Construction of Geometric Outer-Measures and Dimension Theory

by

David Worth

B.S., Mathematics, University of New Mexico, 2003

THESIS

Submitted in Partial Fulfillment of the
Requirements for the Degree of

Master of Science
Mathematics

The University of New Mexico

Albuquerque, New Mexico

December, 2006

DEDICATION

To my lovely wife Meghan, to whom I am eternally grateful for her support and love. Without her I would never have followed my education or my bliss.

ACKNOWLEDGMENTS

I would like to thank my advisor, Dr. Terry Loring, for his years of encouragement and aid in pursuing mathematics. I would also like to thank Dr. Cristina Pereyra for her unfailing encouragement and her aid in completing this daunting task. I would like to thank Dr. Wojciech Kucharz for his guidance and encouragement for the entirety of my adult mathematical career.

Moreover I would like to thank Dr. Jens Lorenz, Dr. Vladimir I Koltchinskii, Dr. James Ellison, and Dr. Alexandru Buium for their years of inspirational teaching and their influence in my life.

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ABSTRACT OF THESIS

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Abstract

Geometric Measure Theory is the rigorous mathematical study of the field commonly known as Fractal Geometry. In this work we survey means of constructing families of measures, via the so-called "Carathéodory construction", which isolate certain small-scale features of complicated sets in a metric space. The construction is explicit and covered in great detail, after which specific instances of constructed measures are investigated in depth. The work then investigates another related by fundamentally different class of measures, the "packing measures", and the two classes are compared. Finally, certain important dimensional ideas are investigated in detail including some oddities of the field.

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Chapter 1

Introduction

1.1 Background

The study of Geometric Measure Theory, often referred to as "Fractal Geometry", has roots in physics, mathematics, and in fields as concrete as geography. The grey area between mathematics and physics is realized in the study of stochastic systems; in geography the study of coastlines led to "fractal structures" as observed by Richardson [BM83, pg. 33]; and in pure mathematics functions of complicated fine structure were realized by Karl Weierstrass (1872) in functions without derivatives anywhere [GAE04, pg. 3], Helge Von Koch (1904) in non-rectifiable curves [GAE04, pg. 25], and later, and in great depth, by A.S. Besicovitch and Felix Hausdorff[1].

[1] To avoid merely copying the table of contents of Gerald Edgar's encyclopedic work documenting the mathematical foundations of modern Fractal Geometry/Geometric Measure Theory the author recommends that the reader new to the area find a copy of Edgar's Classics on Fractals, in which he has collected, cataloged, and annotated the major early papers in the field, and included recommendations for further reading. [GAE04]

1.2 Overview

Geometric measure theory is the name for the modern mathematical framework in which to discuss "fractal geometry" as it is understood by mathematicians and physicists.

The primary goal of this work is to discuss the constructive methods used for generating so-called "Geometric Measures" with some abstract machinery, and to prove in great generality, properties of the constructed measures. There are actually two general families of incompatible measures discussed. The incompatibility is discussed as a side effect of their varying constructions, and comparisons between the two families, as specialized as they are, are discussed.

We begin with a minimal amount of analysis, including topology and measure theory, enough to provide the necessary definitions for the rest of the work but little enough that the subject cannot be learned only from the provided foundations. A basic undergraduate or early graduate background in those subjects is expected and required for the remainder of the text.

The next focus is the Carathéodory construction and the so-called "Carathéodory geometric measures", which are the measures derived from the construction. The most famous of these measures, the family of Hausdorff measures (or Hausdorff-Besicovitch measures) is pursued in some depth, including some explicit calculations and constructions of sets with a given measure. We also discuss two other classes of measures, closedly related to the Hausdorff measure, the spherical and net measures, which provide bounds on the family of Hausdorff outer-measures in terms of more easily visualized and worked with covering sets.

A more active subject is then explored, that of the packing measures, introduced by Taylor and Tricot [SJTCT85]. The family of packing measures is in many respects a family of measures "dual" to the Hausdorff measures but fundamentally different in the sets which they measure effectively.

The final subject of importance in the text is that of dimension theory and two dimensions of sets derived from the Hausdorff and packing outer-measure families. These dimensions are related to the topological dimension in important ways; so much so that Mandelbrot defined a "fractal" in terms of the relationship between the topological dimension and the Hausdorff dimensions [BM83, pg. 15].

Chapter 2

Mathematical Foundations

2.1 Set Theory and Analytic Background

Notation. By *countable* we mean either a finite set or a countable set in the usual sense of a set which can be put in 1-1 correspondence with the natural numbers, denoted by $\mathbb{N}$

Notation. Let X be a set, then by $\mathscr{P}(X)$ we denote the power-set of X.

Definition 2.1.1. Let (X, d) be a metric space. A function $f : X \to X$ is called *Lipschitz with constant c* if $d(f(x), f(y)) \leq cd(x, y)$ for all $x, y \in X$.

Of course there is a more general notion of Lipschitz with f mapping between two different metric spaces but the modification of the definition is obvious. Moreover if the explicit Lipschitz constant is not listed then one assumes it exists as needed.

Definition 2.1.2. Let V be a linear subspace of the normed vector space $(\mathbb{R}^n, ||\cdot||)$, and let $A \subseteq \mathbb{R}^n$. We denote the orthogonal projection of A onto V by $Pr_V(A) = \{Pr_V(a) : a \in A\}$.

Lemma 2.1.3. *Let V be a linear subspace of the complete normed vector space $(\mathbb{R}^n, ||\cdot||)$. Pr_V is Lipschitz of constant 1.*

Proof. By the projection theorem (see Royden [HLR68, pg. 214, #53]) we may write $x = v + w$ where $v \in V$ and $w \in V^{\perp}$. Then

$$||Pr_V(x)|| = ||Pr_V(v+w)|| = ||Pr_V(v) + Pr_V(w)|| = ||v|| \leq ||v+w|| = ||x||$$

Using this result we see that by replacing x with $x - y$ we have

$$||Pr_V(x) - Pr_V(y)|| = ||Pr_V(x-y)|| \leq ||x-y||$$

so Pr_V is Lipschitz with constant 1. □

Definition 2.1.4. A function $\phi : [0,\infty) \to [0,\infty)$ is called *Hausdorff* if it is non-decreasing, continuous, $\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$.

It should be noted that various authors require that a Hausdorff function be continuous from the right at zero but as a matter of convenience we choose continuity everywhere. This approach is consistent with the work of McClure [MM94, pg. 5] and Hasse [HH86] (who only insists that the function be zero at zero and continuous) and presents no clear limitation to the theory. To the contrary, several useful results become untrue if we drop the continuity assumption.

Lemma 2.1.5. *The composition of two Hausdorff functions is Hausdorff.*

Proof. Let h, g be Hausdorff, and let $f = h \circ g$. $f(0) = h(g(0)) = h(0) = 0$ and let $t > 0$ then if $t_0 = g(t)$ then $t_0 > 0$. $f(t) = h(g(t)) = h(t_0) > 0$ since h is Hausdorff so f is Hausdorff. □

Example 2.1.6. Let $f_s : [0,\infty) \to [0,\infty), f_s(t) = t^s$, then f_s is Hausdorff for all $s > 0$.

2.2 Topological Background

Notation. Throughout this section let $(X, ||\cdot||)$ be a normed vector space over some field F, and let $A \subseteq X$, $x \in X$, and $c \in F$, and let Y by a topological space, unless otherwise noted.

For this paper we will only concern ourselves with real vector spaces unless otherwise noted. We state those results which we can in great generality, but when specificity is required the real case is used.

Notation. Let $x \in X$ and let $r > 0$. We denote by $B(x,r) = \{a \in X : d(x,a) < r\}$, the open ball of radius r about the point x. We denote by $\overline{B}(x,r) = \{a \in X : d(x,a) \leq r\}$, the closed ball of radius r about the point x.

Definition 2.2.1. By the *diameter* of a set A in a normed vector space we mean $diam(A) = \sup\{||x-y|| : x, y \in A\}$. For brevity we use the notation $d(A) = diam(A)$. If we are in an arbitrary metric space (X,d) we write $d(A) = \sup\{d(x,y) : x, y \in A\}$ for the diameter.

Definition 2.2.2. By a *translation of* A *by* x, denoted $A+x$, we mean the set $\{a+x : a \in A\}$.

Lemma 2.2.3. $diam(\cdot)$ *is translationally invariant.*

Proof. By definition for any $U \subseteq X$ and any $x \in X$, $U + x = \{u + x : u \in U\}$. So for $u_1, u_2 \in U + x$ there are $u_1', u_2' \in U$ such that $u_1 = u_1' + x$ and $u_2 = u_2' + x$.

So

$$\begin{aligned} d(U+x) &\stackrel{def}{=} \sup\{||u_1 - u_2|| : u_1, u_2 \in U + x\} \\ &= \sup\{||(u_1'+x) - (u_2'+x)|| : u_1', u_2' \in U\} \\ &= \sup\{||u_1' - u_2'|| : u_1', u_2' \in U\} \\ &\stackrel{def}{=} d(U) \end{aligned}$$

□

Definition 2.2.4. Let $A \subseteq X$ and $c \in F$. By a *scaling of A by c*, denoted by cA, we mean the set $\{ca : a \in A\}$.

Lemma 2.2.5. *Let $(X, ||\cdot||)$ be a normed vector space over $\mathbb{R}$. If $c \in (0, \infty)$ then $d(cA) = cd(A)$.*

Proof.

$$\begin{aligned} d(cA) &\overset{def}{=} \sup\{||x - y|| : x, y \in cA\} \\ &= \sup\{||cx_0 - cy_0|| : x_0, y_0 \in A\} \\ &= \sup\{c||x_0 - y_0|| : x_0, y_0 \in A\} \\ &= cd(A) \end{aligned}$$

□

Definition 2.2.6. Fix $\delta \in (0, \infty)$. A *δ-cover of A* is countable collection of subsets of X, $\{U_i\}_{i=1}^{\infty}$, such that $A \subseteq \bigcup_i U_i$ and $d(U_i) \leq \delta$.

Lemma 2.2.7. *Let $\{U_i\}$ be a δ-cover of A, then $\{U_i + x\}$ is a δ-cover of $A + x$.*

Proof. Let $y \in A$, then $y \in U_i$ for some i. Then $y + x \in U_i + x$ and $y + x \in A + x$, both by the definition of translation. Since this holds for every $y \in A$ and $d(U_i) = d(U_i + x)$ by Lemma 2.2.3, $\{U_i + x\}$ is a δ-cover of $A + x$. □

Lemma 2.2.8. *Let $\{U_i\}$ be a δ-cover of A, then $\{cU_i\}$ is a $(c\delta)$-cover of cA for all $c \in (0, \infty)$.*

Proof. Let $y \in A$ then $y \in U_i$ for some i. Then $cy \in cU_i$ and $cy \in cA$, but this is true for all $y \in A$ so $\{cU_i\}$ is a cover of A, and by Lemma 2.2.5 $\{cU_i\}$ is a $(c\delta)$-cover of cA. □

Definition 2.2.9. Two sets $A, B \subset X$ are *positively separated* if $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\} > 0$.

The following theorem is a nice topological result, the proof of which is straightforward (and may be found in Edgar [GAE91, pg. 58]) but the result useful later.

Theorem 2.2.10. *Let (X, d) be a metric space, $A \subseteq X$ be closed, and $B \subseteq X$ be compact. If $A \cap B = \emptyset$ then $dist(A, B) > 0$.*

Proof. Assume $dist(A, B) = 0$ then there exist $\langle x_n \rangle \subset A$, $\langle y_n \rangle \subset B$ such that $d(x_n, y_n) < 1/n$. Since B is compact we may pass to a convergent sub-sequence (also called $\langle y_n \rangle$) with the property that $y_n \to y \in B$ as $n \to \infty$. Then $x_n \to y$ as $n \to \infty$ by assumption but since A is closed $y \in A$ thus $A \cap B \neq \emptyset$, a contradiction.

So $dist(A, B) > 0$. □

Example 2.2.11. It should be noted that the compactness of the second set is necessary in the hypotheses since the following two sets are both closed and disjoint but the distance between them is zero. Let $A = \mathbb{R} \times \{0\}$ (essentially the x-axis in the plane) and $B = \{(x, 1/x) : x \in \mathbb{R}\}$ (essentially the graph of the function $f(x) = 1/x$). Both are closed since their complements are open, but neither are compact since neither is bounded (by the Heine-Borel Theorem). Moreover $dist(A, B) = 0$.

Corollary 2.2.12. *Disjoint compact sets in $\mathbb{R}^n$ are positively separated.*

Proof. By Heine-Borel compact sets are closed and bounded in $\mathbb{R}^n$ thus we simply apply the above theorem directly. □

Definition 2.2.13. A topological space Y is called *Hausdorff* if given two points $x, y \in Y$ with $x \neq y$ then there exist open sets $U, V \subset Y$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

Definition 2.2.14. A Hausdorff topological space Y is called *locally compact* if for every $x \in Y$ there exists an open set $U \subset Y$ such that the closure of U, denoted $\overline{U}$, is compact.

Example 2.2.15. $(\mathbb{R}^n, ||\cdot||_2)$, with $||\cdot||_2$ being the Euclidean norm, is a locally compact normed vector space over $\mathbb{R}$.

Note. For the majority of this work if we discuss the topology of $\mathbb{R}^n$ we assume it is equipped with the Euclidean norm (and subsequently the Euclidean metric).

Lemma 2.2.16. *Let (X, d) be a metric space, and let $A \subset X$ be a bounded set, then for all $a \in A$, $A \subseteq \overline{B}(a, d(A))$.*

Proof. Fix $a_0 \in A$, then for any $a \in A$ $d(a_0, a) < d(A)$ by definition of diameter, thus $A \subseteq \overline{B}(a_0, d(A))$. Since the result is independent of our choice of a_0 the lemma follows. $\square$

2.3 Measure Theoretic Background

Definition 2.3.1. Given a set X, a function $\nu : \mathscr{P}(X) \to [0, \infty]$ is an *outer-measure* if

1. $\nu(\emptyset) = 0$
2. For $A \subseteq A' \subseteq X$, $\nu(A) \leq \nu(A')$
3. For $\{A_i\}_{i=1}^{\infty} \subset \mathscr{P}(X), \nu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \nu(A_i)$

Note. Halmos' definition of an outer-measure is identical [PH64, pg. 42], but stated as "An *outer measure* is an extended real valued, non negative, monotone, and countably subadditive set function ..." It is important to note that countable subadditivity is defined as above for outer measures rather than additivity as one has for measures.

Definition 2.3.2. Given an outer-measure ν, a set B is *ν-measurable in the sense of Carathéodory* if and only if for all $A \subseteq X$ arbitrary we have $\nu(A) = \nu(A \cap B) + \nu(A \setminus B)$.

Note. For the duration of this work when we refer to a set as measurable we mean measurable in the sense of Carathéodory.

Notation. We denote the set of all ν-measurable subsets of X (in the sense of Carathéodory) by $\mathscr{M}(X, \nu)$.

The credit given to Carathéodory in this specific definition of measurability comes from Gerald Edgar [GAE91, pg. 130]. This definition may also be found in Halmos [PH64, pg. 44] and Royden [HLR68, pg. 251], but Carathéodory is not named. Royden provides an alternate definition of measurability [HLR68, pg. 296] using the notion of measurable functions. For further information please see Royden's exposition on the subject.

In the geometric measure theory literature very often an outer-measure is simply referred to as a measure. We hold the convention that a measure is an outer-measure where the sub-additivity condition is stronger in that if $\{A_i\}_{i=1}^{\infty} \subset \mathscr{M}(X, \nu) \subset \mathscr{P}(X)$ are such that $A_i \cup A_j = \emptyset$ for $i \neq j$ then we have $\nu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \nu(A_i)$. Alternatively we may take a measure to be an outer-measure restricted to a σ-algebra of measurable sets where this subadditivity condition holds as the following Theorem of Carathéodory's (as found in Bauer [HB01, pg. 21] and paraphrased here with consistent notation) states:

Theorem 2.3.3. *Let ν be an outer-measure on a set X, then the system $\mathcal{A}$ of ν-measurable subsets of X is a σ-algebra. Moreover the restriction of ν to $\mathcal{A}$ is a measure.*

Definition 2.3.4. Let X be a locally compact Hausdorff topological space. An outer-measure ν on X is a *metric outer-measure* if given two positively separated sets $A, A' \subseteq X$ then $\nu(A \cup A') = \nu(A) + \nu(A')$.

Remark. Since normed vector spaces are metric spaces, and thus locally compact Hausdorff, the definition of metric outer-measure (Def. 2.3.4) applies immediately to any normed vector space

Example 2.3.5. The *n-dimensional Lebesgue Outer-Measure*, $\mathcal{L}^n$ is a metric outer-measure. This outer-measure may be constructed explicitly, and all of its properties proven independently or it may be constructed using abstract machinery, as we do in Example 3.1.16.

The following technical lemma on the continuity of metric outer-measures is necessary:

Lemma 2.3.6 (Carathéodory's Lemma)**.** *Let ν be a metric outer-measure in some metric space (X,d), $\langle A_n \rangle$ an increasing sequence of subsets of X such that $A = \lim_{n\to\infty} A_n$ and $d(A_j, A \setminus A_{j+1}) > 0$ for all j then $\lim_{n\to\infty} \nu(A_n) = \nu(A)$.*

Proof. By the definition of outer-measure we have $\nu(A_j) \leq \nu(A)$ for all j thus $\lim_{j\to\infty} \nu(A_j) \leq \nu(A)$ so we need $\nu(A) \leq \lim_{j\to\infty} \nu(A_j)$.

We begin by setting $B_1 = A_1$ and $B_j = A_{j+1} \setminus A_j$ so we have $A = \bigcup_{j=1}^{\infty} B_j$ and because ν is metric and $d(B_{2i}, B_{2j}) > 0$ by hypothesis (similarly for the odd indices) we have the following equalities:

$$\nu(\bigcup_{i=1}^{\infty} B_{2i}) = \sum_{i=1}^{\infty} \nu(B_{2i}) \quad \text{and} \quad \nu(\bigcup_{i=1}^{\infty} B_{2i-1}) = \sum_{i=1}^{\infty} \nu(B_{2i-1})$$

We assume both series converge otherwise $\nu(A) = \infty$. So the following inequalities hold:

$$\begin{aligned}
\nu(A) &= \nu(\bigcup_{i=1}^{\infty} A_i) \\
&= \nu\left(A_j \cup \left(\bigcup_{i=j+1}^{\infty} B_i\right)\right) \\
&\leq \nu(A_j) + \nu(\bigcup_{i=j+1}^{\infty} B_i) \\
&\leq \nu(A_j) + \sum_{i=j+1}^{\infty} \nu(B_i)
\end{aligned}$$

and since the series converges the tail goes to zero as $j \to \infty$.

The first inequality is by the fact that ν is metric and the second is by monotonicity of ν. So we have $\nu(A) \leq \lim_{j\to\infty} \nu(A_j)$ thus $\nu(A) = \lim_{j\to\infty} \nu(A_j)$. □

Definition 2.3.7. The *Borel Sets in* $\mathbb{R}^n$ are the smallest σ-algebra generated by the open (resp. closed, compact) sets of $\mathbb{R}^n$.

For a definition and formal exposition regarding the Borel sets, see Halmos [PH64, pg. 153] or Bauer [HB01, pg. 27].

Definition 2.3.8. An outer-measure (or measure) ν is *Borel* if the Borel sets are measurable.

Theorem 2.3.9 (Carathéodory's Criterion)**.** *An outer-measure ν on a metric space (X, d) is Borel if and only if ν is metric.*

Proof. The following proof that if ν is metric then ν is Borel is from Falconer [KJF85, Thm. 1.5, pg. 6] and relies heavily on Carathéodory's Lemma [2.3.6].

Let ν be a metric outer-measure on (X, d). Since the Borel σ-algebra is the smallest σ-algebra generated by the closed sets we may consider $E \subset X$ closed and $A \subset X$ arbitrary.

Define the sets $A_j = \{a \in A \backslash E : d(a, E) \geq \frac{1}{j}\}$. $\{A_j\}$ is increasing, i.e. $A_j \subsetneq A_{j+1}$. Since E is closed we have $\lim_{j\to\infty} A_j = \bigcup_{j=1}^{\infty} A_j = A \setminus E$. Moreover $d(A \cap E, A_j) \geq \frac{1}{j}$ so we have, since ν is metric, $\nu(A \cap E) + \nu(A_j) = \nu((A \cap E) \cup A_j) \leq \nu(A)$ where the final inequality is by the monotonicity of ν since $(A \cap E), A_j \subsetneq A$ are disjoint for all j.

To apply Carathéodory's Lemma we need $d(A_j, (A \setminus E) \setminus A_{j+1}) > 0$ for all j. Let $x \in (A \setminus E) \setminus A_{j+1}$, then there exists $z \in E$ such that $d(x, z) < \frac{1}{j+1}$ by definition. So if $y \in A_j$ then

$$d(x, y) + d(y, z) \leq d(x, z) < \frac{1}{j+1}$$

By subtracting and multiplying by -1 we have

$$d(x, y) \geq d(y, z) + d(x, z) \geq \frac{1}{j} - \frac{1}{j+1} > 0$$

So via Carathéodory's Lemma we have $\lim_{j\to\infty} \nu(A_j) = \nu(A \setminus E)$. As above we have

$$\begin{aligned}
\nu(A \cap E) + \nu(A \setminus E) &= \nu(A \cap E) + \lim_{j\to\infty} \nu(A_j) \\
&= \lim_{j\to\infty} \nu((A \cap E) \cup A_j) \\
&= \nu((A \cap E) \cup A \setminus E) \\
&= \nu(A)
\end{aligned}$$

So ν is Borel.

Conversely, let ν be Borel and let $A, A' \subset X$ such that $d(A, A') = \delta > 0$. Define

$$A_\epsilon = \{x \in X : d(x, A) < \epsilon = \delta/3\}$$

then $A \subseteq A_\epsilon$ and $d(A_\epsilon, A') > 0$ which implies $A_\epsilon \cap A' = \emptyset$. By construction A_ϵ is open and thus Borel so since ν is Borel and by the Carathéodory definition of measurability we get the following:

$$\nu(A \cup A') = \nu((A \cup A') \cap A_\epsilon) + \nu((A \cup A') \setminus A_\epsilon) = \nu(A) + \nu(A')$$

So ν is metric. $\square$

Note that to prove that ν Borel implies ν metric one may replace A_ϵ with $\overline{A}$ and the same proof goes through.

Definition 2.3.10. An outer-measure ν is *outer-regular* if given a set A there exists a ν-measurable set U such that $A \subseteq U$ and $\nu(A) = \nu(U)$.

Definition 2.3.11. An outer-measure ν is *Borel-regular* if given a set A there exists a Borel set B such that $A \subseteq B$ and $\nu(A) = \nu(B)$.

Note that Borel regular implies outer-regular for any Borel outer-measure (or measure) ν; in other words, Borel regularity is a stronger condition than outer-regularity.

Definition 2.3.12. An outer-measure (or measure) ν is *Radon* if the following conditions are met:

- ν is Borel
- $\nu(K) < \infty$ for all $K \subset X$ compact.
- $\nu(U) = \sup\{\nu(K) : K \text{ compact }, K \subset U\}$ for all open $U \subseteq X$.
- $\nu(A) = \inf\{\nu(U) : A \subseteq U \subseteq X, U \text{ open }\}$ for $A \subseteq X$.

Definition 2.3.13. Let $(X, ||\cdot||)$ be a normed vector space. An outer-measure ν on X is *translationally invariant* if $\nu(A) = \nu(A + x)$ for all $A \subseteq X$ and $x \in X$.

Example 2.3.14. The n-dimensional Lebesgue outer-measure $\mathcal{L}^n$ on $\mathbb{R}^n$ is translationally invariant and a Radon measure. This is clear since by Heine-Borel compact sets are closed and bounded, and bounded measurable sets have finite measure with respect to $\mathcal{L}^n$. The fact that $\mathcal{L}^n$ is Borel is well understood (see Halmos [PH64, pg. 153]).

Chapter 3

Carathéodory Geometric Measures

This section is entitled "Carathéodory Geometric Measures," but a great deal of the chapter discusses specific details of the Hausdorff measure, a specific instance of a "Carathéodory measure", or more accurately, a family thereof. We first introduce the "Carathéodory construction", an abstract piece of machinery for constructing metric outer-measures on a given metric space. Often that metric space is $\mathbb{R}^n$ but this constraint is not necessary for the general construction which may take place in a more general metric space within the stated constraints.

The best understood, or at least most general, of the outer-measures constructed by the Carathéodory construction is the family of Hausdorff outer-measures. This family is often referred to as the Hausdorff-Besicovitch outer-measures by influential members of the mathematical community, including Benoit Mandelbrot in The Fractal Geometry of Nature [BM83, pg. 15]. This family of outer-measures has a straight-forward construction in terms of the Carathéodory method, but generating useful lower bounds on the outer-measures of certain sets is difficult, so we introduce two other families of outer-measures which provide approachable bounds for the Hausdorff outer-measures in terms of more tractable sets. These two families of measures, the so-called spherical and net measures, are generated out of simple sets and are bounded by the Hausdorff outer-measures with "nice" coefficients.

The use of the term "Carathéodory Geometric Measures" appears unique to this work after a survey of related works. This naming scheme was chosen for two reasons: First, it groups a large class of metric outer-measures under an umbrella term allowing them to be considered together as a family, and secondly it distinguishes them from another important class of outer-measures, the packing measures, and allows one to discuss the difference between the two classes easily.

Throughout this section, unless otherwise noted, let $(X, ||\cdot||)$ be a locally compact normed vector space. In all cases for simplicity X may be simply thought of as $\mathbb{R}^n$ with the standard Euclidean topology.

3.1 Carathéodory Construction

The Carathéodory construction of outer-measures is a general framework with which one can construct many of the standard geometric outer-measures including the Hausdorff measures. For a fairly general discussion of the many outer-measures which may be constructed with this method see Federer [HF69, §2.10, pg 169].

Definition 3.1.1. Let (X, d) be a metric space, $\mathcal{F} \subseteq \mathscr{P}(X)$, and $\zeta : \mathcal{F} \to [0, \infty)$ (potentially Hausdorff) such that

1. For all $\delta > 0$ there exist $\{U_i\} \subset \mathcal{F}$ such that $X \subset \bigcup_i U_i$ and $d(U_i) \leq \delta$.
2. For all $\delta > 0$ there exists $U \in \mathcal{F}$ such that $\zeta(U) \leq \delta$ and $d(U) \leq \delta$.

For $\delta > 0$ we define

$$\psi_\delta : \mathscr{P}(X) \to [0, \infty], \psi_\delta(A) = \inf \left\{ \sum_i \zeta(U_i) : A \subseteq \bigcup_i U_i, d(U_i) \leq \delta, \{U_i\} \subset \mathcal{F} \right\}$$

As Mattila notes, the first condition above guarantees the existence of at least one δ-cover for any subset of X and the second guarantees that $\psi_\delta(\emptyset) = 0$.

Notation. By a *δ-cover* in the context of the Carathéodory Construction we mean a countable collection of sets $\{U_i\} \subset \mathcal{F}$ such that $\zeta(U_i) \leq \delta$ and $d(U_i) \leq \delta$. This definition is dependent on $\mathcal{F}$, if this is ambiguous we will refer to such covers as $(\mathcal{F}, \delta)$-covers.

For brevity we write $\psi_\delta(A) = \inf \sum_i \zeta(U_i)$ where $\{U_i\}$ is understood to be a $(\mathcal{F}, \delta)$-cover of A. In cases where this notation is ambiguous we will use an appropriately descriptive unambiguous version of the definition above.

If there are multiple families of subsets of the metric space (i.e. $\mathcal{F}, \mathcal{F}' \subseteq \mathscr{P}(X)$) or multiple set functions (i.e. $\zeta, \zeta' : \mathcal{F} \to [0, \infty)$) we may specify those objects with the notation: $\psi_\delta(\mathcal{F}, \zeta)$ and may distinguish between two constructed outer-measures such as $\psi_\delta(\mathcal{F}, \zeta)$ and $\psi_\delta(\mathcal{F}', \zeta')$.

Theorem 3.1.2. *ψ_δ is an outer-measure.*

Proof.

- We must show $\psi_\delta(\emptyset) = 0$. $\emptyset \subset U$ for all $U \in \mathcal{F}$ and by condition (2) of the definition there exists $U \in \mathcal{F}$ such that $\zeta(U) \leq \delta$ and $d(U) \leq \delta$ for any $\delta > 0$ so following the definitions we have $\psi_\delta(\emptyset) = \inf\{\zeta(U) : U \in \mathcal{F}\} = 0$.

- (Monotonicity of ψ_δ). We must show if $A \subseteq A' \subseteq X$ then $\psi_\delta(A) \leq \psi_\delta(A')$. Any δ-cover $\{U_i\}$ of A' is also a δ-cover of A so we have

$$\begin{aligned}
\psi_\delta(A) &= \inf\left\{\sum_i \zeta(U_i) : \{U_i\} \text{ a } \delta\text{-cover of } A\right\} \\
&\leq \inf\left\{\sum_i \zeta(V_i) : \{V_i\} \text{ a } \delta\text{-cover of } A'\right\} \\
&= \psi_\delta(A')
\end{aligned}$$

where the inequality is from the definition of infimum.

- (Countable subadditivity of ψ_δ). We must show that if $\{A_i\} \subset \mathscr{P}(X)$ then $\psi_\delta(\bigcup_i A_i) \leq \sum_i \psi_\delta(A_i)$. Moreover, we assume that $\sum_i \psi_\delta(A_i) < \infty$ or we have nothing to prove. So we have

$$
\begin{aligned}
\psi_\delta(\bigcup_i A_i) &\stackrel{def}{=} \inf\left\{\sum_k \zeta(E_k) : \{E_k\} \text{ a } \delta\text{-cover of } \bigcup_i A_i\right\} \\
&\leq \inf\left\{\sum_{i,j} \zeta(U_j^{(i)}) : \{U_j^{(i)}\} \text{ a } \delta\text{-cover of } A_i\right\} \\
&= \inf\left\{\left(\sum_j \zeta(U_j^{(1)})\right) + \left(\sum_j \zeta(U_j^{(2)}\right) + \cdots\right\} \\
&= \sum_i \left(\inf \sum_j \zeta(U_j^{(i)})\right) \\
&\stackrel{def}{=} \sum_i \psi_\delta(A_i) < \infty
\end{aligned}
$$

The inequality is because the set over which the infimum is taken is smaller on the right than the left hand side of the inequality. The following equality is simply a re-ordering of the sum in the previous term by grouping the entries by the set which they cover. □

Lemma 3.1.3. *ψ_δ is non-increasing in δ.*

Proof. Let $\delta_0 \leq \delta_1$.

Define $\mathcal{S}_\epsilon(A) := \{\{U_i\} : \{U_i\} \text{ is an } \epsilon\text{-cover of } A\}$, then $\mathcal{S}_{\delta_0} \subset \mathcal{S}_{\delta_1}$ and

$$
\psi_{\delta_0}(A) = \inf\left\{\sum_i \zeta(U_i) : \{U_i\} \in \mathcal{S}_{\delta_0}\right\} \geq \inf\left\{\sum_i \zeta(U_i) : \{U_i\} \in \mathcal{S}_{\delta_1}\right\} = \psi_{\delta_1}(A)
$$

□

Since the construction above is such that covers exist on all scales (as a function of δ) it is natural to consider the behavior of $\lim_{\delta \downarrow 0} \psi_\delta$.

Definition 3.1.4. We define $\psi(A) = \lim_{\delta \downarrow 0} \psi_\delta(A)$.

Lemma 3.1.5. *$\psi(A) = \sup_{\delta > 0} \psi_\delta(A)$ for all $A \subseteq X$.*

Proof. By Lemma 3.1.3 we know that $\psi_\delta(A)$ is non-increasing as a function of δ so

$$\sup_{\delta>0} \psi_\delta(A) = \lim_{\delta \downarrow 0} \psi_\delta(A) \stackrel{def}{=} \psi(A)$$

□

Corollary 3.1.6. *Let $0 \le s < \infty$. The following are equivalent:*

1. $\psi(A) = 0$.
2. $\psi_\delta(A) = 0$ *for all* $0 < \delta \le \infty$.
3. *For all $\epsilon > 0$ there exists $\{U_i\}$ such that $A \subset \bigcup_i U_i$ and $\sum_i \zeta(U_i) < \epsilon$.*

Proof. $(1 \Leftrightarrow 2)$: By Lemma 3.1.5 we have $0 = \psi(A) = \sup_{\delta>0} \psi_\delta(A)$ which is true if and only if $\psi_\delta(A) = 0$ for all $\delta > 0$.

$(1 \Leftrightarrow 3)$: By definition $\psi(A) = \lim_{\delta \downarrow 0} \inf \sum_i \zeta(U_i)$. If (3) holds then $\inf \sum_i \zeta(U_i) = 0$ so $\psi(A) = 0$ and if (1) holds then $\inf \sum_i \zeta(U_i) = 0 < \epsilon$ □

Theorem 3.1.7. *ψ is an outer-measure*

Proof. The proof follows immediately from the proof that ψ_δ is an outer-measures independent of δ.

- $\psi(\emptyset) \stackrel{def}{=} \lim_{\delta \downarrow 0} \psi_\delta(\emptyset) = \lim_{\delta \downarrow 0} 0 = 0.$

- Let $A \subseteq A' \subseteq X$ then $\psi(A) \overset{def}{=} \lim_{\delta \downarrow 0} \psi_\delta(A) \leq \lim_{\delta \downarrow 0} \psi_\delta(A') \overset{def}{=} \psi(A')$.
- Let $\{A_i\} \subset \mathscr{P}(X)$ then

$$\begin{aligned}
\psi(\bigcup_i A_i) &\overset{def}{=} \lim_{\delta \downarrow 0} \psi_\delta(\bigcup_i A_i) \\
&\leq \lim_{\delta \downarrow 0} \left(\sum_i \psi_\delta(A_i) \right) \\
&\leq \sum_i \lim_{\delta \downarrow 0} \psi_\delta(A_i) \\
&\overset{def}{=} \sum_i \psi(A_i)
\end{aligned}$$

The first inequality holds since $\psi_\delta(\bigcup_i A_i) \leq \sum_i \psi_\delta(A_i)$ for all $\delta > 0$ by the countable subadditivity of ψ_δ whereas the second inequality is due to the fact that $\psi_\delta(A_i) \leq \psi(A_i)$ for all $\delta > 0$.

So ψ is an outer-measure. □

Federer [HF69, Pg. 170] refers to ψ as the "Result of Carathéodory's construction from ζ on $\mathcal{F}$" and ψ_δ as the "size δ approximating measure." We will refer to them as outer-measures and approximating outer-measures throughout this work. In situations where we wish to denote explicitly what family of sets and what function is used to construct an outer-measures via the Carathéodory construction we will write $\psi(\mathcal{F}, \zeta)$ meaning that ψ is the Result of Carathéodory's construction from ζ on $\mathcal{F}$ in the language of Federer.

Theorem 3.1.8. *ψ is a metric outer-measure*

Proof. Let $A, A' \in \mathcal{F}$ such that $d(A, A') > 0$. Let $\delta \leq d(A, A')/3$.

Then if $\{U_i\}$ is a δ-cover of $A \cup A'$ and $U \in \{U_i\}$ we have if $U \cap A \neq \emptyset$ then $U \cap A' = \emptyset$ and by symmetry if $U \cap A' \neq \emptyset$ then $U \cap A = \emptyset$ Thus if I is the index set for a given δ-cover of $A \cup A'$ then we may partition I into I_1 and I_2 such that $I_1 \cap I_2 = \emptyset$ and $I = I_1 \cup I_2$.

Then we have

$$\begin{aligned}
\psi(A \cup A') &\overset{def}{=} \lim_{\delta \downarrow 0} \psi_\delta(A \cup A') \\
&\overset{def}{=} \lim_{\delta \downarrow 0} \inf \left\{ \sum_{i \in I} \zeta(U_i) \right\} \\
&= \lim_{\delta \downarrow 0} \inf \left\{ \sum_{i \in I_1} \zeta(U_i) + \sum_{i \in I_2} \zeta(U_i) \right\} \\
&= \lim_{\delta \downarrow 0} \left(\inf \left\{ \sum_{i \in I_1} \zeta(U_i) \right\} + \inf \left\{ \sum_{i \in I_2} \zeta(U_i) \right\} \right) \\
&= \lim_{\delta \downarrow 0} \inf \left\{ \sum_{i} \zeta(U_i) \right\} + \lim_{\delta \downarrow 0} \inf \left\{ \sum_{i} \zeta(U_i) \right\} \\
&\overset{def}{=} \psi(A) + \psi(A')
\end{aligned}$$

So ψ is metric.

□

Corollary 3.1.9. *ψ is Borel*

Proof. By Carathéodory's Criterion (2.3.9) since ψ is metric it is Borel. □

Corollary 3.1.10. *The outer-measure ψ is a measure when restricted to the Borel σ-algebra.*

Proof. See Theorem 2.3.3. □

This fact is particularly nice in that when we are measuring disjoint Borel sets we have additivity instead of subadditivity! With Borel regularity, if we can construct disjoint Borel sets containing arbitrary sets, we have additivity of the measures of arbitrary sets, making ψ a measure on a wider class of sets than the Borel sets.

Corollary 3.1.11. *Let ψ be the result of the Carathéodory construction on $(\mathbb{R}^n, d)$ with some family of sets $\mathcal{F} \subseteq \mathscr{P}(\mathbb{R}^n)$, and some $\zeta : \mathcal{F} \to [0, \infty)$. Let $A \subseteq \mathbb{R}^n$ be closed, $B \subset \mathbb{R}^n$ be compact, and $A \cap B = \emptyset$ then $\psi(A \cup B) = \psi(A) + \psi(B)$.*

Proof. By Theorem 2.2.10 we know that $dist(A, B) > 0$ and since ψ is metric the result follows. $\square$

Definition 3.1.12. Let $(X, ||\cdot||)$ be a normed vector space. A family of subsets $\mathcal{F}$ of X is *translationally invariant* if for all $U \in \mathcal{F}$ and $x \in X$ we have $(U + x) \in \mathcal{F}$.

Example 3.1.13. Examples of translationally invariant families of subsets of $\mathbb{R}^n$ when equipped with a norm are $\mathscr{P}(X)$, $\{B(x, r) : x \in X, r > 0\}$, $\{\overline{B}(x, r) : x \in X, r > 0\}$, and the Borel sets.

Theorem 3.1.14. *Let $(X, ||\cdot||)$ be a normed vector space. If ζ and $\mathcal{F}$ are translationally invariant then $\psi(\mathcal{F}, \zeta)$ is translationally invariant.*

Proof. Let $A \subseteq X$ and $\{U_i\} \subset \mathcal{F}$ be a cover of A. Then $\{U_i + x\}$ is a cover of $A + x$. By Lemma 2.2.3 we know that $d(U_i) = d(U_i + x)$ and by hypothesis $\zeta(U_i) = \zeta(U_i + x)$.

Thus we have

$$\begin{aligned}
\psi(A) &= \lim_{\delta \downarrow 0} \inf \left\{ \sum_i \zeta(U_i) : \{U_i\} \subset \mathcal{F}, A \subseteq \bigcup_i U_i \right\} \\
&= \lim_{\delta \downarrow 0} \inf \left\{ \sum_i \zeta(U_i + x) : \{U_i + x\} \subset \mathcal{F}, (A + x) \subseteq \bigcup_i (U_i + x) \right\} \\
&= \psi(A + x)
\end{aligned}$$

$\square$

Example 3.1.15. Let $(X, ||\cdot||)$ be a normed vector space. If we let $\mathcal{F} = \mathscr{P}(X)$ then for any function $\zeta : \mathcal{F} \to [0, \infty]$ such that $\zeta(U) = \zeta(U + x)$ for all $x \in X$ we have that all size δ approximating outer-measures and results of Carathéodory's construction using ζ and $\mathcal{F}$ are translationally invariant.

Once we have the Carathéodory Construction to define a new geometric outer-measure one need only specify a metric space (X, d), a family of sets $\mathcal{F}$ satisfying the necessary conditions and a function $\zeta : \mathcal{F} \to [0, \infty)$ as above and the resulting outer-measure has the above properties. It may be the case that given two seemingly different families of sets one may generate the exact same measure, for example the Borel sets may be generated in seemingly different, but equivalent ways.

Example 3.1.16. Lebesgue n-dimensional Outer-Measure.

Let $(\mathbb{R}^n, d)$ be the usual Euclidean metric space, and define $\mathcal{F}$ and ζ for the Carathéodory construction as follows:

$$\mathcal{F} = \{C \subset \mathbb{R}^n : C = [a_1, b_1) \times \cdots \times [a_n, b_n), a_i, b_i \in \mathbb{R}, a_i < b_i\}$$

$$\zeta(C) = V(C) = (b_1 - a_1) \cdot \ldots \cdot (b_n - a_n)$$

Then the result of Carathéodory's construction $\mathcal{L}^n$ is the Lebesgue n-dimensional outer-measure and as such is a Borel, translationally invariant, metric outer-measure by the above results. This construction is particularly nice as it agrees with our next subject for $n = 1$, and differs only by a constant for $n > 1$.

3.2 Hausdorff Measures

Definition 3.2.1. Let (X, d) be a metric space, $\mathcal{F} = \mathscr{P}(X)$, and $\zeta_s(\cdot) = d(\cdot)^s$, then for each $s \in (0, \infty)$ we construct the *s-dimensional size δ approximating measures* $\mathcal{H}^s_\delta$ and the *s-dimensional Hausdorff Measure*, $\mathcal{H}^s$ via the Carathéodory Construction.

One should note immediately that rather than constructing one outer-measure we are actually constructing a family of outer-measures parameterized by $s \in [0, \infty)$. This family has the interesting property, which will be shown in this section, that each outer-measure provides useful information about a different family of subsets of X.

Theorem 3.2.2. *Let (X, d) be a metric space, and $\zeta_s(A) = d(A)^s$ for $A \subseteq X$. If, in the Carathéodory Construction, we replace the standard $\mathcal{F} = \mathscr{P}(X)$ with $\mathcal{F}_1 = \{A \in \mathscr{P}(X) : A \text{ closed}\}$ then the resultant outer-measure $\psi(\mathcal{F}_1, \zeta)$ is the s-dimensional Hausdorff outer-measure $\mathcal{H}^s$. If in addition, $X = \mathbb{R}^n$ and $\mathcal{F}_2 = \{A \in \mathscr{P}(\mathbb{R}^n) : A \subset \mathbb{R}^n \text{ is convex}\}$ then $\psi(\mathcal{F}_2, \zeta) = \mathcal{H}^s$ as well.*

Proof. Fix $A \in \mathscr{P}(X)$. Let $\{U_i\}$ be an arbitrary δ-cover of A, then $d(U_i)^s = d(\overline{U_i})^s$ thus $\psi(\mathcal{F}_1, \zeta) = \mathcal{H}^s$. Moreover, if we denote the convex hull of a set U_i by $U_i^{(c)}$ then $d(U_i)^s = d(U_i^{(c)})^s$ by Krein-Milman (see Royden [HLR68, pg. 205]) and we have $\psi(\mathcal{F}_2, \zeta) = \mathcal{H}^s$ for $X = \mathbb{R}^n$. □

Actually Krein-Milman makes a stronger claim, if X is a locally convex topological vector space and a set K is compact and convex then it is the convex hull of its extreme points. So if X is a locally convex topological vector space and $\mathcal{F}$ is made up of compact sets, we may take their convex hulls without changing their diameters, and thus we would still produce the Hausdorff outer-measures using convex sets.

Another interesting special case is $X = \mathbb{R}^n$. Considering only δ-covers made up of closed sets produces the Hausdorff outer-measure in the limit, those sets are also compact by Heine-Borel. Thus choosing only compact δ-covers also produces the Hausdorff-outer measures in the limit as well.

This theorem does *not* state that each of the size δ approximating outer-measures agree with one another, but merely that the limiting behavior of each is the same.

Moreover, if you insist on a cover family made up entirely of open (resp. closed) *balls* the resulting outer-measures are *not* the s-dimensional Hausdorff outer-measures, but instead the s-dimensional Spherical outer-measures introduced in Section 3.3. Finally, if you choose open sets (not necessarily balls) the resulting outer-measures are the "Carathéodory outer-measures," $\mathcal{C}^s$, which are not discussed in this work. It should be noted though, that like the Spherical measure discussed below, the Carathéodory outer-measures are related to the Hausdorff outer-measures by constants.

The following proof, found in both Falconer [KJF85, pg. 8] and in Mattila [PM95, pg. 57], is standard and provides a natural classification of measurable sets:

Corollary 3.2.3. *$\mathcal{H}^s$ is Borel regular on $\mathbb{R}^n$.*

Proof. Let $A \subseteq \mathbb{R}^n$, $s > 0$ be fixed. If $\mathcal{H}^s(A) = \infty$ then $\mathbb{R}^n$ is an open set of equal measure so suppose $\mathcal{H}^s(A) < \infty$. For each $j \in \mathbb{N}$ choose an open $2/j$-cover $\{U_i^{(j)}\}_i$ such that

$$\sum_i d(U_i^{(j)})^s < \mathcal{H}^s_{1/j}(A) + 1/j$$

The existence of such a cover is guaranteed by Lemma 3.1.3 which states that the size δ approximating measures in the Carathéodory construction are non-increasing in δ.

Set $G = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} U_i^{(j)}$, then $A \subseteq G$ and G is a G_δ set (see Halmos [PH64, pg. 3]) since the infinite union of open sets is open. We also have that $\{U_i^{(j)}\}$ is a $2/j$-cover of G so $\mathcal{H}^s_{2/j}(G) \leq \mathcal{H}^s_{1/j}(A) + 1/j$. Finally, by monotonicity of $\mathcal{H}^s_\delta$ we have the following chain of inequalities:

$$\mathcal{H}^s_{2/j}(A) \leq \mathcal{H}^s_{2/j}(G) \leq \mathcal{H}^s_{1/j}(A) + 1/j$$

By letting $j \to \infty$ we have $\mathcal{H}^s(A) = \mathcal{H}^s(G)$, thus $\mathcal{H}^s$ is Borel regular (and outer-regular). $\square$

Theorem 3.2.4. *$\mathcal{H}^s$ respects the following scaling relation: Given $A, B \subseteq X$ and a surjective map $L : A \to B = L(A)$ Lipschitz with constant c we have $\mathcal{H}^s(L(A)) = \mathcal{H}^s(B) \leq c^s \mathcal{H}^s(A)$.*

Proof. Assume $c \geq 1$. Let $\{U_i\}$ be a $\frac{\delta}{c}$-cover of A, then $\{V_i = L(U_i)\}$ is a δ-cover of B. Note also that $\{U_i\}$ is a δ-cover of A since $c \geq 1$.

$$
\begin{aligned}
\mathcal{H}^s(B) &\overset{def}{=} \liminf_{\delta \downarrow 0} \left\{ \sum_i d(C_i)^s : \{C_i\} \text{ a } \delta\text{-cover of } B \right\} \\
&\leq \liminf_{\delta \downarrow 0} \left\{ \sum_i d(V_i)^s : V_i = L(U_i), \{U_i\} \text{ a } \frac{\delta}{c}\text{-cover of } A \right\} \\
&= \liminf_{\delta \downarrow 0} \left\{ \sum_i d(L(U_i)) : \{U_i\} \text{ a } \frac{\delta}{c}\text{-cover of } A \right\} \\
&\leq \liminf_{\delta \downarrow 0} \left\{ \sum_i (cd(U_i))^s : \{U_i\} \text{ a } \frac{\delta}{c}\text{-cover of } A \right\} \\
&= c^s \liminf_{\delta \downarrow 0} \left\{ \sum_i d(U_i)^s : \{U_i\} \text{ a } \frac{\delta}{c}\text{-cover of } A \right\} \\
&= c^s \mathcal{H}^s(A)
\end{aligned}
$$

Where the last equality is true thanks to the limiting behavior of the definition.

Assume $0 < c < 1$. In this case a $c\delta$-cover is also a δ-cover so we replace instances of $\frac{\delta}{c}$-covers with $c\delta$-covers and the proof goes through. □

Corollary 3.2.5. *Scaling a set A by a constant $c > 0$ produces equality in the above theorem, and thus the following relation $\mathcal{H}^s(cA) = c^s \mathcal{H}^s(A)$.*

Proof. Assume $c \geq 1$. Scaling by c is an invertible Lipschitz map so every δ-cover of cA is a scaled $\frac{\delta}{c}$-cover of A, thus the first inequality in the proof of Theorem 3.2.4 is an equality. Similarly, the second inequality is also an equality as $d(cU)^s = c^s d(U)^s$ for all $U \subseteq X$. Thus we have $\mathcal{H}^s(cA) = c^s\mathcal{H}^s(A)$ for all $A \subseteq X$.

And, as in the case of the proof of Theorem 3.2.4, the same proof goes through for $0 < c < 1$ by replacing $\frac{\delta}{c}$-covers with $c\delta$-covers of A. $\square$

Corollary 3.2.6. *Let V be a linear subspace of the normed vector space $(\mathbb{R}^n, ||\cdot||)$. The projection of a set $A \subseteq \mathbb{R}^n$ onto V produces the following relation:*

$$\mathcal{H}^s(Pr_V(A)) \leq \mathcal{H}^s(A)$$

.

Proof. This follows immediately from Lemma 2.1.3. $\square$

Example 3.2.7. Both strict inequality and equality are possible in the above thorem. Let V be the "x-axis" in $\mathbb{R}^2$ and consider the following sets: $A = \{0\} \times \mathbb{R}$, the "y-axis" in $\mathbb{R}^2$, and $B = \mathbb{R} \times \{1\}$, the graph of $f(x) = 1$. The projection of A onto the "x-axis" is the set $\{0\} = \mathrm{Pr}_V(A)$, a singleton, a set of $\mathcal{H}^s$ measure-zero as we will see in Lemma 3.2.12. On the other hand, B is simply a translation of the "x-axis", and as we see in the following corollary $\mathcal{H}^s$ is translationally invariant so we have equality.

Corollary 3.2.8. *$\mathcal{H}^s$ is translationally invariant.*

Proof. First we notice that translation is invertible and Lipschitz with constant 1.

Let $T_x : \mathscr{P}(X) \to \mathscr{P}(X), T_x(A) = A + x$ be the translation of any set $A \in \mathscr{P}(X)$ by some fixed $x \in X$, then for $y, z \in X$ we have

$$||T_x(y) - T_x(z)|| = ||(y + x) - (z + x)|| = ||y - z||$$

Thus we have

$$\mathcal{H}^s(A) = \mathcal{H}^s(T_{-x}(T_x(A))) \leq \mathcal{H}^s(T_x(A)) \leq \mathcal{H}^s(A)$$

□

An alternate more direct proof is as follows:

Proof. As seen in Example 3.1.15, $\mathcal{F} = \mathscr{P}(X)$ is translationally invariant and by Lemma 2.2.3 we know that $diam(\cdot)$ is translationally invariant, as such $diam(\cdot)^s$ is also translationally invariant, so by Theorem 3.1.14 we see that $\mathcal{H}^s$ is translationally invariant. □

Corollary 3.2.9. *Let $A \subseteq \mathbb{R}^n$, and $T \in \mathcal{O}(n, \mathbb{R})$ where $\mathcal{O}(n, \mathbb{R})$ is the n-dimensional real orthogonal group, or the group of isometries leaving the origin fixed in $\mathbb{R}^n$, then $\mathcal{H}^s(A) = \mathcal{H}^s(T(A))$.*

Proof. Since T is invertible and Lipschitz with constant 1 and surjective we have

$$\mathcal{H}^s(T(A)) \leq \mathcal{H}^s(A) = \mathcal{H}^s(T^{-1}(T(A))) \leq \mathcal{H}^s(T(A))$$

□

Lemma 3.2.10. *$\mathcal{H}^s$ is non-increasing in s.*

Proof. Let $A \subseteq X$, and let $0 \leq s < t < \infty$.

First we show the result for the size δ approximating outer-measures $\mathcal{H}^s_\delta$.

$$\mathcal{H}^s_\delta(A) \stackrel{def}{=} \inf \sum_i d(U_i)^s \geq \inf \sum_i d(U_i)^t \stackrel{def}{=} \mathcal{H}^t_\delta(A)$$

The inequality is a result of the fact that the infimum is taken over the same set of covers whose diameters are less than or equal to δ, which we assume is less than 1. When such diameters are raised to a larger power their value is smaller. Since the inequality is independent of δ we have

$$\mathcal{H}^s(A) \stackrel{def}{=} \lim_{\delta \downarrow 0} \mathcal{H}^s_\delta(A) \geq \lim_{\delta \downarrow 0} \mathcal{H}^t_\delta(A) \stackrel{def}{=} \mathcal{H}^t(A)$$

So $\mathcal{H}^s$ is non-increasing in s. □

Since the Hausdorff outer-measures are constructed with the most general family of covering sets $\mathcal{F} = \mathscr{P}(X)$ it may be compared immediately to any other result of the Carathéodory construction with $\mathcal{F}' \subsetneq \mathscr{P}(X)$.

Lemma 3.2.11. *Let $\psi^s = \psi(\mathcal{F}', \zeta_s)$ be the result of the Carathéodory construction with $\mathcal{F}' \subsetneq \mathscr{P}(X)$ and $\zeta_s(\cdot) = d(\cdot)^s$ then for all $A \subseteq X$, $\mathcal{H}^s(A) \leq \psi^s(A)$.*

Proof. Since $\mathcal{F}' \subsetneq \mathscr{P}(X)$, the infimum in each approximating outer-measure forces $\mathcal{H}^s_\delta(A) \leq \psi^s_\delta(A)$ since it is taken over a subset of the power set of X. Since this is true for all δ the result is proved. □

3.2.1 Calculating Hausdorff Measures

First we study sets of Hausdorff measure zero. Once we introduce the "Hausdorff Dimension" (Def. 5.1) we find that any set in $\mathbb{R}^n$ is of s-dimensional Hausdorff measure zero for all s greater than the "Hausdorff dimension" of the set. For now a more careful exposition of sufficient conditions is presented.

It should be noted that calculation of the s-dimensional Hausdorff outer-measure of a set may be quite difficult as the measure is dependent quite heavily on the specific s in question (actually on the so-called "Hausdorff dimension" of the set as we will see later). Since the notion of a set being of "measure zero" is much more general it is worth investigation.

Lemma 3.2.12. *Countable sets are of $\mathcal{H}^s$ measure zero for all $s \in (0, \infty)$.*

Proof. Fix $s \in (0, \infty)$. Let $\epsilon > 0$, and let $A = \{a_i\}$ be countable with the given enumeration.

Let $\{U_i\}$ be a ϵ-cover defined as follows: $U_n = B(a_n, \epsilon 2^{-n/s})$. Then we have

$$\mathcal{H}^s_\epsilon(A) \leq \sum_{n=1}^{\infty} d(U_n)^s = \sum_{n=1}^{\infty} (\epsilon 2^{-n/s})^s = \epsilon^s$$

and taking the limit as in the definition

$$\lim_{\epsilon \downarrow 0} \mathcal{H}^s_\epsilon(A) \leq \lim_{\epsilon \downarrow 0} \epsilon^s = 0$$

□

Or alternatively a second simple proof:

Proof. Let $A = \{a_i\}$ be countable with the given enumeration, then since $d(\{a_i\}) = 0$ for all singletons, and since A is countable, it is itself a countable δ-cover for all $\delta > 0$ such that $\sum_i d(a_i)^s = 0$. □

The following trivial result will be useful later when attempting to define the meaning of $\mathcal{H}^1$ as a useful measurement of arc-length for Jordan curves.

Lemma 3.2.13. *Fix $s \in (0, \infty)$. Let $Z, U \subseteq X$ such that $\mathcal{H}^s(Z) = 0$. Then $\mathcal{H}^s(U \cup Z) = \mathcal{H}^s(U)$. Moreover $\mathcal{H}^s(U \setminus Z) = \mathcal{H}^s(U)$.*

Proof. Given the above assumptions:

$$\mathcal{H}^s(U) \leq \mathcal{H}^s(U \cup Z) \leq \mathcal{H}^s(U) + \mathcal{H}^s(Z) = \mathcal{H}^s(U)$$

The first inequality is by monotonicity of Carathéodory outer-measures and the second is by subadditvity of those outer-measures. Again, given the assumptions in the statement:

$$\mathcal{H}^s(U \setminus Z) = \mathcal{H}^s((U \setminus Z) \cup (U \cap Z)) = \mathcal{H}^s(U)$$

where the first equality is by the first result in this proof since $U \cap Z$ is a set of measure zero and the second is clear.

□

The “”moreover“” clause in Lemma 3.2.13 is only interesting when $U \cap Z \neq \emptyset$ as otherwise $U \setminus Z = U$ and the statement is completely trivial.

The following “complicated,” but well understood sets possess many of the properties studied in geometric measure theory. In fact the Cantor sets can be used for constructive processes in the field (to construct sets of a given dimension) and are intimately related to the structure theorems for complicated sets.

Example 3.2.14 (Ternary Cantor Set)**.** We denote the Ternary Cantor Set, which we construct below, by $C(1/3)$.

We may construct the set by the following iterated system: Set $I_0 = [0, 1]$ and inductively define $I_n = \left(\frac{1}{3}I_{n-1}\right) \cup \left(\frac{2}{3} + \frac{1}{3}I_{n-1}\right)$. Taking the limit $\lim\limits_{n\to\infty} I_n = C(1/3)$. Since the inductive definition of the set is the union of two similitudes (contractions composed with isometries), it is easily shown that the limit point is unique. It should be noted that this construction realizes the ternary Cantor set as the unique limit point of an iterated function system.

An alternative construction of the set, which is similar in nature, but distinct in composition, is as follows: Set $I_{0,1} = [0, 1]$ and define sub-intervals $I_{1,1} = [0, 1/3]$ and $I_{1,2} = [1 - 1/3, 1]$. The indices of the subintervals in $I_{k,j}$ may be read as the j^{th} component of the k^{th} level of the construction. If given intervals $I_{k-1,1}, \ldots, I_{k-1,2^{k-1}}$ we continue the above procedure by removing the middle thirds to produce intervals

$I_{k,1}, \ldots, I_{k,2^k}$, we produce 2^k intervals of length $(1/3)^k$, we may then define $C(1/3) = \bigcap_{k=0}^{\infty} \bigcup_{j=1}^{2^k} I_{k,j}$.

$C(1/3)$ is uncountable, perfect, compact (thus Borel), totally-disconnected, and of $\mathcal{L}^1$ measure zero. Uncountability derives from the fact that the ternary expansion of elements of the set only contain 0's and 2's, and contain all sequences of 0's and 2's, and thus are in 1-1 correspondence with the reals. Note that in the case of a ternary expansion ending in infinite 2's we associated this with a 1 in the previous position and infinite 0's. Compactness is from Heine-Borel, boundedness is clear since the set is a subset of $[0, 1]$, and the fact that it is closed is based on the fact that it is the complement of an open set.

$$\begin{aligned}
\mathcal{L}^1(C(1/3)) &= \mathcal{L}^1([0,1]) - \mathcal{L}^1([0,1] \setminus C(1/3)) \\
&= 1 - \sum_{k=1}^{\infty} \frac{2^k}{3^{k+1}} \\
&= 1 - \frac{1}{3} \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k \\
&= 1 - \frac{1}{3}\left(\frac{1}{1-\frac{2}{3}}\right) \\
&= 1 - \frac{1}{3}(3) = 0
\end{aligned}$$

To see that $C(1/3)$ is totally disconnected, let $a < b \in C(1/3)$, then considering the ternary expansions of both a and b we know that they contain only 0's and 2's and they differ beginning at some index. Taking their difference we can find a rational q with a finite ternary expansion containing exactly one 1, and everywhere else 0's, where $q < b - a$ and thus $a < a + q < b$. Then the ternary expansion of $a + q$ contains a 1 and thus $a + q \notin C(1/3)$. Thus a, b do not live in any interval, and thus $C(1/3)$ is totally disconnected.

Note in this calculation we are using the 1-dimensional Lebesgue measure and the fact that $C(1/3)$ is Borel, since it is compact, so we use the additivity in disjoint sets of the measure as opposed to the subadditivity of the outer-measure.

Thus $\mathcal{H}^1(C(1/3)) = \mathcal{L}^1(C(1/3)) = 0$.

Example 3.2.15 (General $C(\lambda)$ Cantor Sets)**.** To construct general $C(\lambda), 0 < \lambda < 1/2$ Cantor sets we simply replace $1/3$ in the above construction of the Ternary Cantor Set. Many properties of $C(1/3)$ are invariant under the change: $C(\lambda)$ is still uncountable, compact, and totally disconnected but its Lebesgue measure does change.

$0 < \lambda < 1/2 \Leftrightarrow \lambda = 1/\lambda_0, 2 < \lambda_0 < \infty$ thus

$$\begin{aligned}
\mathcal{H}^1(C(\lambda)) = \mathcal{L}^1(C(\lambda)) &= \mathcal{L}^1([0,1]) - \mathcal{L}^1([0,1] \setminus C(\lambda)) \\
&= 1 - \sum_{k=1}^{\infty} \frac{2^k}{\lambda_0^{k+1}} \\
&= 1 - \lambda \sum_{k=1}^{\infty} \left(\frac{2}{\lambda_0}\right)^k \\
&= 1 - \lambda \left(\frac{1}{1 - \frac{2}{\lambda_0}}\right) \\
&= 1 - \frac{\lambda}{1 - 2\lambda}
\end{aligned}$$

Later we will be able to discuss the calculation of a specific s such that $0 < \mathcal{H}^s(C(\lambda)) < \infty$, but a better framework in which to do so is that of the so-called "Hausdorff Dimension."

Example 3.2.16. The Hausdorff outer-measures are not invariant under closure: $\mathcal{H}^s(\mathbb{Q}) \neq \mathcal{H}^s(\overline{\mathbb{Q}}) = \mathcal{H}^s(\mathbb{R})$. This example is also better discussed in the context of Hausdorff dimension but it is easily explained now. By Lemma 3.2.12 we know that $\mathcal{H}^s(\mathbb{Q}) = 0$ for all $s >$, on the other hand, by the upcoming Lemma 3.2.23 we see that $\mathcal{H}^1(\mathbb{R}) = \infty$ but since $\overline{\mathbb{Q}} = \mathbb{R}$ we see that at least $\mathcal{H}^1$ is not invariant under closure. In fact the statement may be far broader: for $s > 0, \mathcal{H}^s$ is not. To see this we must understand the Hausdorff dimension.

3.2.2 Geometric Interpretation of Integral Dimension Hausdorff Measures

Lemma 3.2.17. *$\mathcal{H}^0$ is the counting measure, i.e. $\mathcal{H}^0(A) = \#A$, the cardinality of A, for all $A \subseteq X$.*

Proof.

Case 1 ($\#A < \infty$). Since $\#A < \infty$ there is a finite disjoint δ-cover $\{U_i\}$ of A such that $e_i \in U_i$ for some enumeration of A. Then $\mathcal{H}^0_\delta \overset{def}{=} \inf \sum_i d(U_i)^0 = \inf \sum_i 1 = \#\{U_i\} = \#A$ for all such δ-covers.

Case 2 (A is countably infinite). Assume A has a limit point $p \in \overline{A}$, then there exists a Cauchy sequence $\langle p_n \rangle$ such that $p_n \to p$ as $n \to \infty$ and, without loss of generality, assume $\#\{p_n\} = \infty$. Let $f : \mathbb{N} \to \mathbb{N}$ define a re-ordering of $\langle p_n \rangle$ such that $d(p_{f(n)}, p) \geq d(p_{f(n+1)}, p)$ and set $\delta_n = \frac{1}{3} d(p_{f(n)}, p_{f(n+1)})$.

Define $\mathcal{I}_{\delta_n} = \{p_{f(m)} : d(p_{f(m)}, p_{f(m+1)}) \geq \delta_n\}$. $\#\mathcal{I}_{\delta_n} \to \infty$ as $n \to \infty$, and $\mathcal{H}^0(\mathcal{I}_{\delta_n}) = \#\mathcal{I}_{\delta_n}$ by Case 1. But $\mathcal{I}_{\delta_n} \subset A$ so $\mathcal{H}^0(\mathcal{I}_{\delta_n}) \leq \mathcal{H}^0(A)$ so $\mathcal{H}^0(A) = \infty$.

If A does not have a limit point then define $\delta_0 = \frac{1}{3} \inf\{d(e_i, e_j) : i \neq j\} > 0$, then for all $\delta \leq \delta_0, \mathcal{I}_\delta = A$ and $\mathcal{H}^0(A) = \infty$.

Case 3 (A is uncountably infinite). Since A is uncountable there is a countably infinite subset $A_0 \subset A$ but $\mathcal{H}^0(A_0) = \infty$ by case 2, and by monotonicity $\mathcal{H}^0(A) = \infty$.

$\square$

We now analyze sets of greater structure curves and continua. We show that the 1-dimensional Hausdorff outer-measure has a well-understood interpretation and agrees with the natural and intuitive definition of length of a curve. Moreover, we show that Jordan curves are well behaved enough that the 1-dimensional outer-measure is actually a measure on them!

Definition 3.2.18. Let $\psi : [a, b] \subset \mathbb{R} \to \mathbb{R}^n$ be a continuous injection. We let $\Gamma = \psi([a, b])$ be the image of ψ in $\mathbb{R}^n$, and refer to Γ as a *Jordan curve.*

Lemma 3.2.19. *Any Jordan curve is a Borel set in $\mathbb{R}^n$ and thus is $\mathcal{H}^s$-measurable.*

Proof. The Borel sets are the smallest σ-algebra containing the compact sets. Since Γ is the continuous image of the compact set $[a, b]$ it too is compact in $\mathbb{R}^n$ so it is Borel. Since $\mathcal{H}^s$ is metric (as a result of the Carathéodory construction) it is also Borel (by the Carathéodory criterion) and thus it is $\mathcal{H}^s$-measurable. $\square$

One may *a priori* define the length of a Jordan curve by

$$L(\Gamma) = \sup \sum_{i=1}^{M} ||\psi(t_i) - \psi(t_{i-1})||$$

where the supremum is taken over all finite partitions of $[a, b]$ of the form

$$a = t_0 < t_1 < \cdots < t_{M-1} < t_M = b$$

.

Definition 3.2.20. A Jordan curve Γ is said to be *rectifiable* if $L(\Gamma) < \infty$. Equivalently, if ψ is of bounded variation (see Royden [HLR68, pg. 99]), then Γ is rectifiable

Example 3.2.21 (Non-rectifiable curve: The Koch Curve)**.** Since each approximation of the curve is itself a piecewise linear continuous approximation of the limit-point, if we can calculate the length of each approximation, and show they increase in n, then the limit of their lengths is a lower bound for the length of the limit-point.

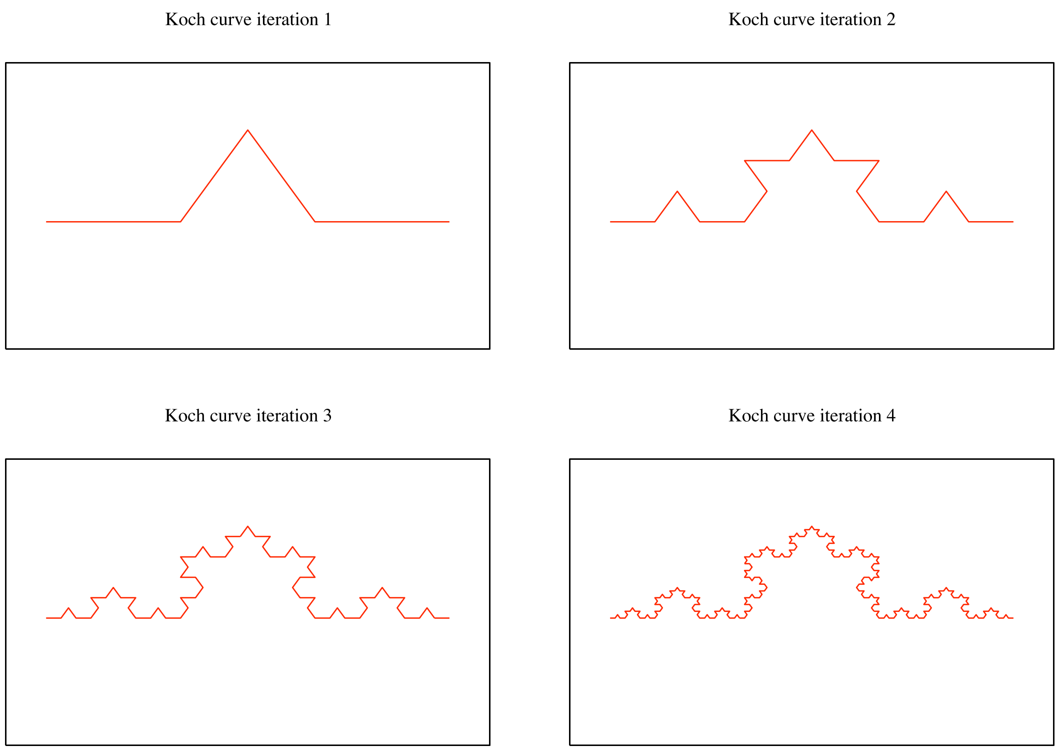

Figure 3.1: Approximations to the Koch Curve

$L(K_0) = L([0,1]) = 1$, $L(K_n) = \frac{4}{3}L(K_{n-1}) = \left(\frac{4}{3}\right)^n$. Since $\frac{4}{3} > 1$ this value diverges as $n \to \infty$ so $L(K) = \infty$.

The Koch-curve is an interesting example in more ways as well. The sequence $\langle K_n \rangle$ converges uniformly to K since $||K_n - K_{n+1}||_\infty = \frac{1}{2\sqrt{3}}\frac{1}{3^n} \to 0$ as $n \to \infty$. Another important property of the Koch curve is that each K_n possesses a tangent at all points excepting a set of measure zero (the "corners") while K_n possesses a tangent at no points.

Example 3.2.22 (Non-rectifiable curve: The Peano Curve)**.** The Peano curve is in many ways more interesting than the Koch curve, but also more regular in its makeup. As before each approximation to the limit curve is piecewise continuous,

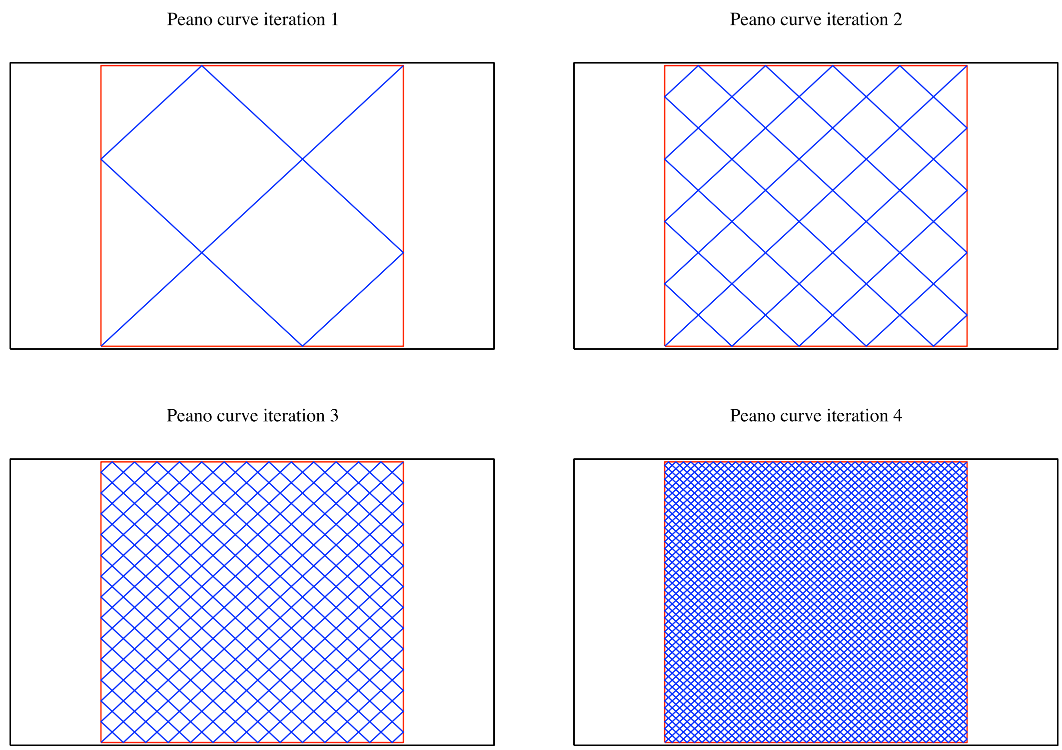

Figure 3.2: Approximations to the Peano Curve

and we will show that their length increases in n.

Consider the sequence of approximating polynomials $P_n : [0,1] \to [0,1]^2$, and their limit curve $P : [0,1] \to [0,1]^2$. $L(P_0) = \sqrt{2}, L(P_n) = \sum_{j=1}^{9^n} \frac{\sqrt{2}}{3^n} = \sqrt{2}\frac{9^n}{3^n} = 3^n\sqrt{2}$, which diverges as $n \to \infty$ so $L(P) = \infty$ and the Peano curve is non-rectifiable. Denote by P the image of the Peano curve (the limit of the P_n). The construction of the Peano curve can be found in Edgar [GAE91, pg. 64], along with some discussion of space-filling curves.

The fact which makes the Peano curve radically different from the Koch curve is that it is "space-filling", or in this case, the image of the curve is the unit squre $[0,1]^2 \subset \mathbb{R}^2$. One may define a space-filling curve as a curve whose image contains a ball $B(x, \delta)$ for some x in the image and some $\delta > 0$.

Claim. *The image of P, denoted by Γ, is dense in* $[0, 1]^2$.

Proof of Claim. We denote the image of P_i by Γ_i. Consider $[0, 1]^2$. First observe that Γ_0 is simply the diagonal of the square, the maximum distance from any point in $[0, 1]^2$ to Γ_0 is $\frac{\sqrt{2}}{2}$. More generally, given a square of side ℓ the maximum distance of any point a in that square to its diagonal is $\frac{\sqrt{2}}{2}\ell$. If one subdivides the unit square into 9^n sub-squares in the obvious way (by dividing each side into 3^n equal lengths), we notice that Γ_n, the n^{th} iteration of the Peano curve, subdivides each of these squares by the diagonal, and thus the maximum distance from any point in $[0, 1]^2$ to the n^{th} iteration of the Peano curve Γ_n is $\frac{\sqrt{2}}{2}3^{-n}$. Since $3^{-n} \to 0$ as $n \to \infty$ we see that the limit Γ is dense in the unit square. □

Claim. $\Gamma = [0, 1]^2$.

Proof of Claim. Let $p \in [0, 1]^2$. Then there exists a sequence $\langle p_n \rangle \subset \Gamma$ such that $p_n \to p$ as $n \to \infty$. Taking the pre-image of the sequence $t_n = P^{-1}(p_n)$, we generate a sequence in $[0, 1]$. Since $[0, 1]$ is compact, the sequence $\langle t_n \rangle$ contains a convergent subsequence, also denoted $\langle t_n \rangle$, such that $t_n \to t \in [0, 1]$ as $n \to \infty$. Since any subsequence of a convergent sequence also converges, and by the continuity of P, the image of $\langle t_n \rangle$ converges to $P(t) = p$ in $[0, 1]^2$. Thus $\Gamma = [0, 1]^2$. □

Since $\Gamma = [0, 1]^2$, it contains a ball around any interior point of $[0, 1]^2$, an important detail later.

The following lemma foreshadows the result described in Theorem 3.2.25 but is important for classification of $\mathcal{H}^1$ specifically.

Lemma 3.2.23. $\mathcal{H}^1(A) = \mathcal{L}^1(A)$ *for all* $A \subseteq \mathbb{R}$.

Proof. The result follows from the same reasoning as in Theorem 3.2.2.

First we observe that both the Lebesgue outer-measure (recall Example 3.1.16) and the Hausdorff outer-measure are the result of the Carathéodory construction with differing functions ($V(\cdot)$ and $d(\cdot)^1$ resp.) and sets ($\{[a,b) \subseteq \mathbb{R} : a,b \in \mathbb{R}, a < b\}$ and $\mathscr{P}(\mathbb{R})$ resp.).

In the special case of $\mathbb{R}^1$ the functions V and d agree on the family of sets used to construct the Lebesgue measure (i.e. $V([a,b)) = d([a,b))^1 = b - a$). So given a δ-cover $\{U_i\}$ of a set $A \subseteq \mathbb{R}$ we may replace each element of the cover with a closed interval of the same diameter which remains a cover (i.e. $U_i \subseteq [a_i, b_i]$ with $a_i = \inf U_i, b_i = \sup U_i$ and $d(U_i) = d([a_i,b_i]) = d([a_i,b_i))$). Thus we may restrict ourselves to half-open intervals in the construction of the Hausdorff outer-measure and $\mathcal{H}^1 = \mathcal{L}^1$. $\square$

Theorem 3.2.24. *Let $\psi : [0,\ell] \to \mathbb{R}^n$ be a rectifiable curve, and let $\Gamma = \psi([0,\ell])$ be the image of ψ then $\mathcal{H}^1(\Gamma) = L(\Gamma)$, the arc-length of Γ.*

Proof. First we let $0 = a_0, \ldots, a_N = \ell$ be a finite dissection of $[0,\ell]$, and let Γ_0 be the piecewise linear approximation of Γ defined by the dissection. Then $\Gamma_0 = \Gamma_0^{(1)} \cup \cdots \cup \Gamma_0^{(N)}$ where $\Gamma_0^{(j)}, j = 1, \ldots, N$ are the line segments connecting a_{j-1} and a_j. Let $0 \leq i < j \leq N$, we note that $\Gamma_0^{(i)} \cap \Gamma_0^{(j)} = \begin{cases} \emptyset & j - i \neq 1 \\ \{a_j\} & j - i = 1 \end{cases}$

We will show that $L(\Gamma_0) = \mathcal{H}^1(\Gamma_0)$.

Claim. $||\psi(a_j) - \psi(a_{j-1})|| = \mathcal{H}^1(\Gamma_0^{(j)})$

Proof of Claim. $||\psi(a_j) - \psi(a_{j-1})|| = ||T_j(\psi(a_j) - \psi(a_{j-1}))||$ where $T_j \in \mathcal{O}(n, \mathbb{R})$ such that $T_j(\psi(a_j) - \psi(a_{j-1})) = (\alpha_j \ 0 \ \cdots \ 0)^T \in \mathbb{R}^n$. Then we have $T_j(\Gamma_0^{(j)}) = [0, \alpha_j]$ then by Corollary 3.2.9 and Lemma 3.2.23 we have

$$
\begin{aligned}
\mathcal{H}^s(\Gamma_0^{(j)}) &= \mathcal{H}^1(T_j(\Gamma_0^{(j)})) \\
&= \mathcal{H}^1([0, \alpha_j]) \\
&= \mathcal{L}^1([0, \alpha_j]) \\
&= \alpha_j \\
&= ||T_j(\psi(a_j) - \psi(a_{j-1}))|| \\
&= ||\psi(a_j) - \psi(a_{j-1})||
\end{aligned}
$$

□

Claim. $\mathcal{H}^1(\Gamma_0) = \sum_{j=1}^{N} \mathcal{H}^1(\Gamma_0^{(j)})$

Proof of Claim. We begin by recalling that an outer-measure ν becomes a measure when restricted to a σ-algebra of ν-measurable sets. Since, as noted above, Jordan curves are Borel sets, and by Corollary 3.1.9 we have that $\mathcal{H}^1$ is a measure (and thus is additive) on Borel sets.

$$
\Gamma_0 = \bigcup_{j=1}^{N} \Gamma_0^{(j)} = \left(\bigcup_{j=1}^{N} \Gamma_0^{(j)} \setminus (\{a_{j-1}\} \cup \{a_j\}) \right) \cup \bigcup_{j=0}^{N} \{a_j\}
$$

This is essentially the fact that the $\Gamma_0^{(j)}$ are disjoint except for the endpoints, a set of measure zero. Removing them from the sum does not change the sum (by Lemma 3.2.13) so we have

$$
\mathcal{H}^1(\Gamma_0) = \mathcal{H}^1 \left(\bigcup_{j=1}^{N} \Gamma_0^{(j)} \right) = \sum_{j=1}^{N} \mathcal{H}^1(\Gamma_0^{(j)})
$$

□

Since all rectifiable curves Γ are the limit points of piecewise linear approximations Γ_0, and $\mathcal{H}^1(\Gamma_0) = L(\Gamma_0)$ on all such approximations we have $\mathcal{H}^1(\Gamma) = L(\Gamma)$ for all rectifiable curves.

$\square$

While we are only considering curves in $\mathbb{R}^n$ it has been suggested by Mattila [PM95, pg. 56] that one may choose to use $\mathcal{H}^1$ as a definition of the arc-length of the image of a Jordan curve in an appropriate metric space where the Hausdorff outer-measures have been defined.

Theorem 3.2.25. *For $n \in \mathbb{N}$, there exists a constant $c(n)$, dependent only on the dimension n, such that $\mathcal{H}^n = c(n)\mathcal{L}^n$.*

This theorem is proved, and the exact constants $c(n)$, are discussed in great detail in Evans and Gariepy [LCERFG92, pg. 65] and rely upon the isoperimetric inequality. A beautiful proof of the isoperimetric inequality based upon harmonic analysis may be found in Stein and Shakarchi [EMSRS03, pg. 103].

Since the Lebesgue measure and Hausdorff outer-measure agree to within a constant for integral dimensions we may normalize the Hausdorff measure to agree exactly with the Lebesgue measure in integral dimensions. Since outer-measures (or measures) form a vector space over $\mathbb{R}^n$ scaling by $c(n)$ is well defined. Often this normalization is ignored as the actual measure of a set is unimportant in many cases, and extremely hard to calculate in most cases. As such we are often interested in the Hausdorff dimension of a set rather than its measure.

Most importantly, the fact that higher dimensional Hausdorff and Lebesgue measures agree to within a constant that may be normalized away means that the Hausdorff measures capture "higher dimensional analogs of length" such as area, volume, etc. Thus the integral dimensional Hausdorff measures have a well-understood meaning and the non-integral dimensional Hausdorff measures may be viewed as a means of interpolating between the Lebesgue measures.

3.2.3 Generalized Hausdorff Measures

The use of the function $\zeta_s(\cdot) = d(\cdot)^s$ in the Carathéodory construction of the Hausdorff outer-measures may be made more general by considering the following class of functions. Let ϕ be Hausdorff then using $\mathcal{F} = \mathscr{P}(X)$, as in the standard Hausdorff measures, we arrive at a different measure $\psi(\mathcal{F}, \phi)$.

Definition 3.2.26. The *Generalized Hausdorff outer-measure with respect to* ϕ is the following result of the Carathéodory construction: $\mathcal{H}^\phi(A) = \lim_{\delta \downarrow 0} \inf \sum_i \phi(U_i)$.

Often the ϕ used is actually a composition of some Hausdorff $\widetilde{\phi}$ with ζ_s (i.e. $\phi = \widetilde{\phi} \circ \zeta_s$). Since ζ_s is Hausdorff by Lemma 2.1.5, ϕ is Hausdorff.

This notation is intimately related to that which will be used in the later discussion of the Packing Measure.

3.3 Spherical Outer-Measures

This section is included for reasons of completeness and provides upper and lower bounds on the s-dimensional Hausdorff outer-measure in terms of the so-called s-dimensional "Spherical-measures", which too are the result of the Carathéodory construction.

Definition 3.3.1. Let $\mathcal{F} = \{B(x_i, r) : r > 0, x_i \in X\}$, and let $\zeta_s = d(\cdot)^s$ as in the construction of the Hausdorff measures. We denote by $\mathcal{S}^s$ the result of the Carathéodory construction from ζ on $\mathcal{F}$ (i.e. $\mathcal{S}^s = \psi(\mathcal{F}, \zeta_s)$).

It should be noted that Federer [HF69, pg. 171] chooses closed balls instead of open in his definition of $\mathcal{F}$ but arrives at the same inequalities below.

Theorem 3.3.2. *For all $A \subseteq X$ and $0 < s < \infty$ the following inequalities hold:*

$$\mathcal{H}^s(A) \leq \mathcal{S}^s(A) \leq 2^s \mathcal{H}^s(A)$$

Proof. The first inequality is by Lemma 3.2.11 while the second must be shown.

Fix $A \subseteq X$. Let $\{U_i\}$ be a δ-cover of A such that $\delta_i = d(U_i)$, then $\{\overline{B}(u_i, \delta_i) : u_i \in U_i\}$ is a (2δ)-cover of A by Lemma 2.2.16.

$$\begin{aligned}
2^s \mathcal{H}^s(A) &\overset{def}{=} 2^s \liminf_{\delta \downarrow 0} \sum_i d(U_i)^s \\
&= \liminf_{\delta \downarrow 0} \sum_i (2d(U_i))^s \\
&\geq \liminf_{\delta \downarrow 0} \sum_i d(\overline{B}(u_i, \delta_i))^s \\
&= \mathcal{S}^s(A)
\end{aligned}$$

Note that the equality after bringing 2^s inside the sum makes the infimum effectively over (2δ)-covers of A, and the inequality follows by definition of infimum since covers by balls are a subset of all possible covers. Since the above is independent of A the result is proved. □

It should be noted that the above inequality is not sharp. In an article by Besicovitch [ASB28], a subset of the plane is constructed whose 1-dimensional Hausdorff outer-measure is 1 while its 1-dimensional spherical outer-measure is $2/\sqrt{3}$. The specific set is a variant on the standard Sierpinski gasket, and is also discussed in Mattila [PM95, pg. 75]. In the same article by Besicovitch it is stated that the 1-dimensional spherical and Hausdorff outer-measures agree on certain "regular" sets while on irregular sets we have

$$\mathcal{H}^1(A) \leq \mathcal{S}^1(A) \leq \frac{2}{\sqrt{3}}\mathcal{H}^1(A)$$

The specific notions of "regularity" to which Besicovitch refers are defined in terms of "densities of measures".

3.4 Net Outer-Measures

The so-called "Net outer-measures" provide "nice" bounds (in that they are in terms of the dimension of the ambient space only) on the Hausdorff outer-measures in that the covering sets are well behaved, as we see in Lemma 3.4.4. In fact, the results proven here are analogous to those proven about the spherical outer-measures in Section 3.3.

Definition 3.4.1. A *net of sets* is a family of sets $\mathcal{F}$ such that if $U, U' \in \mathcal{F}$ then $U \cap U' = \emptyset$ or $U \subseteq U'$ or $U' \subseteq U$ and each element of $\mathcal{F}$ is contained in finitely many others.

Some authors refer to "nets of sets" as "meshes", which while more intuitive, has fallen out of favor in more recent works.

Definition 3.4.2. The *dyadic cubes in* $\mathbb{R}^n$ is the family of sets

$$\mathcal{F} = \{[2^{-j}k_1, 2^{-j}(k_1+1)) \times \cdots \times [2^{-j}k_n, 2^{-j}(k_n+1)) : k_i \in \mathbb{Z}, j \in \mathbb{N}\}$$

Lemma 3.4.3. *The dyadic cubes are a net of sets.*

Proof. The dyadic cubes of side 2^{-m} for all fixed $m \in \mathbb{N}$ partition $\mathbb{R}^n$ and are pairwise disjoint by construction. Given a dyadic cube of side 2^{-m} it may be uniquely decomposed into a finite union of 2^n dyadic cubes of size $2^{-(m+1)}$, thus a given dyadic cube of side 2^{-m} is contained in $m-1$ dyadic cubes, each being a cube of size $2^{-\ell}$ for $\ell = 1, \dots, m-1$. $\square$

Lemma 3.4.4. *Given an arbitrary sub-family $\mathcal{F}' \subsetneq \mathcal{F}$ where $\mathcal{F}$ is the dyadic cubes in $\mathbb{R}^n$, there exists a pairwise disjoint subcollection $\widetilde{\mathcal{F}} \subseteq \mathcal{F}'$ such that $\bigcup_{F_i' \in \mathcal{F}'} F_i' = \bigcup_{F_i \in \widetilde{\mathcal{F}}} F_i$*

Proof. By construction dyadic cubes of the same size are either disjoint or equal. We inductively define a sub-family of sets

$$\mathcal{F}_1 = \{F' \in \mathcal{F}' : F' \text{ is of side } 2^{-1}\}$$

$$\mathcal{F}_m = \{F' \in \mathcal{F}' : F' \text{ is of side } 2^{-m}, F' \cap F_j = \emptyset \text{ for all } F_j \in \bigcup_{1 \leq j < m} \mathcal{F}_j\}$$

Each $\mathcal{F}_i$ is a pairwise disjoint collection of dyadic cubes, each of which is not contained in a larger dyadic cube in $\mathcal{F}'$. The first property is by definition of the dyadic cubes while the second is by construction. We then define $\widetilde{\mathcal{F}} = \bigcup_{i \in \mathbb{N}} \mathcal{F}_i$. $\square$

The same result is true of any net of sets but the dyadic cubes provide a tangible context in which to prove the result.

Definition 3.4.5. The *s-dimensional Net Outer-Measures,* $\mathcal{N}^s$ are the result of the Carathéodory construction with $\mathcal{F}$ being the dyadic cubes and $\zeta_s(\cdot) = d(\cdot)^s$.

Theorem 3.4.6. *For all $A \subseteq \mathbb{R}^n, n \geq 2$ the following inequalities hold:*

$$\mathcal{H}^n(A) \leq \mathcal{N}^n(A) \leq 4^n n^{n/2} \mathcal{H}^n(A)$$

Proof. As in the proof of the bounds for spherical outer-measures the first inequality is by the definition of the Hausdorff outer-measures as shown in Lemma 3.2.11.

To prove the second inequality we start as in the proof of the Spherical measures:

$$4^n n^{n/2} \mathcal{H}^n(A) = \lim_{\delta \downarrow 0} \inf \sum_i \left(4\sqrt{n} \cdot d(U_i)\right)^n \geq \lim_{\delta \downarrow 0} \inf \sum_i d(D_i)^n = \mathcal{N}^{n(A)}$$

where D_i is a δ-cover of Dyadic cubes.

The inequality above requires explanation: given a $\frac{\delta}{\sqrt{n}}$-cover of A, we may replace each element of the cover with 4^n dyadic cubes whose diameters are 2^{-k} for appropriate $k \in \mathbb{N}$, since for every $0 < \delta < 1$ there exists $k \in \mathbb{N}$ such that $2^{-k-1} < \frac{\delta}{\sqrt{n}} \leq 2^{-k}$. But, once fixed, those dyadic cubes may not satisfy the infimum over all possible covers by dyadic cubes of diameter less that $\frac{\delta}{\sqrt{n}}$, hence the inequality. $\square$

This bound is similar to the result for spherical outer-measures above (in fact, the parallel is the reason for the discussion of spherical measures). In and of themselves, neither of these families of outer-measures provides terribly more information than the Hausdorff outer-measures do, but due to their use of more tractable sets, they are useful in computation of the measures of sets. Moreover, they provide information about the "Hausdorff dimension" of a given set, as we will see shortly.

Chapter 4

Packing Outer-Measures

Our next focus, the packing outer-measures, provide natural lower-bounds for sets where the Hausdorff outer-measures provide natural upper-bounds. They are not derived from the Carathéodory construction above, but instead are constructed from an apparently "dual" construction of packings rather than coverings. In this section we follow the general construction of McClure [MM94] in his Dissertation work. As in the case of Hausdorff outer-measures there is a family of packing outer-measures parameterized by $s \in [0, \infty)$, each of which provides a meaningful measure for a restricted family of sets.

Definition 4.0.7. Let (X, d) be a metric space, $A \subseteq X$. A *centered δ-packing of A* is a collection $\{\overline{B}(a_i, \delta_i)\}$ of disjoint closed balls centered about $a_i \in A$ of radius $\delta_i \leq \delta$.

Lemma 4.0.8. *Let (X, d) be a metric space where $d(\overline{B}(x, r)) = d(B(x, r))$ and $A \subset X$. Fix $\delta > 0$. Let $\{B(a_i, \delta_i)\}$ be a collection of disjoint open balls centered about $a_i \in A$ with radius $\delta_i < \delta$. This collection of disjoint open balls is the limit point of a sequence of centered δ-packings consisting only of closed balls (as in the definition above).*

Proof. $\lim_{j\to\infty} \overline{B}(a_i, \delta_i - 1/j) = B(a_i, \delta_i)$. Note that we want $1/j < \delta_i$ which is always true in the limit. Moreover, since $\overline{B}(a_i, \delta_i - 1/j) \subsetneq B(a_i, \delta_i)$ and the collection $\{B(a_i, \delta_i)\}$ is pairwise disjoint the collection $\{\overline{B}(a_i, \delta_i - 1/j)\}$ is a centered δ-packing as in the first definition. □

This lemma shows that in a given metric space where closed balls and open balls of the same radius are of the same diameter we may use either open or closed packings for our centered δ-packings. This is the case in $\mathbb{R}^n$, so in instances where the use of open balls is advantageous we are free to consider such packings.

Definition 4.0.9. Let B be centered δ-packing of a set $A \subseteq X$. A centered δ-packing B' of A is called an *extension of* B if $B \subsetneq B'$.

Lemma 4.0.10. *Let ϕ be Hausdorff. Let $B = \{\overline{B}(a_i, \delta_i)\}$ be a centered δ-packing of $A \subseteq X$, and $B' = \{\overline{B}(a'_i, \delta'_i)\}$ be an extension of B, then $\sum_i \phi(2\delta_i) < \sum_i \phi(2\delta'_i)$.*

Proof. Let $B = \{\overline{B}(a_i, \delta_i)\}_{i\in I}$ and by definition of extension $B' = B \cup \{\overline{B}(a'_i, \delta'_i)\}_{i\in I'}$ where I, I' are countable index sets. Then

$$\sum_{i\in I} \phi(2\delta_i) < \sum_{i\in I} \phi(2\delta_i) + \sum_{i\in I'} \phi(2\delta'_i) = \sum_{i\in I\cup I'} \phi(2\delta'_i)$$

The first inequality follows from ϕ being Hausdorff (and thus positive), and the second equality is by the construction of B'. □

Definition 4.0.11. Let (X, d) be a metric space, ϕ Hausdorff. We define the *Packing pre-measure* by

$$P_\delta^\phi(A) \overset{def}{=} \sup\left\{\sum_i \phi(2\delta_i) : \{\overline{B}(a_i, \delta_i)\} \text{ a centered } \delta\text{-packing of } A\right\}$$

Note. In much of the literature we find the definitions of the Packing outer-measures P^s constructed of $\phi_s(A) = d(A)^s$ for $s \in [0, \infty)$. This has technical problems in notation so often authors choose a "radius definition" where instead of $\phi_s(B(x,\delta)) = d(B(x,\delta))^s$ we find $\phi_s(B(x,\delta)) = (2\delta)^s$ where δ is the radius of a closed ball in a given δ-packing of A. These pre-measures and associated outer-measures (defined below) are denoted by P^s and $\mathcal{P}^s$ respectively, where it is to be understood that the superscript $s \in [0,\infty)$ is associated to ϕ_s. The case $s = 0$ is used by making the assumption that $0^0 = 0$ and $t^0 = 1$ for $t \neq 0$. By choosing a more complicated function $\phi_s(2\delta_i)$ we generate more complicated outer-measures in a similar manner to the Generalized Hausdorff outer-measures above. This is the distinction made by McClure, as noted in the introduction to this section, which we follow here.

Notation. As in the Carathéodory construction we will, for brevity, write $P^\phi(A) = \sup \sum_i \phi(2\delta_i)$ where the $\{\delta_i\}$ being summed over are understood to come from a centered δ-packing $\{B(a_i,\delta_i)\}$ of A.

As in the case of the outer-measures constructed via the Carathéodory construction we now consider the limiting behavior of the approximating size δ packing pre-measures.

Definition 4.0.12. Let $A \subseteq X$ then $P^\phi(A) \overset{def}{=} \lim_{\delta\downarrow 0} P^\phi_\delta(A)$.

Lemma 4.0.13. *P^ϕ_δ is non-decreasing in δ*

Proof. Fix $A \subseteq X$. Define $S_\delta(A) = \big\{\{\overline{B}(a_i,\delta_i)\}$ a centered δ-packing of $A\big\}$ and $\widetilde{S}_\delta(A) = \left\{\sum_i \phi(2\delta_i) : \{\overline{B}(a_i,\delta_i)\} \in S_\delta(A)\right\}$.

Let $0 < \delta_0 < \delta_1$ then $S_{\delta_0}(A) \subset S_{\delta_1}(A)$ and thus $\widetilde{S}_{\delta_0}(A) \subset \widetilde{S}_{\delta_1}(A)$. We then have $P^\phi_{\delta_0}(A) = \sup \widetilde{S}_{\delta_0}(A) \leq \sup \widetilde{S}_{\delta_1}(A) = P^\phi_{\delta_1}(A)$.

Since the inequality is independent of A we see that P^ϕ_δ is non-decreasing in δ. □

Corollary 4.0.14. $P^\phi(A) = \inf_{\delta>0} P^\phi_\delta(A)$.

Proof. Proof is analogous to the proof of Lemma 3.1.5. □

It is clear that $P^\phi(\emptyset) = 0$ since there are no centered δ-packings of the empty set. To show monotonicity of P^ϕ, notice that given sets $A \subseteq A' \subseteq X$, any centered δ-packing of A is also a centered δ-packing of A', so taking the appropriate suprema we find that P^ϕ is monotonic. Though these results are promising the packing pre-measure has one major flaw, P^ϕ is not subadditive!

Lemma 4.0.15. *P^ϕ is finitely subadditive.*

Proof. Let $A, A' \subseteq X$. Every packing of $A \cup A'$ may be partitioned into two packings of A and A' repectively but not every packing of general A, A' may be derived this way (consider the case when $A \cap A'$ is non-trivial). Let $\{\overline{B}(a_i, \delta_i)\}$ be a packing of $A \cup A'$, then the partition the packing such that $\{\overline{B}(a_i', \delta_i)\}$ is a packing of A' (i.e. $a_i' \in A'$), and similarly for A. $\square$

Lemma 4.0.16. *P^ϕ is not countably subadditive.*

Proof. Proof by example: we consider the packing outer-measure of $\mathbb{N} \subset \mathbb{R}$ with respect to the Euclidean metric on $\mathbb{R}$ and claim $P^\phi(\mathbb{N}) > \sum\limits_{i=0}^{\infty} P^\phi(\{i\})$.

First we note that the packing outer-measure of a singleton is zero

$$\begin{aligned} P^\phi(\{i\}) &= \inf_{\delta>0} P^\phi_\delta(\{i\}) \\ &= \inf_{\delta>0} \sup \sum_i \phi(2\delta_i) \\ &= \inf_{\delta>0} \sup \phi(2\delta_i) \\ &= \inf_{\delta>0} \phi(2\delta) = 0 \end{aligned}$$

where the final equality is by the Hausdorff property of ϕ. Thus we have

$$\sum_{i=0}^{\infty} P^\phi(\{i\}) = 0$$

On the other hand

$$P^{\phi}(\mathbb{N}) = \inf_{\delta>0} P^{\phi}_{\delta}(\mathbb{N}) = \inf_{\delta>0} \sup \sum_{i=0}^{\infty} \phi(2\delta) = \inf_{\delta>0} \phi(2\delta) \sup \sum_{i=0}^{\infty} 1 = \infty$$

The above divergence of the packing outer-measure holds for all $\delta > 0$ so the infimum is also infinite.

Thus we have $P^{\phi}(\mathbb{N}) > \sum\limits_{i=0}^{\infty} P^{\phi}(\{i\})$. □

There are also compact examples of sets whose packing pre-measures dominate their packing outer-measure. For example $\{1/n : n \in \mathbb{N}\}$.

Definition 4.0.17. The *ϕ-induced packing outer-measure of A* is defined as

$$\mathcal{P}^{\phi}(A) = \inf \left\{ \sum_i P^{\phi}(A_i) : A = \bigcup_i A_i \right\}$$

This definition is very similar to the definition found in Halmos [PH64, pg. 42] in defining the extension of a measure to an outer-measure. Notice that this definition fixes the above problem of infinite subadditivity failing since simply taking the singleton decomposition of the naturals makes $\mathcal{P}^{\phi}(\mathbb{N}) = 0$.

Lemma 4.0.18. *Every countable set A is of measure zero with respect to the Packing outer-measures.*

Proof. Take as a cover of A the collection of singletons making up A. Their respective packing pre-measures is zero, so the sum of their pre-measures is zero, so their Packing measure is zero. □

Lemma 4.0.19. *$\mathcal{P}^0$ is the counting measure.*

Note that in this case both P^0 and $\mathcal{P}^0$ are the packing pre-measure and outer-measure associated to the function $\phi_0(\cdot) = d(\cdot)^0$ with the assumption that $0^0 = 0$.

Proof. The proof that P^0 is the counting measure is almost identical to the proof that $\mathcal{H}^0$ is the counting measure in Lemma 3.2.17. To prove that $\mathcal{P}^0$ is also the counting measure, consider a set A and any collection $\{A_i\}$ such that $A = \bigcup_i A_i$. As we will see shortly (in Lemma 4.0.21), $\mathcal{P}^0(A) \leq P^0(A)$ so we need $P^0(A) \leq \mathcal{P}^0(A)$. One need only note that $P^0(A) \leq \sum_i P^0(A_i)$ for any cover of A, thus $P^0(A) \leq \inf \sum_i P^0(A_i) = \mathcal{P}^0(A)$. Thus $\mathcal{P}^0$ is the counting measure. □

Lemma 4.0.20. *Let $A \subseteq X$. An equivalent definition of the packing outer-measure is*

$$\mathcal{P}^\phi(A) = \inf \left\{ \sum_i P^\phi(A_i) : A \subseteq \bigcup_i A_i \right\}$$

Proof. Since P^ϕ is monotonic we know that $P^\phi(A \cap A_i) \leq P^\phi(A_i)$. Let $\{A_i\}$ be such that $A \subseteq \bigcup_i A_i$ then $\sum_i P^\phi(A \cap A_i) \leq \sum_i P^\phi(A_i)$, independent of the specific cover $\{A_i\}$ chosen. Moreover every partition of the form $A = \bigcup_i A_i$ is also of the form $A \subseteq \bigcup_i A_i$, and the result follows. □

This equivalent definition is found in Hasse [HH86], where it is the only definition.

Lemma 4.0.21. *Let $A \subseteq X$. $\mathcal{P}^\phi(A) \leq P^\phi(A)$.*

Proof. Notice that in the definition of $\mathcal{P}^\phi$ we take infimum over all coverings $\{A_i\}$ of A such that $A = \bigcup_i A_i$, and as such A is a covering of itself, thus the result follows. □

Theorem 4.0.22. *$\mathcal{P}^\phi$ is an outer-measure.*

Proof.

- Since $P^\phi(\emptyset) = 0$ and $\mathcal{P}^\phi(\emptyset) \leq P^\phi(\emptyset) = 0$ we have $\mathcal{P}^\phi(\emptyset) = 0$.

- (Monotonicity) Let $A \subseteq A' \subseteq X$ then any centered δ-packing $\{U_i\}$ of A is also a centered δ-packing of A' so we have $P^\phi(A) \leq P^\phi(A')$. Moreover, every partition of A' induces a partition of A. Given a partition of $A' = \bigcup_i A'_i$ the induced partition of A is $A = \bigcup_i A_i$, where $A_i = (A'_i \cap A)$. Thus $A_i \subseteq A'_i$ and by the monotonicity of P^ϕ we have $\mathcal{P}^\phi(A) = \inf \left\{ \sum_i P^\phi(A_i) \right\} \leq \inf \left\{ \sum_i P^\phi(A'_i) \right\} = \mathcal{P}^\phi(A')$.

- (Countable sub-additivity) Let $\{A_i\} \subset \mathscr{P}(X)$. Define the following sets:

$$U = \left\{ \{U_i\} : \bigcup_i A_i = \bigcup_i U_i \right\}$$

$$V^{(i)} = \left\{ \{V_j^{(i)}\} : A_i = \bigcup_j V_j^{(i)} \right\}$$

$$W = \left\{ \bigcup_i \{V_j^{(i)}\} : \{V_j^{(i)}\} \in V^{(i)} \right\}$$

U is the set of all covers of $\bigcup\limits_i A_i$, each $V^{(i)}$ is the set of all covers of A_i, and W is the set of all covers of $\bigcup\limits_i A_i$ constructed of unions of all covers of the individual A_i as defined in $V^{(i)}$.

Claim. $W \subsetneq U$.

Proof of Claim. Let $\{W_i\} \in W$ then $\{W_i\} = \bigcup\limits_k \{V_j^{(k)}\}$ where $A_k = \bigcup\limits_j V_j^{(k)}$. $\bigcup\limits_i W_i = \bigcup\limits_j \bigcup\limits_k V_j^{(k)} = \bigcup\limits_k \bigcup\limits_j V_j^{(k)} = \bigcup\limits_k A_k$ thus $\{W_i\} \in U$ so $W \subset U$. Now to see that every cover of $\bigcup\limits_i A_i$ is not in W, simply consider $A_1, A_2 \in \mathscr{P}(X)$ such that $d(A_1, A_2) > 0$ and a partition $\{U_1, U_2\}$ of $A_1 \cup A_2$ such that $U_i \cap A_j \neq \emptyset$ for $i, j = 1, 2$, and $A_1 \cup A_2 = U_1 \cup U_2$. Then neither U_j can be in either $V^{(i)}$ for $i, j = 1, 2$. $\square$

With that small technicality the result follows.

$$
\begin{aligned}
\mathcal{P}^{\phi}(\bigcup_i A_i) &\stackrel{def}{=} \inf\left\{\sum_i P^{\phi}(U_i) : \{U_i\} \in U\right\} \\
&\leq \inf\left\{\sum_i P^{\phi}(W_i) : \{W_i\} \in W\right\} \\
&= \inf\left\{\sum_j \sum_k P^{\phi}(V_k^{(j)}) : \{V_k^{(j)}\} \in V^{(j)}\right\} \\
&= \sum_j \inf\left\{\sum_k P^{\phi}(V_k^{(j)}) : \{V_k^{(j)}\} \in V^{(j)}\right\} \\
&= \sum_j \mathcal{P}^{\phi}(A_j)
\end{aligned}
$$

The inequality follows from the fact that the infimum in the latter set is taken over a smaller set (i.e. $W \subsetneq U$). The following equality follows from the fact that since P^{ϕ} is positive re-ordering the summands does not change the value of the series. The next equality is due to the fact that the infimum is taken over covers from a fixed $V^{(j)}$ which are completely independent at each stage of the sum. The final equality is simply by the definition of the packing measure.

□

Lemma 4.0.23. *$\mathcal{P}^{\phi}$ is translationally invariant over $\mathbb{R}^n$.*

Proof. Let $A \subset \mathbb{R}^n$. If $\{\overline{B}(a_i, \delta_i)\}$ is a δ-packing of A then $\{\overline{B}(a_i + x, \delta_i)\}$ is a δ-packing of $A + x$. Since translation of a packing does not change the radii of the closed balls in the packing the result follows. □

Lemma 4.0.24. *$\mathcal{P}^{\phi}$ is metric and Borel.*

Proof. By Carathéodory's Criterion (Theorem 2.3.9) it is enough to show that $\mathcal{P}^{\phi}$ is metric. Let $A, A' \subset X$ such that $d(A, A') \geq \delta > 0$ Then for all $\frac{\delta}{3}$-packings of A (resp. A') no element of the packing intersects A' (resp. A).

Thus, given a $\frac{\delta}{3}$-packing $\{V_i\}$ of $A \cup A'$, we may partition the packing into two packings $\{U_i\}$ and $\{U_i'\}$ where $\{U_i : U_i = \overline{B}(a_i, \delta_i), a_i \in A\}$ (and similarly for A' define $\{U_i' : U_i' = \overline{B}(a_i', \delta_i', \})$. Then we have $\{V_i\} = \{U_i\} \cup \{U_i'\}$. So following the definitions we have

$$\begin{aligned}
P^{\phi}(A \cup A') &= \lim_{\delta \downarrow 0} \sup \left\{ \sum_i \phi(2\delta_i) : \{\overline{B}(a_i, \delta_i)\} \right\} \\
&= \lim_{\delta \downarrow 0} \sup \left\{ \left(\sum_i \phi(2\delta_i) \right) + \left(\sum_i \phi(2\delta_i') \right) \right\} \\
&= \lim_{\delta \downarrow 0} \sup \sum_i \phi(2\delta_i) + \lim_{\delta \downarrow 0} \sup \sum_i \phi(2\delta_i') \\
&= P^{\phi}(A) + P^{\phi}(A')
\end{aligned}$$

Thus P^{ϕ} is metric.

Let $\{A_i\}$ be a partition of $A \cup A'$, then we define $A_i^{(1)} = A_i \cap A$ and $A_i^{(2)} = A_i \cap A'$, then $A_i = A_i^{(1)} \cup A_i^{(2)}$ so $\{A_i\} = \{A_i^{(1)} \cup A_i^{(2)}\}$. Since P^{ϕ} is metric we have $P^{\phi}(A_i) = P^{\phi}(A_i^{(1)} \cup A_i^{(2)}) = P^{\phi}(A_i^{(1)}) + P^{\phi}(A_i^{(2)})$ since $d(A_i^{(1)}, A_j^{(2)}) > 0$ for all $i, j \in \mathbb{N}$.

So since every partition of A (resp. A') can be obtained from the restriction of a partition of $A \cup A'$ in the above way we have

$$\begin{aligned}
\mathcal{P}^{\phi}(A \cup A') &= \inf \left\{ \sum_i P^{\phi}(A_i) : A \cup A' = \bigcup_i A_i \right\} \\
&= \inf \left\{ \sum_i (P^{\phi}(A_i^{(1)}) + P^{\phi}(A_i^{(2)})) \right\} \\
&= \inf \left\{ \left(\sum_i P^{\phi}(A_i^{(1)}) \right) + \left(\sum_i P^{\phi}(A_i^{(2)}) \right) \right\} \\
&= \inf \left\{ \sum_i P^{\phi}(A_i^{(1)}) \right\} + \inf \left\{ \sum_i P^{\phi}(A_i^{(2)}) \right\} \\
&= \mathcal{P}^{\phi}(A) + \mathcal{P}^{\phi}(A')
\end{aligned}$$

So $\mathcal{P}^\phi$ is metric, and thus by Carathéodory's Criterion $\mathcal{P}^\phi$ is Borel. $\square$

We now roughly follow the opening notes of Edgar [GAE94] and the properties listed in Hasse [HH86] and discuss in detail certain properties of the Packing outer-measure (and pre-measures). The difference being that Edgar only explicitly discusses $\mathcal{P}^s$ instead of $\mathcal{P}^\phi$, where as Hasse discusses the topic in generality using different notation.

Lemma 4.0.25. *Finite packings suffice in the following sense:*

$$P^\phi_\delta(A) = \sup_{M\in\mathbb{N}} \left\{ \sum_{i=1}^{M} \phi(2\delta_i) : \{\overline{B}(a_i,\delta_i)\}_{i=1}^M \textit{ a finite } \delta\textit{-packing of } A \right\}$$

One must be careful to interpret this lemma correctly. It does not say that given a set A there exists a finite δ-packing $B = \{\overline{B}(a_i,\delta_i)\}_{i=1}^M$ of A such that $P^\phi_\delta(A) = \sum_{i=1}^{M} \phi(2\delta_i)$ for some fixed finite M. Instead it means that if $P^\phi_\delta(A)$ is finite then one may arbitrarily estimate its value by a finite sum.

Proof. If A does not admit an infinite packing (for example $\#A < \infty$) then the lemma is clear so we assume A admits an infinite packing. If A admits a countably infinite centered δ-packing $B = \{\overline{B}(a_i,\delta_i)\}$ then it may be enumerated. Then since $\lim_{j\to\infty} \sum_{i=1}^{j} \phi(2\delta_i) = \sum_{i=1}^{\infty} \phi(2\delta_i)$ we have $\sup_j \sum_{i=1}^{j} \phi(2\delta_i) = \sum_{i=1}^{\infty} \phi(2\delta_i)$ since the finite sum strictly increases in j. So we have

$$\sup_{M\in\mathbb{N}} \left\{ \sum_{i=1}^{M} \phi(2\delta_i) : \{\overline{B}(a_i,\delta_i)\}_{i=1}^M \text{ a finite } \delta\text{-packing of } A \right\} = \sup \sum_i \phi(2\delta_i)$$

And the result is proved. $\square$

Lemma 4.0.26. *Let $A \subseteq \mathbb{R}^n$, then $\mathcal{P}^\phi(A) = \mathcal{P}^\phi(\overline{A})$.*

Proof. Let $\{B(a_i,\delta_i)\}$ be a centered δ-packing of $\overline{A}$. If $a_i \in \overline{A} \setminus A$ then there exists a sequence $\langle a^{(j)} \rangle$ such that $a^{(j)} \to a_i$ as $j \to \infty$ by definition of closure. Thus we may replace $B(a_i, \delta_i)$ in the packing with $B(a^{(j)}, \delta^{(j)})$ such that for any ϵ, $d(a_i, a^{(j)}) < \epsilon$ and thus $\phi(2\delta_i) - \phi(2\delta^{(j)}) < \epsilon$ since ϕ is Hausdorff.

So any δ-packing of $\overline{A}$ may be arbitrarily approximated by a δ-packing of A, thus $P^\phi(A) = P^\phi(\overline{A})$. Since every collection $\{A_i\}$ such that $A \subseteq \bigcup_i$ may be seen as being induced by a similar cover of $\overline{A}$ the result holds for $\mathcal{P}^\phi$. $\square$

One should note that we use the continuity of ϕ to guarantee that

$$\phi(2\delta_i) - \phi(2\delta^{(j)}) < \epsilon$$

in the above proof!

Corollary 4.0.27. *Let $A \subseteq \mathbb{R}^n$. Since $P^\phi(A) = P^\phi(\overline{A})$ we may take the definition of $\mathcal{P}^\phi$ as*

$$\mathcal{P}^\phi(A) = \inf\left\{\sum_i P^\phi(A_i) : A \subseteq \bigcup_i A_i, A_i \text{ Borel}\right\}$$

Corollary 4.0.28. *$\mathcal{P}^\phi$ is Borel-regular in $\mathbb{R}^n$.*

Proof. Since given $A \subseteq X$, by definition $\overline{A}$ is closed, and thus Borel, by the last lemma $\mathcal{P}^\phi$ is Borel-regular. $\square$

One should note that our decision to limit ourselves to $\mathbb{R}^n$ is motivated by the opening notes in Joyce's paper describing a space on which a certain class of packing measures is not Borel-regular [HJ99]. Moreover his opening prose discuss a number of issues which may arise in non-Euclidean spaces or with weak definitions of our function ϕ.

4.1 A "Dual" Construction?

If one were able to generate a family of packing outer-measures in much the same way we generate geometric outer-measures using the Carathéodory construction, there would be a satisfying symmetry in the theory. Let us pursue such a construction and what would be necessary for it to be valid.

In the Carathéodory construction one requires a family of sets $\mathcal{F} \subseteq \mathscr{P}(\mathbb{R}^n)$ (we will restrict ourselves in this discussion to $\mathbb{R}^n$), and a function $\zeta : \mathcal{F} \to [0, \infty)$ with appropriate properties. In the explicit construction of the Packing outer-measures one uses packings of balls centered about points in a given set. We consider packings of a set $A \subseteq \mathbb{R}^n$. We may set $\mathcal{F} = \{\overline{B}(a_i, \delta), \delta \in (0, \infty))\}$ but this set is dependent upon the set being packed so instead we denote it $\mathcal{F}_A$. If instead we replace $\mathcal{F}_A$ with $\mathcal{F} = \{\overline{B}(x, \delta) : x \in \mathbb{R}^n, \delta \in (0, \infty)\}$, the set of all centered δ-packings of $\mathbb{R}^n$, this set is not dependent upon that being packed but it is fairly unmanageable and $\mathcal{F}_A \subseteq \mathcal{F}$. Taking motivation from the separability of the space one may wish to consider packings from the sets $\mathcal{F}_{\mathbb{Q}} = \{\overline{B}(x, \delta), x \in \mathbb{Q}^n, \delta \in (0, \infty)\}$ or even $\mathcal{F}_2 = \{\overline{B}(x, \delta) : x \in \mathbb{Q}^n, x = a/2^n, a \in \mathbb{Z}, \delta \in (0, \infty)\}$ since both $\mathbb{Q}^n$ and the dyadic rationals, $\{x \in \mathbb{Q}^n, x = a/2^n, a \in \mathbb{Z}\}$, are countable and dense in $\mathbb{R}^n$. This yields little since the constructed packing using either family of sets measures $\mathbb{R}^n \setminus \mathbb{Q}^n$ as zero. Moreover, we can take any set, remove the rational points, and the resulting set has a maximal packing of size zero!

One may also want to generate basic bounded relations using easier to calculate packings similarly to what was done with the net outer-measures above. To do so one needs to consider packings by some net of sets. The dyadic cubes seem a reasonable candidate if we consider "centered dyadic cubes", since each dyadic cube has a center point that is also a dyadic rational. We immediately run into the problem above that not every set contains a dyadic rational (or even a rational at all)!

Thus we find that there is no clear construction, “dual” to the Carathéodory construction, for packing outer-measures in $\mathbb{R}^n$, or any other metric space.

Chapter 5

Dimension Theory

5.1 Hausdorff Dimension

The following theorem has many equivalent statements:

Theorem 5.1.1. *Let* $s, t \in \mathbb{R}$ *such that* $0 \leq s < t < \infty$ *then for any* $A \subseteq X$ *we have* $\mathcal{H}^s(A) < \infty$ *implies* $\mathcal{H}^t(A) = 0$*. Equivalently* $\mathcal{H}^t(A) > 0$ *implies* $\mathcal{H}^s(A) = \infty$*.*

Proof. The following is the standard proof as found in Falconer [KJF85, pg. 7], Mattila [PM95, Theorem 4.7, pg. 58], and Evans and Gariepy [LCERFG92, Lemma 2, pg. 65]:

Let $0 < s < t$, then we have $\mathcal{H}^s_\delta(A) \geq \mathcal{H}^t_\delta(A) \geq \delta^{s-t}\mathcal{H}^t_\delta(A)$ where the first inequality is by by Lemma 3.2.10 and the final inequality is from the assumption that $0 < \delta < 1$ and thus $0 < \delta^{s-t} < 1$.

So we have $\mathcal{H}^s(A) \overset{def}{=} \lim\limits_{\delta \downarrow 0} \mathcal{H}^s_\delta(A) \geq \lim\limits_{\delta \downarrow 0} \frac{1}{\delta^{t-s}} \mathcal{H}^t_\delta(A) = \begin{cases} 0 & \text{if } \mathcal{H}^t_\delta(A) = 0 \\ \infty & \text{if } \mathcal{H}^t_\delta(A) > 0 \end{cases}$

□

This theorem may be re-phrased as follows:

Theorem 5.1.2. *Let $A \subset X$. There exists a unique $s \in [0, \infty)$ such that $\mathcal{H}^t(A) = \infty$ if $t < s$ and $\mathcal{H}^t(A) = 0$ if $t > s$.*

Definition 5.1.3. The *Hausdorff Dimension* (or *Hausdorff-Besicovitch Dimension*) of a set A is the unique $s \in [0, \infty)$ such that

$$\mathcal{H}^t(A) = \begin{cases} \infty & \text{for all } 0 \leq t < s \\ 0 & \text{for all } t > s \end{cases}$$

Notation. We denote the Hausdorff Dimension of a set A by $\dim_{\mathcal{H}}(A)$.

Corollary 5.1.4. *If $A \subseteq X$ and $\dim_{\mathcal{H}}(A) = s$ then $\mathcal{H}^t(A) = 0$ for all $t > s$, and $\mathcal{H}^t(A) = \infty$ for $t < s$.*

Notice that this is merely a re-statement of Theorem 5.1.1 in terms of "Hausdorff dimension".

Lemma 5.1.5. *Let $A \subseteq X$, then the following equalities are true:*

$$\begin{aligned} \dim_{\mathcal{H}}(A) &= \sup\{s : \mathcal{H}^s(A) > 0\} = \sup\{s : \mathcal{H}^s(A) = \infty\} \\ &= \inf\{s : \mathcal{H}^s(A) < \infty\} = \inf\{s : \mathcal{H}^s(A) = 0\} \end{aligned}$$

Proof. Since $\dim_{\mathcal{H}}(A) = s$ is the unique number such that $\mathcal{H}^t(A) = 0$ for all $t > s$ and $\mathcal{H}^t(A) = \infty$ for all $t \leq s$ all of the equalities hold by definition. □

It should be noted that if $\dim_{\mathcal{H}}(A) = t$ then the specific value of $\mathcal{H}^t(A)$ may be zero, finite, or infinite.

Lemma 5.1.6. $\dim_{\mathcal{H}}$ *is monotonic i.e. if $A \subseteq A' \subseteq X$ then $\dim_{\mathcal{H}}(A) \leq \dim_{\mathcal{H}}(A')$.*

Proof. If $\dim_{\mathcal{H}}(A) = s$ then we have $\mathcal{H}^s(A) \leq \mathcal{H}^s(A')$ by monotonicity of $\mathcal{H}^s$. If $\mathcal{H}^s(A) = \infty$ then $\mathcal{H}^s(A') = \infty$ and thus $\dim_{\mathcal{H}}(A') \geq s$. If, on the other hand, $\mathcal{H}^s(A) = \alpha < \infty$, then $\mathcal{H}^s(A')$ is either finite or infinite. If it is infinite then we are in the previous case, if it is finite then $\dim_{\mathcal{H}}(A') = s$ by definition of Hausdorff dimension.

So $\dim_{\mathcal{H}}$ is monotonic. ☐

Theorem 5.1.7. *Let $A \subseteq \mathbb{R}^n$, Consider $L : A \to \mathbb{R}^m$ Lipschitz with Lipschitz constant c, then* $\dim_{\mathcal{H}}(L(A)) \leq \dim_{\mathcal{H}}(A)$.

Proof. Let $s = \dim_{\mathcal{H}}(A)$. Recalling Theorem 3.2.4 we know that $\mathcal{H}^s(L(A)) \leq c^s\mathcal{H}^s(A)$ and the result is immediate by the definition of $\dim_{\mathcal{H}}$. ☐

And as we saw in Example 3.2.7, strict inequality of $\dim_{\mathcal{H}}$ under Lipschitz maps is possible.

Corollary 5.1.8. *$\mathcal{H}^s$ is not Radon.*

Proof. Let $A \subset X$ be a compact set such that $\dim_{\mathcal{H}} = t$, then if $s < t$ then $\mathcal{H}^s(A) = \infty$ so $\mathcal{H}^s$ is not locally finite. ☐

While the definition of Hausdorff dimension and its corollaries provide powerful machinery, in the theory an explicit calculation is not amiss.

Lemma 5.1.9. *Let A be countable then* $\dim_{\mathcal{H}}(A) = 0$.

Proof. We know that $\mathcal{H}^s(A) = 0$ for all $s > 0$ by Lemma 3.2.12 and that $\mathcal{H}^0$ is the counting measure by Lemma 3.2.17, so $\mathcal{H}^0(A) > 0$, thus zero satisfies the definition of Hausdorff dimension for A and $\dim_{\mathcal{H}}(A) = 0$. ☐

Lemma 5.1.10. *For* $s > n > 0, \mathcal{H}^n(\mathbb{R}^n) = \infty, \mathcal{H}^s(\mathbb{R}^n) = 0$. *Equivalently* $\dim_{\mathcal{H}}(\mathbb{R}^n) = n$.

Proof. This proof is by Falconer [KJF85, pg. 8]. Consider the n-dimensional cube $C = [0,1]^n \subset \mathbb{R}^n$. Let $\delta > 0$, then there exists a $k \in \mathbb{N}$ such that $k^{-1}n^{1/2} < \delta$. We may divide C into k^n subcubes of size k^{-1} in the obvious way. Then $\mathcal{H}^n(C) \leq k^n(k^{-1}n^{1/2})^n = n^{n/2} < \infty$. Thus, since $s > n$ and by Theorem 5.1.1 $\mathcal{H}^s(C) = 0$, and since $\mathbb{R}^n$ is a countable union of translated cubes and by the monotonicity of $\mathcal{H}^s$ we have $\mathcal{H}^s(\mathbb{R}^n) = 0$ and $\mathcal{H}^n(\mathbb{R}^n) = \infty$. □

Corollary 5.1.11. $\mathcal{H}^s$ *is* σ*-finite.*

Lemma 5.1.12. *Let* $A = B(x, r)$ *be the open ball of radius* r *about* $x \in \mathbb{R}^n$*, then* $\dim_{\mathcal{H}}(A) = n$.

Proof. By Lemma 3.2.25 we know that $0 < \mathcal{H}^n(A) = c(n)\mathcal{L}^n(A) < \infty$ since A is bounded, thus $\mathcal{L}^n(A) < \infty$, and $c(n)$ is finite. So $\dim_{\mathcal{H}}(A) = n$. □

Corollary 5.1.13. *Let* $A \subseteq \mathbb{R}^n$ *such that for some* $a_0 \in A$ *and some* $r > 0$ *the ball* $B(a_0, r) \subseteq A$ *then* $\dim_{\mathcal{H}}(A) = n$

Proof. By Lemma 5.1.12 we know $\dim_{\mathcal{H}}(B(a_0, r)) = n$ and by Lemmas 5.1.6 and 5.1.10 we know that

$$n = \dim_{\mathcal{H}}(B(a_0, r)) \leq \dim_{\mathcal{H}}(A) \leq \dim_{\mathcal{H}}(\mathbb{R}^n) = n$$

as was to be shown. □

Corollary 5.1.14. *Any space-filling curve in* $\mathbb{R}^n$ *is of Hausdorff dimension* n.

Proof. By definition a space-filling curve contains a ball, and by Lemma 5.1.13 we are done. □

Corollary 5.1.15. *Let $A \subseteq \mathbb{R}$, if $0 < \dim_{\mathcal{H}}(A) < 1$ then A is totally disconnected.*

Proof. If A contains an interval (a connected non-singleton subset) then $\dim_{\mathcal{H}}(A) = 1$ by Corollary 5.1.13, so A does not possess any non-singleton connected components, and thus A is totally disconnected. $\square$

An interesting feature of the Hausdorff dimension (and subsequently the Hausdorff outer-measures indexed by s) is that it partitions the set of Hausdorff-measurable sets. These sets are precisely those which meet the definition of Carathéodory measurability for some $\mathcal{H}^s$. The partition of these sets is trivial in that two sets are in the same class in the partition if they are of the same Hausdorff dimension. There are sets which are not Hausdorff measurable so those sets are excluded in the partition, and those sets which remain lie in the partition nicely.

Lemma 5.1.16. *Let $0 < \lambda < 1/2$ then $\dim_{\mathcal{H}}(C(\lambda)) = \log(2)/\log(1/\lambda)$ where $C(\lambda)$ is the generalized Cantor set defined in Example 3.2.14.*

Proof. This proof is also by Falconer [KJF85, pg. 14], though all details of the general result for $C(\lambda)$ are shown where as Falconer only shows the result for $C(1/3)$ and provides a hint as to the general result.[1] This proof is similar to a proof in Mattila but provides a sharp inequality which thus computes that actual measure of the Cantor set with respect to the s-dimensional Hausdorff outer-measure (for s defined below).

By the nature of the Hausdorff outer-measures it is generally easier to bound the measure of a set from above than below so we begin there. Set $s = \log(2)/\log(1/\lambda)$.

[1] For the careful reader it should be noted that in the 2002 paperback printing of Falconer's book there is a typo on pg. 15. The author provides a set of inequalities similar to those used here but the second equality shown should be greater-than-or-equal and the last inequality should be equality.

Recalling the construction of $C(\lambda)$, we begin with $I_{0,1} = [0,1]$ then define the sub-intervals $I_{j,k}$ and $C(\lambda) = \bigcap_{k=0}^{\infty} \bigcup_{j=1}^{2^k} I_{k,j}$. Now we notice that every finite intersection $C_M(\lambda) = \bigcap_{k=0}^{M} \bigcup_{j=1}^{2^k} I_{k,j}$ is a finite cover of $C(\lambda)$ (containing 2^k intervals) and $I_{k,j}$ is an interval of length λ^k so, given δ, for sufficiently large M, $C_M(\lambda)$ is a δ-cover of $C(\lambda)$. Then, rewriting $\lambda^k = \lambda_0^{-k}, 2 < \lambda_0 < \infty$, we have

$$\mathcal{H}^s_{\lambda_0^{-k}}(C(\lambda)) \leq 2^k \lambda_0^{-sk} = 2^k 2^{-k} = 1$$

since $\lambda_0^s = \lambda_0^{\log(2)/\log(\lambda_0)} = 2$ by the properties of the logarithm. So s is an upper-bound on the Hausdorff dimension of $C(\lambda)$ since $\mathcal{H}^s(C(\lambda)) < \infty$.

To show that $\mathcal{H}^s(C(\lambda)) > 0$, and thus s is the Hausdorff dimension of $C(\lambda)$, we need to show that $1 \leq \sum_i d(I_i)^s$ for any cover of $C(\lambda)$ by intervals $\{I_i\}$.

We begin by noting that since $C(\lambda)$ is compact, for any cover of open intervals there is a finite sub-cover $\{I_j\}$ consisting of open intervals. We then take the closure of each $\{I_j\}$ and reduce each interval to make each one the smallest possible interval containing all of the $I_{j,k}$ used in the construction of $C(\lambda)$.

Now let $I_{j,k}, I_{\ell,m}$ be two disjoint intervals from the construction of $C(\lambda)$ (Falconer refers to these as "net intervals") contained in I_i (potentially from different iterations of the construction), then

$$d(I_i) > d(I_{j,k}) + d(U_i) + d(I_{\ell,m})$$

where U_i is an open interval in the complement of $I_{j,k} \cup I_{\ell,m}$ contained in I_i. Then if

$$d(I_{j,k}) + d(I_{\ell,m}) \leq 2d(U_i)\frac{\lambda}{1-2\lambda}$$

where $1 - 2\lambda$ is the length removed from the unit interval in the first iteration of the construction, we get the following string of inequalities:

$$
\begin{aligned}
d(I_i)^s &= (d(I_{j,k}) + d(U_i) + d(I_{\ell,m}))^s \\
&\geq \left(\frac{1}{2\lambda}(d(I_{j,k}) + d(I_{\ell,m}))\right)^s \\
&= \left(\frac{1}{\lambda}\right)^s \left(\frac{d(I_{j,k}) + d(I_{\ell,m})}{2}\right)^s \\
&= 2\left(\frac{d(I_{j,k}) + d(I_{\ell,m})}{2}\right)^s \\
&\geq 2\left(\frac{1}{2}d(I_{j,k})^s + \frac{1}{2}d(I_{\ell,m})^s\right) \\
&= d(I_{j,k})^s + d(I_{\ell,m})^s
\end{aligned}
$$

The first inequality is by the condition stated before the equations and the second is by the concavity of $f(t) = t^s$ (since a function f is concave down if and only if $f(\frac{x+y}{2}) \geq \frac{f(x)+f(y)}{2}$). Thus we may replace I_i with the two sub-intervals $I_{j,k}$ and $I_{\ell,m}$ in the above sum: $\sum\limits_j d(I_j)^s = \sum\limits_j (d(I_{j,k})^s + d(I_{\ell,m})^s)$. Each time we do this we increase the number of intervals covering $C(\lambda)$ by two but the covering set remains finite. We may repeat this process until each element of this finite covering is of length λ^j which then must cover all of the $\{I_{j,\ell}\}_\ell$ thus the sum above must be greater than $2^j(\frac{1}{\lambda^j})^s = 1$.

Thus $\dim_{\mathcal{H}}(C(\lambda)) = s = \log(2)/\log(1/\lambda)$. □

Corollary 5.1.17. *Let $0 < \lambda < 1/2$, $C(\lambda)$ be the associated Cantor set, and $s = \log(2)/\log(1/\lambda)$, then $\mathcal{H}^s(C(\lambda) = 1$.*

Proof. In the calculation of the Hausdorff dimension of $C(\lambda)$ we bound $\mathcal{H}^s(C(\lambda))$ from both above and below by 1. □

Corollary 5.1.18. *For every $s \in [0,1]$ there exists a set $A_s \subseteq [0,1]$ such that $\dim_{\mathcal{H}}(A_s) = s$.*

Proof. We begin with the two boundary conditions $s = 0, 1$: in the case $s = 0$ simply take A_0 to be any finite or countable subset of $[0, 1]$, and $\mathcal{H}^0$ is the counting measure, so by combining Lemma 3.2.12 and the definition of Hausdorff outer-measures we find that $\dim_{\mathcal{H}}(A_0) = 0$. On the other hand if $s = 1$ then let $A_1 = [0, 1]$, since $\mathcal{H}^1([0, 1]) = \mathcal{L}^1([0, 1]) = 1$ we see that by definition of the Hausdorff outer-measures $\dim_{\mathcal{H}}([0, 1]) = 1$ and we are done.

We now show that the map $\lambda \mapsto \log(2)/\log(1/\lambda)$ is onto $(0, 1)$. We know that $\lim_{\lambda\to 0} \log(\lambda) = -\infty$ so $\lim_{\lambda\to 0} \log(2)/\log(1/\lambda) = \lim_{\lambda\to 0} \log(2)/(-\log(\lambda)) = 0$. On the other hand, $\lim_{\lambda\to 1/2} \log(2)/\log(1/\lambda) = 1$ by the continuity of log and by the intermediate value theorem the function takes on all values between the two limits. In fact, the map is bijective since the derivative of log is strictly positive on $(0, \infty)$.

Since $\dim_{\mathcal{H}}(C(\lambda)) = \log(2)/\log(1/\lambda)$ for $\lambda \in (0, 1/2)$ the above argument shows that there is a Cantor set $C(\lambda)$ realizing every Hausdorff dimension in $(0, 1)$. Moreover, the bijectivity of above map shows that there is a unique Cantor set realizing each dimension. $\square$

Theorem 5.1.19. *Let $A, B \subseteq X$ then $\dim_{\mathcal{H}}(A \cup B) = \max\{\dim_{\mathcal{H}}(A), \dim_{\mathcal{H}}(B)\}$ and $\dim_{\mathcal{H}}(A \cap B) \leq \min\{\dim_{\mathcal{H}}(A), \dim_{\mathcal{H}}(B)\}$.*

Proof. First we set $\dim_{\mathcal{H}}(A) = s$ and $\dim_{\mathcal{H}}(B) = t$, and assume without loss of generality that $s < t$. Then $\mathcal{H}^t(A) = 0$ by definition of Hausdorff dimension, and $\mathcal{H}^t(B) = \beta \in (0, \infty)$ so we have

$$\beta = \mathcal{H}^t(B) \leq \mathcal{H}^t(A \cup B) \leq \mathcal{H}^t(A) + \mathcal{H}^t(B) = \mathcal{H}^t(B) = \beta$$

The first inequality is by monotonicity of outer-measures and the second is by subadditivity. So

$$\dim_{\mathcal{H}}(A \cup B) = t = \max\{\dim_{\mathcal{H}}(A), \dim_{\mathcal{H}}(B)\} = \max\{s, t\}$$

Similarly to the first half of the proof we have $\mathcal{H}^s(A) = \alpha$ and $\mathcal{H}^s(B) = \infty$. Moreover by definition $A \cap B \subseteq A$ so

$$0 \leq \mathcal{H}^s(A \cap B) \leq \mathcal{H}^s(A) = \alpha$$

where the first inequality is by definition and the second is by monotonicity of outer-measures. Thus we have

$$\dim_{\mathcal{H}}(A \cap B) \leq s = \min\{\dim_{\mathcal{H}}(A), \dim_{\mathcal{H}}(B)\} = \min\{s, t\}$$

□

Corollary 5.1.20. *Let* $A = \{A_i\} \subset \mathscr{P}(X)$ *be countable, then* $\dim_{\mathcal{H}}(\bigcup_i A_i) = \sup_i \dim_{\mathcal{H}}(A_i)$. *Moreover if* $\dim_{\mathcal{H}}(X) < \infty$ *then* $\sup_i \dim_{\mathcal{H}}(A_i) < \dim_{\mathcal{H}}(X)$. *Finally* $0 \leq \dim_{\mathcal{H}}(\bigcap_i A_i) \leq \inf_i \dim_{\mathcal{H}}(A_i)$.

Proof. By Theorem 5.1.19 that $\dim_{\mathcal{H}}(\bigcup_i A_i) = \sup_i \dim_{\mathcal{H}}(A_i)$ is clear since $\dim_{\mathcal{H}}$ can only increase under union. Any by monotonicity of $\dim_{\mathcal{H}}$ that $\sup_i \dim_{\mathcal{H}}(A_i) < \dim_{\mathcal{H}}(X)$ is clear. Actually the moreover condition in the statement holds if $\dim_{\mathcal{H}}(X) = \infty$ but the conclusion is then automatic and vacuous.

Finally, since $\dim_{\mathcal{H}}(A) \geq 0$ for all $A \subseteq X$, and since $\dim_{\mathcal{H}}$ decreases under intersections, we have $0 \leq \dim_{\mathcal{H}}(\bigcap_i A_i) \leq \inf_i \dim_{\mathcal{H}}(A_i)$, so the result is proved. □

Theorem 5.1.21. *There exists Borel sets* $A, B \subset \mathbb{R}$ *of Hausdorff dimension 0 such that* $\dim_{\mathcal{H}}(A \times B) > 0$.

For a proof please see Falconer [KJF85, pg. 73, Thm. 5.11]. The proof is constructive and interesting but outside of the scope of this work.

The primary motivation for including the Spherical and Net outer-measures is that they are "comparable measures", meaning that they are bounded above and below by the Hausdorff measures, and thus may be used to analyze the Hausdorff dimension of a given set with respect to more tractable families of sets.

5.2 Packing Dimension

In this section we choose the specific family of packing outer-measures $\mathcal{P}^s$ indexed by a real parameter $s \in [0, \infty)$ as discussed earlier.

Lemma 5.2.1. *$P^s, \mathcal{P}^s$ are non-increasing in s.*

Proof. Let $0 \leq s < t < \infty$, $A \subseteq X$. First we prove the lemma for the packing pre-measures.

Since the set of δ-packings of the set A does not vary with s, and thus the pre-measure being applied to the set we have

$$P^s_\delta(A) \overset{def}{=} \sup \sum_i (2\delta_i)^s \geq \sup \sum_i (2\delta_i)^t \overset{def}{=} P^t_\delta(A)$$

But this inequality is independent of δ so we have

$$P^s(A) \overset{def}{=} \lim_{\delta \downarrow 0} P^s_\delta(A) \geq \lim_{\delta \downarrow 0} P^t_\delta(A) \overset{def}{=} P^t(A)$$

So P^s is non-increasing in s. It follows that we have

$$\mathcal{P}^s(A) \overset{def}{=} \inf \left\{ \sum_i P^s(A_i) : A = \bigcup_i A_i \right\} \geq \inf \left\{ \sum_i P^t(A_i) : A = \bigcup_i A_i \right\} \overset{def}{=} \mathcal{P}^t(A)$$

The inequality above is because the covers of A are shared between the two infimums and P^s is non-increasing in s so $\mathcal{P}^s$ is non-increasing in s. $\square$

Just as in the case of the Hausdorff dimension there exists a critical value of s at which the family of packing outer-measures provides meaningful information about a given set.

Theorem 5.2.2. *Let $s, t \in \mathbb{R}$ such that $0 \leq s < t < \infty$ then for any $A \subseteq X$ we have $P^s(A) < \infty$ implies $P^t(A) = 0$. Equivalently, if $P^t(A) > 0$ implies $P^s(A) = \infty$.*

Proof. This proof is nearly identical to the makeup of the proof of Theorem 5.1.1, which is an analogous statement about the family of Hausdorff measures.

Let $0 < s < t$, then by Lemma 5.2.1 we have $P^s_\delta(A) \geq P^t_\delta(A) \geq \delta^{s-t} P^t_\delta(A)$ where the final inequality follows from the assumption that $0 < \delta < 1$ and thus $0 < \delta^{s-t} < 1$.

So we have $P^s(A) \stackrel{def}{=} \lim_{\delta \downarrow 0} P^s_\delta(A) \geq \lim_{\delta \downarrow 0} \frac{1}{\delta^{t-s}} P^t_\delta(A) = \begin{cases} 0 & \text{if } P^t_\delta(A) = 0 \\ \infty & \text{if } P^t_\delta(A) > 0 \end{cases}$

□

Definition 5.2.3. Let $A \subseteq X$. The unique s defined above is known as the *Packing Index of* A which we will denote $\mathrm{Ind}_P(A)$.

The packing index of a set is a stepping stone to defining the "Packing Dimension" of a set, our next goal:

Theorem 5.2.4. *Let* $A \subseteq X$. *There exists a unique* $s \in [0, \infty)$ *such that* $\mathcal{P}^t(A) = \infty$ *for* $t < s$ *and* $\mathcal{P}^t(A) = 0$ *for* $t > s$.

Proof. Let $A \subseteq X$, and let $\{A_i\}$ be any cover of A.

Define the map $\{A_i\} \mapsto \alpha(\{A_i\}) = \sup_i \{\mathrm{Ind}_P(A_i)\}$.

Set $s = \inf\{\alpha(\{A_i\})\}$ where the infimum is taken over all covers of A. We claim that if $t > s$ then $\mathcal{P}^t(A) = 0$ and if $t < s$ then $\mathcal{P}^t(A) = \infty$.

Let $t < s$. If $\{A_i\}$ is any cover of A then for any $A' \in \{A_i\}$ we have $\mathrm{Ind}_P(A') > t$, since $t < s$, so $P^t(A') = \infty$ and thus $\sum_i P^t(A_i) = \infty$. Since this is true of any cover of A we have that if $t < s$ then $\mathcal{P}^t(A) = \infty$.

Let $t > s$. There exists a cover $\{A_i\}$ of a A such that $s < \alpha(\{A_i\}) < t$. This is true because either the infimum is a minimum (i.e. the infimum is attained by some cover) or the infimum is a limit of a sequence of covers. So $P^t(A_i) = 0$ for all i, thus $\sum_i P^t(A_i) = 0$ so the infimum in the definition of the packing outer-measures is also zero. □

Definition 5.2.5. The *Packing dimension* of a set A is the unique $s \in [0, \infty)$ such that

$$\mathcal{P}^t(A) = \begin{cases} \infty & \text{for all } 0 \leq t < s \\ 0 & \text{for all } t > s \end{cases}$$

Notation. We denote the Packing dimension of a set A by $\dim_{\mathcal{P}}(A)$.

There are four equivalent definitions for the packing dimension which are identical to the equivalent definitions of the Hausdorff dimension found in Lemma 5.1.5. Simply replace the Hausdorff outer-measures with Packing outer-measures and the statement is identical, as is the proof.

5.3 Comparison of Dimensions

Theorem 5.3.1. *Let* $A \subseteq X$*, then* $\dim_{\mathcal{H}}(A) \leq \dim_{\mathcal{P}}(A)$

This result is a remark in Mattila [PM95, pg. 85] and a proposition with proof in Edgar [GAE91, pg. 182]. Catastrophic inequality may exist in that there are sets for which their Hausdorff outer-measure is zero and their Packing outer-measure is infinite! As the next result states, this is a fairly representative example as equality occurs fairly rarely.

Theorem 5.3.2. *If* $d = \dim_{\mathcal{H}}(A) = \dim_{\mathcal{P}}(A)$ *then the* $d \in \mathbb{N}$.

The proof of this result may be found in Mattila [PM95, pg. 247] and requires machinery beyond the scope of this work. The Hausdorff and Packing measures can only agree in the case of integral dimensions, but there is certainly no guarantee that they do agree for such dimensions.. Earlier we noted that the Lebesgue outer-measure agreed (to within a constant multiple) of the Hausdorff dimension for all integral dimensions. Such a relation was nice as one could calculate the n-dimensional Lebesgue measure for n-dimensional objects, a computation which is quite natural relative to using the Hausdorff measure. On the other, hand this theorem essentially states that the Hausdorff measure and Packing measure are so radically different that they can only agree in fairly specific cases, and even there they need not. In this case, these statements about the outer-measures lead directly to the statements about the dimensions in question, the translation between the two statements being straight-forward.

The following result is by Hasse [HH86].

Theorem 5.3.3. *There do not exist Hausdorff functions ϕ, ψ such that $\mathcal{H}^\phi = \mathcal{P}^\psi$.*

In other words, the class of Packing outer-measures is distinct from the class of (generalized) Hausdorff measures. The statement of Hasse's theorem is stronger than the above but the spirit is consistent with Hasse's statement. If this were not the case the great deal of work above to show that the Packing outer-measures have properties analogous to those of the Hausdorff outer-measures should simply be reduced to defining a ζ and $\mathcal{F}$ and using the properties of the Carathéodory construction.

Chapter 6

Conclusion

The apparent incompatibility between Carathéodory Geometric outer-measures and Packing measures makes this area of research interesting, and it continues to grow under the work of mathematical physics and pure mathematics. A coherent theory enveloping both families of measures does not clearly exist, but its potential is a compelling motivation for further research.

Other outer-measures, measures, and dimensions have been defined in this field as well. For example the "Similarity Dimension" [GAE91, pg. 106], which is defined for the limits of iterated function systems, is related to the Hausdorff dimension through appropriate bounds. One may even want to consider the "fractal dimension" of other mathematical objects such as graphs (made into metric spaces) [GAE91] or measures. The idea of the dimension of a measure is investigated deeply in the literature of mathematical physics. In the paper by Barbaroux, Germinet, and Tcheremchantsev the definition of the Hausdorff dimension of a measure is introduced [JMBFGST01, pg. 23, Rem. 4.1]. In the same paper, further dimensions, the Generalized Fractal Dimensions, the Rényi dimension, and the Entropy Dimension, all applied to measures, are discussed in a unified framework.

Of interest for further research is looking specifically at the geometric outer-measures and packing measures defined in this work on function spaces. Many functions spaces are well understood and possess the topological properties necessary for further research in those areas. Of particular interest is the study of fractal dimension on the spaces $\ell^2(\mathbb{N})$ or $\mathcal{L}^2(\mathbb{R})$ equipped with the appropriate norms. This work was begun, at least in part, by Mark McClure in his dissertation work, where he investigated infinite dimensional sets and their associated packing measures. Little work, if any, has been done in the specific case of $\mathcal{L}^2(\mathbb{R})$, which is of particular interest in functional analysis.

Notation	Description	Page
$B(x,r), \overline{B}(x,r)$	The open (resp. closed) ball of radius r about the point x	6
$\#A$	The cardinality of the set A	34
$\overline{A}$	The closure of the set A	9
$C(\lambda)$	The Cantor set defined by $\lambda \in (0, 1/2)$	31
$A^{(c)}$	The convex hull of a set A	24
$d(A)$	The diameter of the set A	6
$\dim_{\mathcal{H}}(A)$	The Hausdorff Dimension of the set A	61
$\dim_{\mathcal{P}}(A)$	The Packing Dimension of the set A	71
$\mathcal{H}^s(A)$	The s-dimensional Hausdorff Outer-Measure of A	23
$\mathscr{M}(X,\nu)$	The set of all ν-measurable subsets of X	10
$\mathcal{N}^s(A)$	The s-dimensional Net Outer-Measure of A	45
$\mathscr{P}(X)$	The Power-set of the set X	4
$P^s(A)$	The s-dimensional Packing pre-measure of A	48
$\mathcal{P}^s(A)$	The s-dimensional Packing Outer-Measure of A	51
$\mathcal{S}^s(A)$	The s-dimensional Spherical Outer-Measure of A	42

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