

In[1]:= **n = 5;**

In[2]:= **sigmax = {{0, 1}, {1, 0}};**

In[3]:= **sigmay = {{0, -i}, {i, 0}};**

In[4]:= **sigmaz = {{1, 0}, {0, -1}};**

In[5]:= **I2 = IdentityMatrix[2];**

In[6]:= **u = DiagonalMatrix[ConstantArray[1, n - 1], 1] +
DiagonalMatrix[ConstantArray[1, 1], -n + 1];**

In[7]:= **v = DiagonalMatrix[Exp[2 Pi I / n]^Range[n]];**

In[8]:= **AA = (1/2) * (ConjugateTranspose[u] + u);**

In[9]:= **BB = (i/2) * (ConjugateTranspose[u] - u);**

In[10]:= **CC = (1/2) * (ConjugateTranspose[v] + v);**

In[11]:= **DD = (i/2) * (ConjugateTranspose[v] - v);**

In[12]:= **loclzr = KroneckerProduct[i * sigmax, AA - w * IdentityMatrix[n]] +
KroneckerProduct[i * sigmay, BB - x * IdentityMatrix[n]] +
KroneckerProduct[i * sigmaz, CC - y * IdentityMatrix[n]] +
KroneckerProduct[I2, DD - z * IdentityMatrix[n]];**

In[13]:= **charpoly = Det[loclzr];**

In[14]:= **impoly = FullSimplify[Im[charpoly],
{Element[w, Reals], Element[x, Reals], Element[y, Reals], Element[z, Reals]}]**

Out[14]=
$$\frac{1}{2\sqrt{2}} 5 (w^2 + x^2 - y^2 - z^2)$$
$$\left(\sqrt{65 + 29\sqrt{5}} + \sqrt{5 + \sqrt{5}} w^4 + \sqrt{5 + \sqrt{5}} x^4 + \sqrt{5 + \sqrt{5}} y^4 + \sqrt{5 - \sqrt{5}} z^2 + \right.$$
$$\left. \sqrt{5 + \sqrt{5}} z^2 (3 + z^2) + y^2 \left(\sqrt{50 + 20\sqrt{5}} + 2\sqrt{5 + \sqrt{5}} z^2 \right) + \right.$$
$$\left. x^2 \left(\sqrt{50 + 20\sqrt{5}} + 2\sqrt{5 + \sqrt{5}} (y^2 + z^2) \right) + w^2 \left(\sqrt{50 + 20\sqrt{5}} + 2\sqrt{5 + \sqrt{5}} (x^2 + y^2 + z^2) \right) \right)$$

In[15]:= **therest = ExpandAll[impoly / (w^2 + x^2 - y^2 - z^2)]**

Out[15]=
$$\frac{5}{2} \sqrt{\frac{1}{2} (65 + 29\sqrt{5})} + \frac{5}{2} \sqrt{\frac{1}{2} (50 + 20\sqrt{5})} w^2 + \frac{5}{2} \sqrt{\frac{1}{2} (5 + \sqrt{5})} w^4 + \frac{5}{2} \sqrt{\frac{1}{2} (50 + 20\sqrt{5})} x^2 +$$
$$5 \sqrt{\frac{1}{2} (5 + \sqrt{5})} w^2 x^2 + \frac{5}{2} \sqrt{\frac{1}{2} (5 + \sqrt{5})} x^4 + \frac{5}{2} \sqrt{\frac{1}{2} (50 + 20\sqrt{5})} y^2 + 5 \sqrt{\frac{1}{2} (5 + \sqrt{5})} w^2 y^2 +$$
$$5 \sqrt{\frac{1}{2} (5 + \sqrt{5})} x^2 y^2 + \frac{5}{2} \sqrt{\frac{1}{2} (5 + \sqrt{5})} y^4 + \frac{5}{2} \sqrt{\frac{1}{2} (5 - \sqrt{5})} z^2 + \frac{15}{2} \sqrt{\frac{1}{2} (5 + \sqrt{5})} z^2 +$$
$$5 \sqrt{\frac{1}{2} (5 + \sqrt{5})} w^2 z^2 + 5 \sqrt{\frac{1}{2} (5 + \sqrt{5})} x^2 z^2 + 5 \sqrt{\frac{1}{2} (5 + \sqrt{5})} y^2 z^2 + \frac{5}{2} \sqrt{\frac{1}{2} (5 + \sqrt{5})} z^4$$

In[16]:= **realpoly** = FullSimplify[Re[charpoly], {w^2 + x^2 == y^2 + z^2,
Element[w, Reals], Element[x, Reals], Element[y, Reals], Element[z, Reals]}];

In[17]:= **realpoly** = ReplaceAll[realpoly, {x → Sqrt[-w^2 + y^2 + z^2]}];

In[18]:= **realpoly** =
FullSimplify[realpoly, {Element[w, Reals], Element[y, Reals], Element[z, Reals]}]

Out[18]=
$$-32 w^5 + y^4 \left(5 \left(3 + \sqrt{5} \right) + y \left(-2 + 5 \left(13 + \sqrt{5} \right) y + 80 y^3 + 32 y^5 \right) \right) +$$

$$5 y^2 \left(6 + 2 \sqrt{5} + 4 y + 3 \left(13 + \sqrt{5} \right) y^2 + 64 y^4 + 32 y^6 \right) z^2 +$$

$$5 \left(3 + \sqrt{5} + y \left(-2 + 3 \left(13 + \sqrt{5} \right) y + 96 y^3 + 64 y^5 \right) \right) z^4 +$$

$$5 \left(13 + \sqrt{5} + 64 \left(y^2 + y^4 \right) \right) z^6 + 80 \left(1 + 2 y^2 \right) z^8 + 32 z^{10} + 40 w^3 \left(y^2 + z^2 \right) -$$

$$10 w \left(y^2 + z^2 \right)^2 + \frac{1}{2} \left(-11 - 5 \sqrt{5} - 5 \left(3 + \sqrt{5} \right) y^2 - 5 \left(3 + \sqrt{5} \right) z^2 \right)$$

In[19]:= **altpoly** = ReplaceAll[realpoly, {w → r * Cos[th], y → r * Cos[phi], z → r * Sin[phi]}];

In[20]:= **altpoly** = FullSimplify[altpoly, {r > 0, Element[th, Reals], Element[phi, Reals]}];

In[21]:= **Collect[ExpandAll[altpoly], r]**

Out[21]=
$$-\frac{11}{2} - \frac{5\sqrt{5}}{2} + \left(-\frac{15}{2} - \frac{5\sqrt{5}}{2} \right) r^2 + \left(15 + 5\sqrt{5} \right) r^4 +$$

$$\left(65 + 5\sqrt{5} \right) r^6 + 80 r^8 + 32 r^{10} + r^5 \left(-2 \cos[5 \text{ phi}] - 2 \cos[5 \text{ th}] \right)$$

In[22]:= **estLeft** =

$$-\frac{11}{2} - \frac{5\sqrt{5}}{2} + \left(15 + 5\sqrt{5} \right) \left(6/10 \right)^4 + \left(65 + 5\sqrt{5} \right) \left(6/10 \right)^6 + 80 \left(6/10 \right)^8 + 32 \left(6/10 \right)^{10}$$

Out[22]=
$$-\frac{77398739}{19531250} - \frac{5\sqrt{5}}{2} + \frac{81}{625} \left(15 + 5\sqrt{5} \right) + \frac{729 \left(65 + 5\sqrt{5} \right)}{15625}$$

In[23]:= **N[estLeft]**

Out[23]= -2.60574

In[24]:= **Collect[Expand[D[altpoly, r]], r]**

Out[24]=
$$\left(-15 - 5\sqrt{5} \right) r + \left(60 + 20\sqrt{5} \right) r^3 + \left(390 + 30\sqrt{5} \right) r^5 +$$

$$640 r^7 + 320 r^9 + r^4 \left(-10 \cos[5 \text{ phi}] - 10 \cos[5 \text{ th}] \right)$$

In[25]:= **estRight** =
$$\left(-15 - 5\sqrt{5} \right) 1 + \left(60 + 20\sqrt{5} \right) \left(6/10 \right)^3 +$$

$$\left(390 + 30\sqrt{5} \right) \left(6/10 \right)^5 + 640 \left(6/10 \right)^7 + 320 \left(6/10 \right)^9 - 20$$

Out[25]=
$$-\frac{5413763}{390625} - 5\sqrt{5} + \frac{27}{125} \left(60 + 20\sqrt{5} \right) + \frac{243 \left(390 + 30\sqrt{5} \right)}{3125}$$

In[26]:= **N[estRight]**

Out[26]= 33.1229

```

In[27]:= ContourPlot3D[realpoly == 0, {w, -1, 1}, {y, -1, 1},
  {z, -1, 1}, RegionFunction -> Function[{w, y, z}, 0 < y^2 + z^2 - w^2],
  ColorFunction -> Function[{w, y, z},
    ColorData["TemperatureMap"][0.5 + 0.5 Sqrt[y^2 + z^2 - w^2]]],
  ColorFunctionScaling -> False, ViewPoint -> {-5, -8, 4},
  BoxRatios -> Automatic, PlotPoints -> 25, MaxRecursion -> 3]
ContourPlot3D[realpoly == 0, {w, -1, 1}, {y, -1, 1},
  {z, -1, 1}, RegionFunction -> Function[{w, y, z}, 0 < y^2 + z^2 - w^2],
  ColorFunction -> Function[{w, y, z},
    ColorData["TemperatureMap"][0.5 - 0.5 * Sqrt[y^2 + z^2 - w^2]]],
  ColorFunctionScaling -> False, ViewPoint -> {-5, -8, 4},
  BoxRatios -> Automatic, PlotPoints -> 25, MaxRecursion -> 3]

```



