work in the larger algebra $M_{2^n} \oplus M_{2^n}$. The approximate isomorphism that Pimsner defines will send the function $x \mapsto e^{2\pi i x}$ to the matrix V_n and will send the identity in $\mathbf{Z}$ to U_n, where we define U_n and V_n as follows:

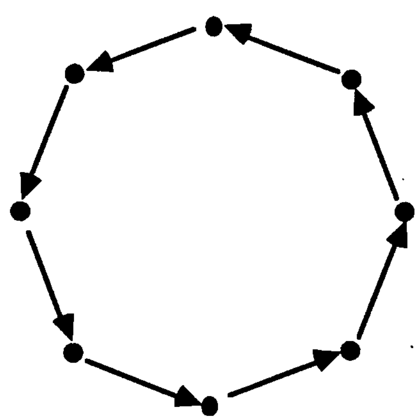
Let $V_n = {}_lV_n \oplus {}_rV_n$ where

$${}_lV_n = {}_rV_n = \begin{pmatrix} \Omega_{2^n} & & & \\ & \Omega_{2^n} \ (2^n \text{ times}) & & \\ & & \ddots & \\ & & & \Omega_{2^n} \end{pmatrix}$$

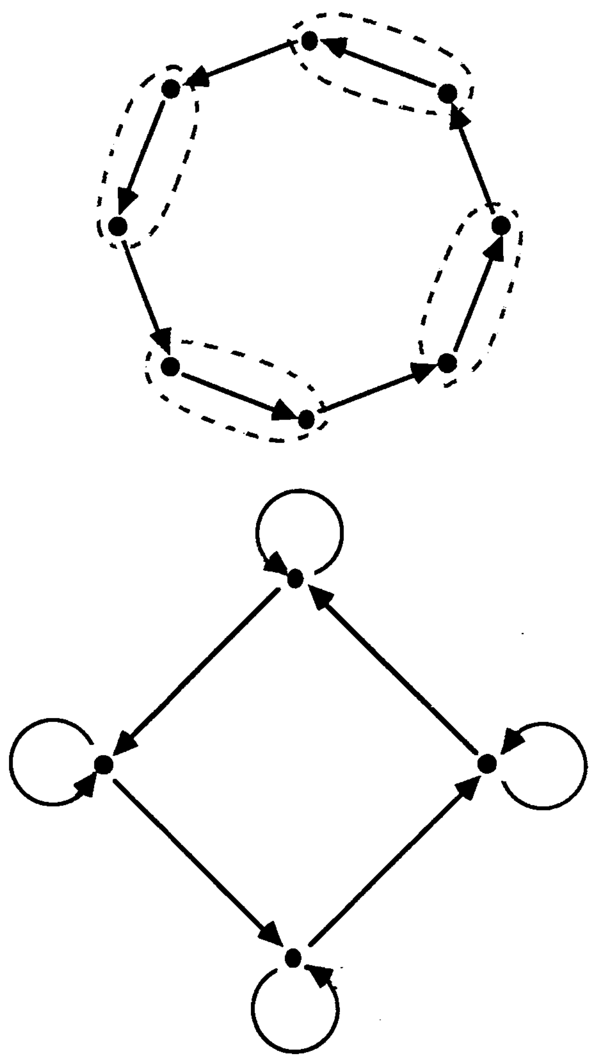
Let $U_n = {}_lU_n \oplus {}_rU_n$ where

$$
{}_lU_n = \begin{pmatrix} 0 & & & & S_{2^n} \\ I & 0 & & & \\ & I & 0 & & \\ & & I & \ddots & \\ & & & \ddots & \\ & & & I & 0 \end{pmatrix} \qquad {}_rU_n = \begin{pmatrix} 0 & & & & I \\ I & 0 & & & \\ & I & 0 & & \\ & & & & \\ & & & & \\ & & & I & 0 \end{pmatrix}
$$

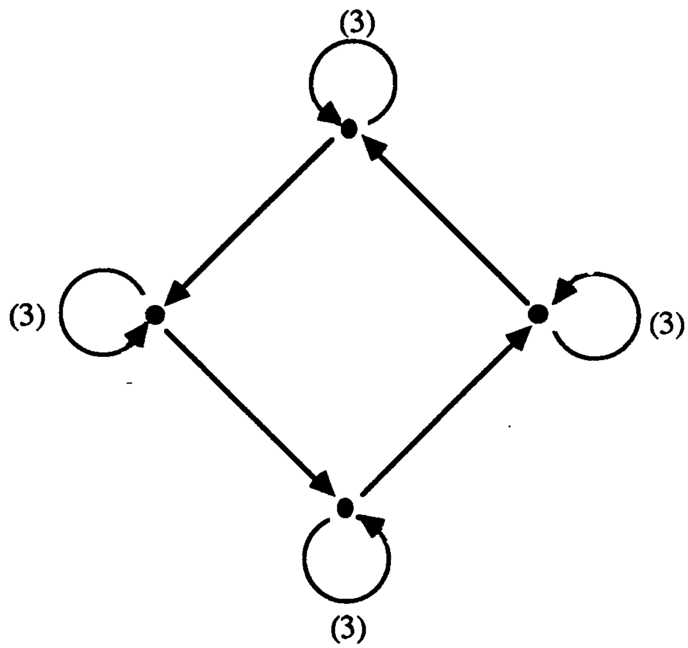
The embedding into the next stage is completely determined by choosing a way to decompose the pseudo-orbits of $T^{m(n+1)}$ with respect to a cover by $m(n+1) = 2^{n+1}$ intervals into pseudo-orbits of $T^{m(n)}$ with respect to a cover by $m(n) = 2^n$ intervals. The pseudo-orbit that runs around the circle can be viewed as a circular graph on 2^{2n} points.



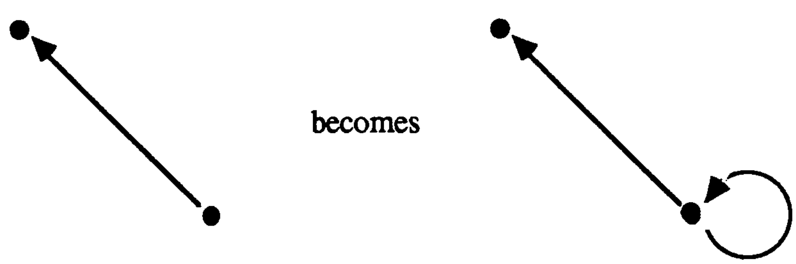
Regarded as a pseudo-orbit for the coarser covering, we get a path around 2^n points that alternately stops at a point and then moves on to the next.



A pseudo-orbit for $T^{m(n+1)} = (T^{m(n)})^2$ gives a pseudo-orbit for $T^{m(n)}$ by filling in intermediate steps, so that by looking at every other point we recover the original pseudo-orbit. In this case, it once more means alternately staying put and then following the given pseudo-orbit.

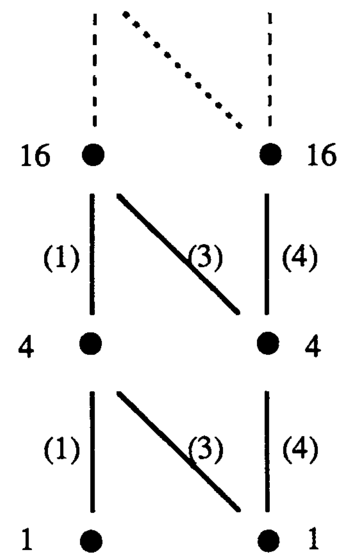


since



and

We may now conclude that the Brattelli diagram shown below is (a modified version of) the one obtained via Pimsner's techniques.



Since the matrix

$$\begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}$$

is the square of the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

the same AF algebra comes from the diagram

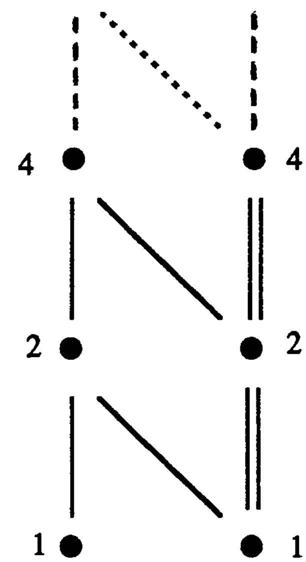


figure 3.1

This is the preferred Brattelli diagram for defining the AF algebra. One small problem is that the matrices U_n and V_n live in every other level of the inductive limit. To correct this, we introduce matrices to fill in the gaps.

3.2 Definition: Let $V_{k,m} = {}_lV_{k,m} \oplus {}_rV_{k,m}$ where ${}_lV_{k,m} = {}_rV_{k,m}$ is the $k{\cdot}m$ by $k{\cdot}m$ matrix

$$ {}_lV_{k,m} = {}_rV_{k,m} = \begin{pmatrix} \Omega_k & & & \\ & \Omega_k & & \text{(m times)} \\ & & \ddots & \\ & & & \Omega_k \end{pmatrix} \quad \text{where } \Omega_k = \begin{pmatrix} \omega & & & & \\ & \omega^2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \;\; \omega^k \end{pmatrix} $$

Define $U_{k,m}$ in $M_{k \cdot m} \oplus M_{k \cdot m}$ as ${}_lU_{k,m} \oplus {}_rU_{k,m}$

$$ {}_{l}U_{k,m} = \begin{pmatrix} 0 & & & & S_k \\ I & 0 & & & \\ & I & 0 & & \\ & & I & \cdot & \\ & & & \cdot & \\ & & & \cdot & \\ & & & I & 0 \end{pmatrix} \qquad {}_{r}U_{k,m} = \begin{pmatrix} 0 & & & I \\ I & 0 & & \\ & I & 0 & \\ & & & \\ & & & \\ & & I & 0 \end{pmatrix} $$

Of course, we will mostly be interested in the case where m and n are powers of two. In particular, we are interested in the sequences $U_{1,1}$, $U_{2,1}$, $U_{2,2}$, $U_{4,2}$ ... and $V_{1,1}$, $V_{2,1}$, $V_{2,2}$, $V_{4,2}$ ... .

§4 The inclusions: Motivated by the last section, we look for ways to put the $U_{k,m}$ and $V_{k,m}$ into an AF algebra that has figure 3.1 for Bratteli diagram so that convergence is forced. Notice that the righthand matrices ${}_{r}U_{k,m}$ and ${}_{r}V_{k,m}$ commute, so the interesting K-theory will come about on the left side.

The standard embeddings that correspond to the Bratteli diagram do not map the $U_{k,m}$ anywhere near the corresponding matrices in the next level. The major work in this section will be in replacing the inclusion with ones that define the same K-theory while turning the U's and V's into convergent sequences. The proof of Proposition 4.1 is just a specialization of the proof given in [P1].

The calculation of the K-theory of the embedding (Theorem 5.2) will not depend on the exact inclusions obtained, only on the existence of inclusions which force the convergence of the $U_{k,m}$ and $V_{k,m}$. Different inclusions will result in isomorphic AF algebras, but after identifying the algebras, the embeddings of $C(T^2)$ will be different, possibly even non-homotopic.

4.1 Proposition: There exist embeddings $\varphi_{2^n} : M_{2^n} \oplus M_{2^n} \to M_{2^{n+1}} \oplus M_{2^{n+1}}$ for which the associated Bratteli diagram is figure 3.1 and for which both $U_{2^n,2^n}$, $U_{2^{n+1},2^n}$, $U_{2^{n+1},2^{n+1}}$, ... and $V_{2^n,2^n}$, $V_{2^{n+1},2^n}$, $V_{2^{n+1},2^{n+1}}$, ... form convergent sequences in the inductive limit.

Proof: Any embeddings $\varphi_n : M_{2^n} \oplus M_{2^n} \to M_{2^{n+1}} \oplus M_{2^{n+1}}$ that have the K-theory of the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

will define the same AF algebra. As a first approximation, let $\psi_{2^n} : M_{2^n} \oplus M_{2^n} \to M_{2^{n+1}} \oplus M_{2^{n+1}}$ be the map that shuffles the standard bases together; more generally define $\psi_\mu : M_\mu \oplus M_\mu \to M_{2\mu} \oplus M_{2\mu}$. Specifically, let $\{{}_l e^\mu{}_j\}$ and $\{{}_r e^\mu{}_j\}$ be the standard bases of the left and right factors of $M_\mu \oplus M_\mu$. Let $\psi = \psi_\mu = {}_l\psi \oplus {}_r\psi$ where

$$ {}_l\psi({}_l e^\mu{}_j) = {}_l e^{2\mu}(2j) \qquad {}_l\psi({}_r e^\mu{}_j) = {}_l e^{2\mu}(2j+1)$$
$$ {}_r\psi({}_l e^\mu{}_j) = 0 \qquad {}_r\psi({}_r e^\mu{}_j) = {}_r e^{2\mu}(2j) + {}_r e^{2\mu}(2j+1) .$$

We would like to find unitaries by which to conjugate to bring the images of the unitaries at one stage near the unitaries at the next. We begin with the easier case of going from $U_{2^n,2^n}$ and $V_{2^n,2^n}$ to $U_{2^{n+1},2^n}$ and $V_{2^{n+1},2^n}$. (This corresponds to passing to a finer cover while keeping the power of the homeomorphism the same.) Slightly more generally, we consider the matrices $\psi(U_{k,m})$ and $\psi(V_{k,m})$, and how to make them close to $U_{2k,m}$ and $V_{2k,m}$.

There are four pairs of matrices to consider, ${}_l\psi(U_{k,m})$ and ${}_lU_{2k,m}$; ${}_r\psi(U_{k,m})$ and ${}_rU_{2k,m}$; ${}_l\psi(V_{k,m})$ and ${}_lV_{2k,m}$; ${}_r\psi(V_{k,m})$ and ${}_rV_{2k,m}$.

$$ {}_r\psi(U_{k,m}) = \begin{pmatrix} 0 & & & & & I \\ I & 0 & & & & \\ & I & 0 & & & \\ & & I & \cdot & & \\ & & & & \cdot & \\ & & & & \cdot & \\ & & & & I & 0 \end{pmatrix} \in M_{2k} $$

$$ {}_l\psi(U_{k,m}) = \begin{pmatrix} 0 & & & & & W_{2k} \\ I & 0 & & & & \\ & I & 0 & & & \\ & & I & \cdot & & \\ & & & & \cdot & \\ & & & & \cdot & \\ & & & & I & 0 \end{pmatrix} \quad \text{where } W_{2k} = \begin{pmatrix} 0 & & & & & 1 & 0 \\ 0 & 1 & & & & & \\ 1 & 0 & 0 & & & & \\ & & 0 & 1 & & & \\ & & 1 & & & & \\ & & & & & & \\ & & & & & 1 & \end{pmatrix} \in M_{2k} $$

$$ {}_l\psi(V_{k,m}) = {}_r\psi(V_{k,m}) = \begin{pmatrix} \Omega'_{2k} & & & \\ & \Omega'_{2k} & & \\ & & \ddots & \\ & & & \Omega'_{2k} \end{pmatrix} \text{(m times)} $$

$$ \text{where } \Omega'_{2k} = \begin{pmatrix} \omega & & & & & & & \\ & \omega & & & & & & \\ & & \omega^2 & & & & & \\ & & & \omega^2 & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & \omega^k & \\ & & & & & & & \omega^k \end{pmatrix} $$

ω = kth root of unity

One pair is exactly equal; ${}_r\psi(U_{k,m}) = {}_rU_{2k,m}$. The other right-hand pair is almost equal;

$$ \| {}_r\psi(U_{k,m}) - {}_rU_{2k,m} \| < \pi/k . $$

On the left side, however, we find that $\| {}_l\psi(U_{k,m}) - {}_lU_{2k,m} \| = 2$ so ψ must be modified. The modification is done by conjugating by a unitary in $M_{2km} \oplus M_{2km}$.

Let Y be the solution to the equation $W_{2k}Y = S_{2k}$. Specifically,

$$Y = \begin{pmatrix} 0 & 1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & 1 & 0 & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \cdot \\ & & & & & & & 0 & 1 \\ & & & & & & & 1 & 0 \end{pmatrix} .$$

Since the spectrum of Y is discrete, the functional calculus produces an m^{th} root $Y^{1/m}$ for which $\| I - Y^{1/m} \| \leq 2\pi/m$. Since Y commutes with Ω'_k, so does $Y^{1/m}$. The unitary needed to conjugate by is then

$$Y' = Y'_{2k,m} = \begin{pmatrix} Y^{1/m} & & & & & \\ & Y^{2/m} & & & & \\ & & Y^{3/m} & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \; Y \end{pmatrix} .$$

Since Y' commutes with ${}_1\psi(V_{k,m})$,

$$\begin{aligned} & \| Y'^{*} \, {}_1\psi(V_{k,m}) Y' - V'_{2k,m} \| \\ = & \| {}_1\psi(V_{k,m}) - V'_{2k,m} \| \\ = & \max_{\rho=1,\ldots,2mk} | \omega_0^{\rho} - \omega_0^{\rho+1} | \leq \pi/k . \end{aligned}$$

The definition of Y implies that $Y^{-1/m}W_{2k}Y = Y^{-1/m}S_{2k}$ so

$$Y'^{*}\varphi'_1(U_{k,m})Y' = \begin{pmatrix} 0 & & & & & Y^{-1/m}S_{2k} \\ Y^{-1/m} & 0 & & & & \\ & Y^{-1m} & 0 & & & \\ & & Y^{-1/m} & 0 & & \\ & & & \ddots & & \\ & & & & Y^{-1/m} & 0 \end{pmatrix}.$$

It follows that $\| \; Y'^{*}{}_1\psi(U_{k,m})Y' - U'_{2k,m} \| \leq 2\pi/m$.

The required embeddings are then $\varphi_{2n} = Ad_{(Y'\oplus I)}\circ\psi_{2^n}$. So far, we have calculated that

$$\| \varphi_{2n}(U_{2^n,2^n}) - U_{2^{n+1},2^n} \| < \max\{ \pi 2^{-n+1}, 0 \} = \pi 2^{-n+1}$$

and $$\| \varphi_{2n}(V_{2^n,2^n}) - V_{2^{n+1},2^n} \| < \max\{ \pi 2^{-n}, \pi 2^{-n} \} = \pi 2^{-n}.$$

The case of the maps φ_{2n+1} is mostly the same. The main thing to notice is that there exists a change of basis that takes the matrices $U_{k,2m}$ and $V_{k,2m}$ over to

$$\begin{pmatrix} \Omega'_k & & & \\ & \Omega'_k & & \text{(m times)} \\ & & \ddots & \\ & & & \Omega'_k \end{pmatrix} \oplus \begin{pmatrix} \Omega'_k & & & \\ & \Omega'_k & & \\ & & \ddots & \\ & & & \Omega'_k \end{pmatrix}$$

$$\begin{pmatrix} 0 & & & & & S_{2k} \\ I & 0 & & & & \\ & I & 0 & & & \\ & & I & \cdot & & \\ & & & & \cdot & \\ & & & & I & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & & & & Y \\ I & 0 & & & \\ & I & 0 & & \\ & & & \cdot & \\ & & & I & 0 \end{pmatrix}$$

The required change of basis takes e_g to e_h where

$$g = b(mk) + ak + c \qquad \text{and} \qquad h = a(2k) + c(2) + b$$

for all $0 \le a < m$, $0 \le b < 1$, $0 \le c < k$. The unitary to conjugate by is $Y'_{2k,m} \oplus Y'_{2k,m}$ followed by the basis change. The result is

$$\| \varphi_{2n}(U_{2^{n+1},2^n}) - U_{2^{n+1},2^{n+1}} \| < \pi 2^{-n+1}$$

$$\text{and} \qquad \varphi_{2n}(V_{2^{n+1},2^n}) = V_{2^{n+1},2^{n+1}} .$$

These estimates show that in the inductive limit algebra determined by the φ_n the U and V sequences are Cauchy. Q.E.D.

§5 The K-theory:

5.1 Proposition : Let A be the AF algebra defined in the last proposition. Then $K_0(A) \cong \mathbf{Q}(2^\infty) \oplus \mathbf{Z}$ where $\mathbf{Q}(2^\infty)$ denotes the group of diadic rationals

$\{p\cdot 2^{-n} \mid p \in \mathbf{Z}, n \in \mathbf{Z}^+\}$. There is an exact sequence

$$0 \to \mathcal{K} \to A \to \mathrm{UHF}(2^\infty) \to 0$$

where $\mathcal{K}$ denotes the compact operators and $\mathrm{UHF}(2^\infty)$ is the type 2^∞ UHF algebra. The positive cone of $K_0(A)$ is $\{ (r, k) \mid r > 0 \text{ or } (r = 0 \text{ and } k \geq 0) \}$ and the order unit is $(1, 0)$.

Proof: Let G_n denote $K_0(M_{2^n} \oplus M_{2^n})$. The K_0 group of A is the limit of the sequence

$$\begin{array}{ccccccccc} G_1 & \to & G_2 & \to & G_3 & \to & G_4 & \to & \\ \mathbf{Z}\oplus\mathbf{Z} & \to & \mathbf{Z}\oplus\mathbf{Z} & \to & \mathbf{Z}\oplus\mathbf{Z} & \to & \mathbf{Z}\oplus\mathbf{Z} & \to & \ldots \end{array}$$

where the inclusion to the next stage sends (a,b) to $(a+b, 2b)$. The isomorphism to $\mathbf{Q}(2^\infty)\oplus\mathbf{Z}$ is given by sending $(a,b) \in G_n$ to $(b\cdot 2^{-n}, a-b)$. This is well-defined since $(a+b, 2b) \in G_{n+1}$ is sent to $(2b\cdot 2^{-n-1}, (a+b) - 2b)$.

Under this identification, $K_0(A)^+ = \bigcup G_n^+$. Since

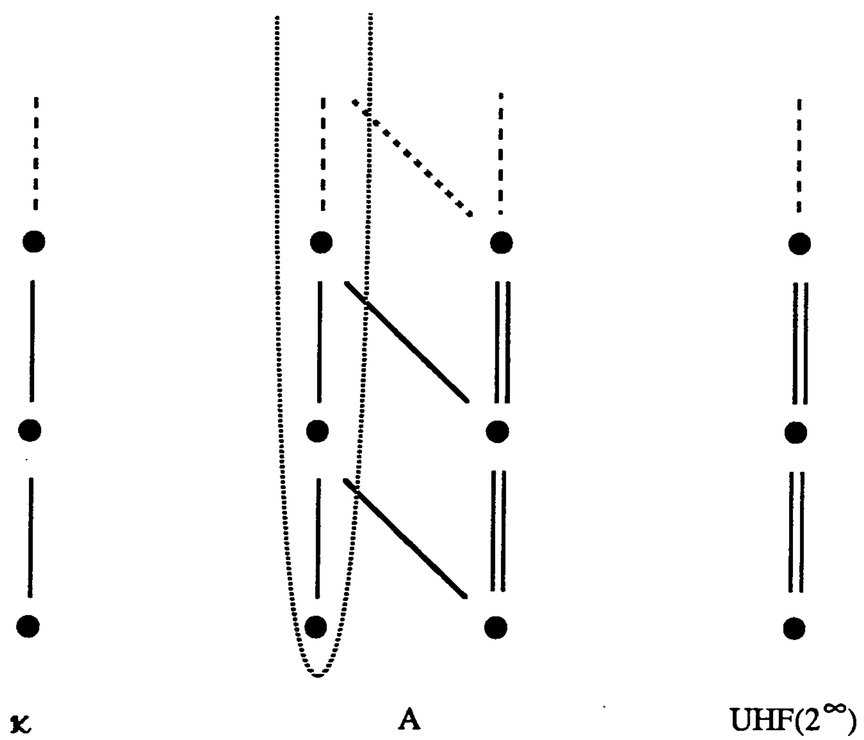
$$\begin{aligned} G_n^+ &= \{ (b\cdot 2^{-n}, a - b) \mid a \geq 0 ,\ b \geq 0 \} \\ &= \{ (r, k) \mid r \geq 0,\ r \in \mathbf{Q}(2^{-n}) ,\ k \geq -r\cdot 2^n \}, \end{aligned}$$

we see that

$$\begin{aligned} K_0(A)^+ &= \bigcup \{ (r, k) \mid r \geq 0,\ r \in \mathbf{Q}(2^{-n}) ,\ k \geq -r\cdot 2^n \} \\ &= \bigcup \{ (r, k) \mid r \geq 0,\ \exists M \in \mathbf{N} \text{ such that } k \geq -r\cdot M \} \end{aligned}$$

$$= \{ (r, k) \mid r = 0,\ k \geq 0 \} \cup \{ (k, r) \mid r > 0,\ k > -\infty \}.$$

The exact sequence can be seen directly from the Bratteli diagram. The left side has no mappings leaving it, and so forms an ideal isomorphic to the compact operators. The right side of the diagram, which determines the quotient, is the diagram for the type 2^∞ UHF algebra. One can also obtain $K_0(A)$ as an unordered group from this exact sequence. Q.E.D.



The unitaries $U = \lim U_{2^n,2^n}$ and $V = \lim V_{2^n,2^n}$ define a homomorphism from $C(T^2)$ into the AF algebra A. Call this map Ψ. In order to describe the K-theory of Ψ we need to choose generators of $K_0(T^2)$. We make almost the usual choices, namely [1] and $[1] - [e(e^{2\pi ix}, e^{2\pi iy})]$. This defines an isomorphism $\mathbf{Z}^2 \to K_0(C(T^2))$ and we have already defined an isomorphism $K_0(A) \to \mathbf{Q}(2^\infty)\oplus\mathbf{Z}$. These are the identifications we have in mind in the statement of 5.2.

5.2 Theorem: The homomorphism $\Psi_* : K_0(C(T^2)) \rightarrow K_0(A)$ sends $(s,t) \in \mathbf{Z}^2 \cong K_0(C(T^2))$ to $(s \cdot 2^{-0}, t) \in K_0(A)$, i.e. Ψ_* is the natural inclusion of $\mathbf{Z}^2$ into $\mathbf{Q}(2^\infty) \oplus \mathbf{Z}$.

Proof: Since Ψ is unital, it carries the order unit of $K_0(C(T^2))$ to the order unit of $K_0(A)$, i.e. $(1,0) = [1]$ is sent to $[1] = (1,0) \in \mathbf{Q}(2^\infty) \oplus \mathbf{Z}$. The element $(1, -1)$ is sent to

$$(m_\Psi , n_\Psi) = [e(U , V)] = [\,\chi(e(U_{2^n,2^n} , V_{2^n,2^n}))] \quad \text{(for large n)}$$

where, as before, χ is the characteristic function of the interval $[1/2, 2]$. Since the algebra $\mathrm{UHF}(2^\infty)$ has a faithful trace, we know by proposition 2.5 that $m_\Psi = 1 \in \mathbf{Q}(2^\infty)$ and so

$$n_\Psi = \lim \,(\,\dim \chi(\,e(\,{}_1U_{2^n,2^n} , {}_1V_{2^n,2^n})) - 2^{2n} \,) .$$

Here τ is the regular trace on M_k with value one on a rank one projection. Of course, the individual traces do not fit together to form a trace on A . We wish to show $n_\Psi = -1$. This will follow from the next lemma, and I.4.2. Q.E.D.

5.3 Lemma: For large n,

$$\begin{aligned}\dim \chi(\,e(\,{}_1U_{2^n,2^n} , {}_1V_{2^n,2^n})) &= \dim \chi(\,e(\,{}_1U_{2^{2n},1} , {}_1V_{2^{2n},1})) \\ &= \dim \chi(\,e(\,S_{2^{2n}} , \Omega_{2^{2n}})) .\end{aligned}$$

Proof: We begin by looking at the matrices ${}_lU_{k,m}$ and ${}_rU_{k,m}$ in a different basis. Let W be the unitary corresponding to the permutation

$$a\cdot k + b \;\rightarrow\; b\cdot m + a \qquad\qquad (0 \le a < m\,,\, 0 \le b < k).$$

Conjugating by W, we find that $W^{*}{}_lU_{k,m}W = S_{km} = {}_lU_{km,1}$ and

$$W^{*}\,{}_lV_{k,m}W = \begin{pmatrix} \omega & & & & & & & & & \\ & \ddots \text{ (m times)} & & & & & & & & \\ & & \omega & & & & & & & \\ & & & \omega^2 & & & & & & \\ & & & & \ddots \text{ (m times)} & & & & & \\ & & & & & \omega^2 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & \omega^k & & \\ & & & & & & & & \ddots \text{ (m times)} & \\ & & & & & & & & & \omega^k \end{pmatrix}$$

where $\omega = k^{th}$ root of unity. This is homotopic to $\Omega_{km} = {}_lV_{km,1}$ via a path of diagonal unitaries $V^{(t)}$ for which adjacent diagonal elements are within $2\pi/k$, i.e., unitaries which commute with S_{km} to within $2\pi/k$. For large k, proposition I.3.5 implies that $1/2$ is not in the spectrum of $e(S_{km}, V^{(t)})$. Applying χ proves that

$$\begin{aligned}(W\otimes I)^{*}\chi(e({}_lU_{k,m}, {}_lV_{k,m}))(W\otimes I) &= \chi(e(W^{*}{}_lU_{k,m}W, W^{*}{}_lV_{k,m}W)) \\ &= \chi(e({}_lU_{km,1}, W^{*}{}_lV_{k,m}W))\end{aligned}$$

is homotopic to

$$\chi(e({}_1U_{km,1}\,,{}_1V_{km,1}))\,.$$

Therefore, for large k, $\chi(e({}_1U_{k,m}\,,{}_1V_{k,m}))$ is equivalent to $\chi(e({}_1U_{km,1}\,,{}_1V_{km,1}))$

Q.E.D.

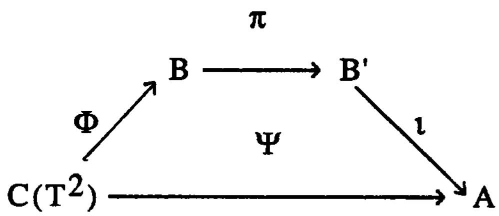
The mapping Ψ was found by a close examination of Pimsner's work. In order to avoid the details of his construction, the existence of Ψ was proven directly in proposition 4.1. It is interesting to note that Ψ factors through Pimsner's map. Unfortunately, the proof of this requires bringing in all the complexities of Pimsner's general construction. In order to avoid this, I offer only a sketch of the proof below. I hope to prove this more rigorously in a future paper.

5.4 Corollary: Let $\Phi : C(S^1) \rtimes \mathbf{Z} \to B$ be the embedding into an AF algebra defined by Pimsner in the special case of a trivial action where the open covers and multiplicities selected are those from section 2 . The induced map $\Phi_* : K^0(T^2) \to K_0(B)$ is faithful.

Proof: Let $B_n = \bigoplus M_{|\sigma|}$ where the sum is over all pseudo-orbits of $(id)^{m(n)}$ for the cover by 2^n sets. Here $|\sigma|$ denotes the length of the pseudo-orbit. Let B'_n be the corresponding sum, but taken only over the orbits that run in the positive direction. The projections $\pi_n : B_n \to B'_n$ commute with the inclusion maps because any pseudo-orbit that runs in the positive direction can only be decomposed into positive pseudo-orbits. Let $\pi : B \to B'$ denote the limit of the π_n .

We have $B'_n \cong M_{2^n} \oplus M_1 \oplus \ldots \oplus M_1$ and in section 2 we defined $A_n \cong$

$M_2 n \oplus M_2 n$. Let $\iota_n : B'_n \to A_n$ denote the inclusion with $\iota : B' \to A$ the limit. By the way that U and V were defined, the map Ψ factors as



Since Ψ_* is faithful, so is Φ_* . Q.E.D.

Chapter III: Semiprojectivity

§1 Definition of semiprojectivity: Following Effros and Kaminker [EK1], we define the notion of semiprojectivity for C*-algebras. Let $\mathcal{C}_0$ denote the category of C*-algebras and arbitrary homomorphisms and let $\mathcal{C}_1$ denote the category of unital C*-algebras and unit-preserving homomorphisms. Let $[A,B]_i$ denote the set of homotopy classes of homomorphism in $\mathcal{C}_i$ from A to B.

1.1 Definition: Let A be a unital C*-algebra. If $[A, \lim B_n]_i = \lim [A, B_n]_i$, for every system of C*-algebras in $\mathcal{C}_i$

$$B_1 \xrightarrow{\varphi_1} B_2 \xrightarrow{\varphi_2} B_3 \xrightarrow{\varphi_3} \cdots \qquad \varphi_n \text{ injective,}$$

then we say that A is semiprojective in $\mathcal{C}_i$.

Unless stated otherwise, all algebras and homomorphisms will be unital, and the term semiprojective will refer to semiprojectivity in $\mathcal{C}_1$.

Finding semiprojective C*-algebras is greatly simplified by the following equivalence [EK1]:

1.2 Theorem: A C*-algebra A is semiprojective in $\mathcal{C}_i$ if for every system of C*-algebras in $\mathcal{C}_i$

$$B_1 \xrightarrow{\varphi_1} B_2 \xrightarrow{\varphi_2} B_3 \xrightarrow{\varphi_3} \cdots \qquad \varphi_n \text{ injective,}$$

every homomorphism $\psi : A \to \lim B_n$ is homotopic to a homomorphism into some B_n.

§2 **The rotation algebras**: Blackadar [B1] has shown that $C(T^2)$ does not satisfy his definition of semiprojectivity, which is more restrictive than the definition above. We show that $C(T^2)$ does not satisfy the less restrictive definition either.

2.1 Proposition: $C(T^2)$ is not semiprojective.

Proof: Proposition II.2.3 shows that any homomorphism from $C(T^2)$ to a finite-dimensional C*-algebra is not injective on K_0. As a consequence, the homomorphism $\Psi : C(T^2) \to A$ defined in chapter 2 is not homotopic to a map with finite-dimensional range. Q.E.D.

2.2 Corollary: $C(S^2)$ is not semiprojective.

Proof: Let $\rho : T^2 \to S^2$ be the surjection defined by collapsing the one-skeleton of T^2 to a point. This induces a map, which we also call ρ, from $C(S^2)$ to $C(T^2)$, which induces an isomorphism on the level of K_0. Then $\Psi \circ \rho : C(S^2) \to A$ is a homomorphism to an AF algebra that is injective on K_0.

To finish the proof, we must only show that any homomorphism $\eta : C(S^2) \to F$ is non-injecitve on K_0 whenever F is finite dimensional. The range of η will be a finite-dimensional commutative C*-algebra, and so has zero second-cohomology. The Chern character now shows us that $\eta_* : K_0(C(S_2)) \to K_0(F)$ kills the second cohomology of S^2, and so is not injective. Q.E.D.

For any real θ, let A_θ denote the rotation algebra

$$C^*\langle U,V \mid UV = e^{2\pi i\theta}VU\, ,\, U^{-1} = U^*\, ,\, V^{-1} = V^* \rangle .$$

If θ is irrational, A_θ is simple so we can use any U and V that satisfy these relations to generate A_θ. If θ is rational, we must choose universal unitaries for which $UV = e^{2\pi i\theta}VU$. When $\theta = 0$, $A_\theta = C(T^2)$.

2.3 Theorem: For any irrational number θ, A_θ is not semiprojective.

Proof: Since A_θ is simple and infinite-dimensional, the only homomorphism from A_θ into a finite-dimensional C*-algebra is the zero map. By the results of Pimsner and Voiculescu [PV1] there is a unital homomorphism of A_θ into an AF algebra that is an isomorphism on K_0. This homomorphism cannot be homotopic to the zero map, and so is not homotopic to a map with finite-dimensional range. Therefore A_θ is not semiprojective. Q.E.D.

Since the rational rotation algebras are strongly Morita equivalent to $C(T^2)$, it would be nice if we could show that unital C*-algebras which are strongly Morita equivalent to a semiprojective C*-algebra are semiprojective. I believe that it is unknown whether or not this is true. However, when semiprojectivity fails due to a K-theoretic obstruction, it is possible to show that non-semiprojectivy is preserved under strong Morita equivalence. I am indebted to Marc Rieffel for pointing out this line of reasoning to me.

2.4 Lemma: Let B be a unital separable C*-algebra, and suppose that $x \in K_0(B)$. Let C be any separable C*-algebra which is stably isomorphic to B, and let x' be the element of $K_0(C)$ which corresponds to x (for some fixed choice of isomorphism between $B \otimes \mathcal{K}$ and $C \otimes \mathcal{K}$). If x is in the kernal of $K_0(\varphi)$ for every *-homomorphism $\varphi : B \to F$ with F finite-dimensional, then the same property holds for C and x'.

Proof: Assume that $\varphi : C \to F$ is given, with F finite-dimensional. The given choice of a stable isomorphism defines an injection $\iota : B \to C \otimes \mathcal{K}$ which induces an isomorphism of K-theory, and $\iota_*(x) = x'$ by assumption. (We will everywhere identify $K_0(D)$ with $K_0(D \otimes \mathcal{K})$.) Consider the composition $\psi = (\varphi \otimes id) \circ \iota : B \to F \otimes \mathcal{K}$. The projection $p = \psi(1)$ must be finite-dimensional, since $F \otimes \mathcal{K}$ is isomorphic to a finite direct sum of copies of $\mathcal{K}$. Therefore ψ has finite-dimensional range, and so by assumption $\psi_*(x) = 0$. (We use the fact that equivalent projections in a subalgebra remain equivalent in the larger algebra.) Now $\varphi_*(x') = (\varphi \otimes id)_*(x') = \psi_*(x) = 0$.

Q.E.D.

2.5 Lemma: Let B and C be unital C*-algebras that are stably isomorphic, and let x and x' be elements of $K_0(B)$ and $K_0(C)$ which correspond under some fixed choice of stable isomorphism. If there is a unital *-homomorphism φ of B to an AF algebra A for which $\varphi_*(x)$ is nonzero, then the same holds for C and x'.

Proof: As before, the choice of stable isomorphism determines a map $\iota : C \to B \otimes \mathcal{K}$. Let $p \in B \otimes \mathcal{K}$ denote the projection $((\varphi \otimes id) \circ \iota)(1)$ and let $\psi = p((\varphi \otimes id) \circ \iota)p$. Then pAp is a unital AF algebra and ψ is a unital *-homomorphism. If $\psi_*(x')$ were zero in pAp, then it would be zero in A, but it cannot be since $((\varphi \otimes id) \circ \iota)_*(x') =$

$(\varphi \otimes \mathrm{id})_*(x) \neq 0$ by assumption. Q.E.D.

This in particular applies to the rational rotation algebras.

2.6 Corollary: For any rational θ, there exists a unital $*$-homomorphism $\varphi : A_\theta \to A$ such that A is AF and φ_* is an injection on K_0.

Proof: This follows from the last lemma and theorem **II**.5.2. Q.E.D.

2.7 Theorem: Suppose that A is a seperable unital C*-algebra, and that $x \in K_0(A)$. If there is a $*$-homomorphism φ into an AF algebra for which $\varphi_*(x) \neq 0$ and yet $\psi_*(x) = 0$ for every $*$-homomorphism $\psi : A \to F$ with finite-dimensional range, then every separable unital C*-algebra which is strongly Morita equivalent to A is not semiprojective.

Proof: This follows immediately from the last two lemmas and the fact that for separable C*-algebras, strong Morita equivalence implies stable isomorphism.

Q.E.D.

2.8 Corollary: The rational rotation algebras are not semiprojective.

Proof: Theorem **II.5.2** and proposition **II.2.3** imply that $C(T^2)$ satisfies the conditions of theorem 2.7. The rotation algebras are all strongly Morita equivalent to $C(T^2)$ and so are not semiprojective. Q.E.D.

§3 Bouquets of circles: Given two unitaries U and V, the relation UV=VU is not stable in the sense that if this relation is only approximately true, there need not be a path through approximately commuting unitaries to a pair that commute exactly. More precisely, for some C*-algebra A, the set

$$C_\varepsilon(A) = \{(U,V) \in \mathcal{U}(A) \times \mathcal{U}(A) \mid \|UV - VU\| \leq \varepsilon\}$$

contains elements that cannot be connected by a path in C_ε to C_0. For if this were true, $C(T^2)$ would be semprojective.

The commutativity of two unitaries implies that they are simultaneously diagonalizable. In a heuristic sense, there are two opposing ways that unitaries can approximately commute. The first is that they are nearly simultaneously diagonalizable. In this case it should be possible to "push" the spectral measure of one unitary to the spectral measure of the other. So pairs of nearly simultaneously diagonalizable unitaries should be in the connected component of C_0 in C_ε.

On the other hand, one of the unitaries can take the eigenspace of the second unitary, say with eigenvalue λ, to the eigenspace of nearby but distinct eigenvalue $\sigma(\lambda)$. The only obvious way to obtain a homotopy of these matrices to unitaries which commute is to bring the appropriate eigenvaues together, that is to move λ over to $\sigma(\lambda)$. This is not always going to be possible to do for all the eigenvalues simultaneously. This is exactly what was going on with the unitaries in chapter II.

The quotients of $C(T^2)$ can be semiprojective even though $C(T^2)$ is not. These are all homotopic to $C(X)$ where X is one or zero dimensional. We concentrate on the simplest interesting quotient, $C(S^1 \vee S^1)$. The key to working with homotopies of maps from this C*-algebra is to express it in terms of generators and relations. There are many possible presentations, and the ones in proposition 3.1 are perhaps not the most natural. However, expressions such as

$$(2 - U - U^*)^{1/2}(2 - V - V^*)(2 - U - U^*)^{1/2}$$

arise when looking for projections in subalgebras of $C(T^2)$. It is possible to show that the Noetherian algebra generated by U, V and $(2 - V - V^*)^{1/2}$ has the full K-theory of the torus.

For notation, let $C^*\langle T_1, T_2, \ldots, T_n \mid p_1(T)=0, p_2(T)=0, \ldots, p_k(T)=0 \rangle$ denote the universal C*-algebra generated by n operators $T_1, T_2, \ldots, T_n$ that satisfy the k polynomial relations $p_1(T)=0, p_2(T)=0, \ldots, p_k(T)=0$, $T = (T_1, T_2, \ldots, T_n)$, assuming that such an algebra exists. See [B1] for a precise definition.

3.1 Proposition: Let $S^1 \vee S^1$ denote a bouquet of two circles. Then

i) $C(S^1 \vee S^1) \cong \mathfrak{B}_1 = C^*\langle U, V \mid U \text{ and } V \text{ are commuting unitaries and } (2 - U - U^*)(2 - V - V^*) = 0 \rangle$

ii) $C(S^1 \vee S^1) \cong \mathfrak{B}_2 = C^*\langle U, V \mid U \text{ and } V \text{ are unitaries such that } (2 - U - U^*)(2 - V - V^*) = 0 \rangle$

iii) $C(S^1 \vee S^1) \cong \mathfrak{B}_3 = C^*\langle U, V \mid U \text{ and } V \text{ are unitaries such that } (2 - U - U^*)^{1/2}(2 - V - V^*)(2 - U - U^*)^{1/2} = 0 \rangle$

Proof: i) Since $C(T^2) \cong C^*\langle U, V \mid U \text{ and } V \text{ are commuting unitaries} \rangle$, $\mathfrak{B}_1$ is a quotient of $C(T^2)$ and so corresponds to a closed set Z in T^2. More precisely, Z is equal to the zero set of $(2 - U - U^*)(2 - V - V^*)$ where $U = e^{2\pi ix}$ and $V = e^{2\pi iy}$. Now

$$\begin{aligned}(2 - U - U^*)(2 - V - V^*) &= (2 - 2\cdot\mathrm{Re}(U))(2 - 2\cdot\mathrm{Re}(V)) \\ &= (2 - 2\cdot\cos(2\pi x))(2 - 2\cdot\cos(2\pi y)) \\ &= (4\cdot\sin(\pi x))^2(4\cdot\sin(\pi x))^2,\end{aligned}$$

which is equal to zero only if x or y is an integer. Therefore, $Z = \{(w,z) \in \mathbf{C}^2 \mid w = 1$ or $z = 1\} = S^1 \vee S^1$.

ii) It is clear that the relations for $\mathfrak{B}_1$ imply those for $\mathfrak{B}_2$; what is not clear is that the generators for $\mathfrak{B}_2$ commute. Let U and V be any unitaries on a Hilbert space $\mathcal{H}$ which satisfy $(2 - U - U^*)(2 - V - V^*) = 0$. We must show that U and V commute.

Taking adjoints, we find that $(2 - V - V^*)(2 - U - U^*) = (2 - U - U^*)(2 - V - V^*)$. Multiplying this out and cancelling terms shows $(V + V^*)(U + U^*) = (U + U^*)(V + V^*)$ so that the real parts U_r and V_r of U and V commute. Let $D = C(Y)$ denote the commutative C*-algebra generated by U_r, V_r and the identity. Let f and g denote the functions that correspond to U_r and V_r. We know that $(1 - f(y))(1 - g(y)) = 0$ for all points $y \in Y$. Therefore, for every point y, either $f(y) = 1$ or $g(y) = 1$.

Let p denote the projection corresponding to the characteristic function of the set $f^{-1}(1)$ and let $q = 1-p$. Notice that p may not be in $\mathfrak{B}_2$. We have shown that $qV_rq = q$, and certainly $pU_rp = p$. One can easily show, by working in p, q coordinates and multiplying out $U^*U = 1$ and $V^*V = 1$, that $pUp = p$ and $qVq = q$, so $\mathcal{H}$ decomposes into two subspaces such that $U = I \oplus U'$ and $V = V' \oplus I$. These operators clearly commute.

iii) For any operators $h, k \geq 0$,

$$\| h^{1/2}kh^{1/2} \| = \| (h^{1/2}k^{1/2})^*(h^{1/2}k^{1/2}) \| = \| h^{1/2}k^{1/2} \|^2.$$

Therefore, if $h^{1/2}kh^{1/2} = 0$, $h^{1/2}k^{1/2} = 0$ and so $hk = h^{1/2}(h^{1/2}k^{1/2})k^{1/2} = 0$. Taking $h = (2 - U - U^*)$ and $k = (2 - V - V^*)$ this shows that the relations for $\mathcal{B}_3$ imply those for $\mathcal{B}_2$. Since the relations for $\mathcal{B}_1$ clearly imply those for $\mathcal{B}_3$, we are done.

Q.E.D.

For any C*-algebra A, we define

$$C'_\varepsilon(A) = \{(U,V) \in \mathcal{U}(A) \times \mathcal{U}(A) \mid \| (2 - U - U^*)^{1/2}(2 - V - V^*)(2 - U - U^*)^{1/2} \| \leq 16\varepsilon\}.$$

3.2 Proposition: There exists an $\varepsilon_0 > 0$ so that for all $\varepsilon < \varepsilon_0$ and for all C*-algebras A, there is a collection of mappings $H_{\varepsilon,A} : C'_\varepsilon(A) \to C'_0(A)$ which is natural with respect to homomorphisms $A \to B$ and leaves $C'_0(A)$ invariant up to homotopy.

Proof: Let U and V be unitaries acting on $\mathcal{H}$ such that

$$\| (2 - U - U^*)^{1/2} (2 - V - V^*) (2 - U - U^*)^{1/2} \| \leq 16\varepsilon .$$

Let E denote the spectral measure for V. For convenience we assume E is defined on subsets of the unit interval, identified in the usual way with $\{ z : |z| = 1 \}$. In the last proposition we used the projection $E((1,1))$ to find the subspace where U was the identity. When the relation only holds approximately, we need to choose a smaller projection $p = E([\delta, 1-\delta])$, $\delta > 0$. This will pick out a subspace where U is approximately the identity. We will choose δ later.

The operator $(2 - V - V^*)$ corresponds to the function $4\sin^2(\pi x)$, so is bounded below by $4\sin^2(\pi\delta)$ on the inteval $[\delta, 1-\delta]$. Therefore,

$$4\sin^2(\pi\delta)p \leq (2 - V - V^*) \Rightarrow p \leq (4\sin^2(\pi\delta))^{-1}(2 - V - V^*)$$

Multiplying on both sides by $(2 - U - U^*)^{1/2}$, we obtain

$$\begin{aligned}(2 - U - U^*)^{1/2}&p(2 - U - U^*)^{1/2} \\ &\leq (4\sin^2(\pi\delta))^{-1}(2 - U - U^*)^{1/2}(2 - V - V^*)(2 - U - U^*)^{1/2} \\ &\leq 4\varepsilon / \sin^2(\pi\delta) .\end{aligned}$$

Since, for a scalar η, $h^*h \leq \eta$ implies $hh^* \leq \eta$,

$$p(2 - U - U^*)p \leq 4\varepsilon / \sin^2(\pi\delta)$$

Let $M = 2\varepsilon / \sin^2(\pi\delta)$. Then

$$p(2 - U - U^*)p \leq 2Mp$$

and

$$p - \mathrm{Re}(pUp) \leq Mp . \qquad (*)$$

We next must calculate how close U is to the identity on $p\mathcal{H}$. Write U in p, $1\text{-}p$ coordinates, breaking up the p,p coefficient into its real and imaginary parts:

$$U = \begin{pmatrix} a = h + ik & b \\ c & d \end{pmatrix}.$$

Writing out $UU^* = 1$ leads to the equality $k^2 + bb^* = 1 - h^2$. In terms of these coordinates, (*) says that $h \geq 1 - M$. Since k^2 and bb^* are positive, $k^2, bb^* \leq 1 - (1 - M)^2 = 2M - M^2$ hence $\|k\| \leq (2M - M^2)^{1/2}$ and $\|b\| \leq (2M - M^2)^{1/2}$. Similarly, $\|c\| \leq (2M - M^2)^{1/2}$ so that $p + (1-p)U(1-p)$ will be near U. It is tempting to use the polar part of $p + (1-p)U(1-p)$ as the unitary that is to replace U. Unfortunately this is no longer in the C*-algebra A.

In order to stay within the C*-algebra we approximate the characteristic function of $[\delta, 1-\delta]$ by a continuous function. So let δ' be yet another constant, to be chosen later, which is between δ and $1/2$. Let f be the function

$$f(t) = \begin{cases} 0 & \text{if } t < \delta \text{ or } 1-\delta < t \\ 1 & \text{if } t \in [\delta', 1-\delta'] \\ \dfrac{t-\delta}{\delta' - \delta} & \text{if } t \in [\delta, \delta'] \\ \dfrac{t-1-\delta}{\delta - \delta'} & \text{if } t \in [1-\delta', 1-\delta] \end{cases}$$

Define $U' = f(V) + (1 - f(V))^{1/2} U (1 - f(V))^{1/2}$. ($U'$ will not be unitary.)

We need to perturb V as well so that it is the identity on $(1 - p)\mathcal{H}$. Let g be the function

$$g(t) = \begin{cases} 1 & \text{if } 1 \leq t \leq \delta' \text{ or } 1 - \delta' \leq t \leq 1 \\ \exp(2\pi i(\dfrac{t - \delta'}{1 - 2\delta'})) & \text{if } \delta' < t < 1 - \delta' \end{cases}$$

and define a new unitary by $V' = g(V)$.

Since $fg = gf$ and $g(1 - f)^{1/2} = (1 - f)^{1/2} = (1 - f)^{1/2} g$,

$$\begin{aligned} U'V' &= f(V)g(V) + (1 - f(V))^{1/2}U\,(1 - f(V))^{1/2}g(V) \\ &= f(V)g(V) + (1 - f(V))^{1/2}U\,(1 - f(V))^{1/2} \\ &= g(V)f(V) + g(V)(1 - f(V))^{1/2}U\,(1 - f(V))^{1/2} \\ &= V'U' \end{aligned}$$

and similarly, U'^* commutes with V'. The unitary we want to use to replace U is $U'(U'^*U')^{-1/2}$. For this to work, U' must be invertible. This in turn will follow whenever $\|U - U'\| < 1$. It remains to find restrictions on ε and δ that force this to hold.

To simplify expressions, write f and $(1 - f)^{1/2}$ for $pf(V)p$ and $p(1 - f)^{1/2}p$.

$$U' = \begin{pmatrix} f & \\ & 0 \end{pmatrix} + \begin{pmatrix} (1-f)^{1/2} & \\ & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} (1-f)^{1/2} & \\ & 1 \end{pmatrix}$$

$$= \begin{pmatrix} f + (1-f)^{1/2}a(1-f)^{1/2} & (1-f)^{1/2}b \\ c(1-f)^{1/2} & d \end{pmatrix}$$

In the 1,1 corner of $U - U'$ we find

$$a - f - (1-f)^{1/2}a(1-f)^{1/2} = (a - 1) + (1-f)^{1/2}(1 - a)(1-f)^{1/2}$$

which, in norm, is less than or equal to $2\|a - 1\| \le 2(\|1 - h\| + \|k\|)$. The error in the 1,2 corner is bounded in norm by $\|b\|$, and in the 2,1 corner the error is bounded by $\|c\|$. By the previous estimates, we find that

$$\|U - U'\| \le 2M + 4(2M - M^2)^{1/2} .$$

If ε and δ are any pair for which $2M + 4(2M - M^2)^{1/2}$ is less than one, then define $H_{\varepsilon,\delta}$ on any pair $(U,V) \in C'_\varepsilon$ by $H_{\varepsilon,\delta}(U,V) = (U'',V')$ where U' and V' are defined as above and $U'' = U'(U'^*U')^{-1/2}$.

If we let $q = 1 - \chi_{[\delta', \delta']}$ then q decomposes $\mathcal{H}$ in such a way that U'' and V' equal the identity on opposing factors. Therefore $H_{\varepsilon,\delta}$ maps into C'_0. The functional calculus is natural with respect to homomorphisms so it is easy to check that $H_{\varepsilon,\delta}$ is natural. If $(U, V) \in C'_0$ then $U'' = U' = U$. There is clearly a homotopy from g to $\exp(2\pi it)$ such that $(U, g_t(V))$ is a path of unitaries in C'_0. Thus the $H_{\varepsilon,\delta}$ satisfy the desired conditions; all that remains is to select choices of δ and δ' for each ε.

Let M_0 be the smallest real such that $2M_0 + 4(2M_0 - M_0^2)^{1/2} = 1$. For $\varepsilon < \varepsilon_0 = M_0/2$, let $\delta = \delta_\varepsilon = \pi^{-1}\arcsin((2\varepsilon/M_0)^{1/2})$ and let $\delta' = (1+ 2\delta)/4$. We define $H_\varepsilon = H_{\varepsilon,\delta\varepsilon}$. For any C*-algebra $A \subset \mathbf{B}(\mathcal{H})$, $H_{\varepsilon,A}$ is the restriction of H_ε to A.

Q.E.D.

3.3 Theorem: $C(S^1 \vee S^1)$ is semiprojective.

Proof: By Proposition 3.1(iii) and Theorem 1.2 it suffices to show that, for every limit C*-algebra $B = \lim B_n$ and any pair $(U, V) \in C'_0(B)$, there exists n such that (U,V) can be connected by a path, in $C'_0(B)$, to $C'_0(B_n)$.

Let $H_{\varepsilon,-}$ denote any collection of mappings satisfying the conditions of proposition 3.2. Choose any $\varepsilon < \varepsilon_0$. Choose a pair of unitaries U_1 and V_1 in some B_n which are very close to U and V. Take the polar part of the linear paths from U to U_1 and V to V_1. This produces a path of unitaries $t \longmapsto (U_t, V_t) \in C'_\varepsilon(B)$ such that $(U_0, V_0) = (U,V)$ and $(U_1, V_1) \in B_n$ for some n. Let $(U'_t, V'_t) = H_{\varepsilon,IB}(U_t, V_t)$ where IB denotes $C([0,1], B)$. The naturality of $H_{\varepsilon,-}$ shows that $(U'_0, V'_0) = H_\varepsilon(U, V)$, which is homotopic to (U,V). On the other hand, $(U'_1,V'_1) = H_\varepsilon(U_1, V_1) \in B_n$.

Q.E.D.

References

[B4] B. Blackadar, *K-Theory for Operator Algebras*, MSRI publication series, Springer Verlag, to appear.

[B1] __________, *Shape theory for C*-algebras*, Math. Scand. **56**(1985) 249-275.

[B2] O. Brattelli, *Inductive limits of finite dimensional C*-algebras*, Trans. Amer. Math. Soc., **171**(1972), 195-234.

[B3] _________, *The center of approximately finite-dimensional C*-algebras*, J. Funct. Anal., **21**(1976), 195-202.

[C1] A. Connes, *C*-algèbres et géométrie différentielle*, C.R. Acad. Sc. Paris **290**(1980),599-604.

[C2] _________, *The Chern character in K homology*, preprint, IHES, 1982.

[EK1] E. Effros and J. Kaminker, *Homotopy continuity and shape theory for C*-algebras*, in Geometric Methods in Operator Algebras, U.S.-Japan Joint Seminar at Kyoto, 1983, Pitman, to appear.

[EK2] __________________________, *Some homotopy and shape calculations for C*-algebras*, preprint, MSRI, 1984.

[MS1] S. Mardesic, J.Segal, *Shapes of compacta and ANR-systems*, Fund. Math. 72(1971), 41-59.

[P1] M. V. Pimsner, *Embedding some transformation group C*-algebras into AF-algebras*, Ergodic Theory and Dynamical Systems, **3**(1983), 613-626.

[PV1] M. Pimsner, D. Voiculescu, *Imbedding the irrational rotation C*-algebra into an AF-algebra*, J. Operator Theory, **4**(1980), 201-210.

[R1] M. A. Rieffel, *C*-algebras associated with irrational rotations*, Pacific J. Math. **93**(1981), 415-429.

[S1] C. Schochet, *Topological methods for C*-algebras II: Geometric resolutions and the Künneth formula*, Pacific J. Math, **98**(1982), 443-458.

[S3] __________, *Topological methods for C*-algebras III: axiomatic homology*, preprint.

[S2] J. S. Spielberg, *Extensions of Subalgebras of AF algebras*, thesis, University of California, Berkeley, 1985.

[T1] J. Taylor, *Banach algebras and topology*, in Algebras in Analysis (J. Williamson, ed.), Academic Press, New York, 1975.

[V1] D. Voiculescu, *Asymptotically commuting finite rank unitary operators without commuting approximants*, Acta Sci. Math., **45**(1983), 429-431.

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