Two topics in particle accelerator beams

by

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Diplom Physiker, University of Hamburg, 1986

Committee Chair

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DISSERTATION

Submitted in Partial Fulfillment of the
Requirements for the Degree of

Doctor of Philosophy
Mathematics

The University of New Mexico

Albuquerque, New Mexico

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Dedication

To my family

Acknowledgments

I would like to thank my advisor, Professor James Ellison, for his advisement and continued support of my work. I would also like to thank Dr. Gabriele Bassi, Professor Georg Hoffstaetter, Dr. Mathias Vogt, Dr. Robert Warnock, and Dr. Jack Zhang for the many discussions we had over the years. I am also grateful to Dr. Desmond Barber and Professors Charles Boyer, Thomas Hagstrom, and Stephen Lau for their time and efforts on my dissertation committee. Additional thanks are extended to Professors Alex Buium, Todd Kapitula, and Cristina Pereyra for their support as Graduate Chairs. Moreover I am indebted to Dr. Desmond Barber for his support over the years. Finally many thanks to everybody else from the Mathematics and Statistics Department at UNM, who helped me directly or indirectly. In particular I have to thank Dr. Oksana Guba for her advice concerning several aspects of my dissertation.

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ABSTRACT OF DISSERTATION

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Abstract

This thesis has two parts. In the first part I present results from my studies of the Vlasov-Maxwell system which was developed, together with a code, in collaboration with Bassi, Ellison and Warnock. The emphasis is on the link between the theory and the self-consistent numerical computations performed by the code. The Vlasov-Maxwell system models electron beams, typically in synchrotron light sources. In the second part I present results from my studies of the dynamics of spin polarized beams. Here the emphasis is on improvements of the theoretical basis of beam simulations by using topological methods.

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Chapter 1

General Introduction

This thesis has two parts. In the first part, consisting of Chapters 2-4 and Appendix A, I present results from my studies of the Vlasov-Maxwell system (VMS) and in the second part, consisting of Chapters 5-10 and Appendices B-G, I present results from my studies of spin polarized beams. Both parts deal with beam dynamics issues for particle accelerators. A good title for the first part is: "Vlasov-Maxwell treatment of coherent synchrotron radiation" and a good title for the second part is: "Topological treatment of spin polarized beams".

In the first part (Chapters 2-4 and Appendix A) I discuss the Vlasov-Maxwell system which was developed, together with a code, in collaboration with Bassi, Ellison and Warnock. Here the emphasis is on the link between the theory and the self-consistent numerical computations performed by the code whence no attempt at extreme rigor is aimed at. For my publications on the Vlasov-Maxwell system, see [EPAC06, PAC07-1, PAC07-2, EPAC08-1, EPAC08-2, MICRO, PAC09, ICAP09].

The second part (Chapters 5-10 and Appendices B-G) presents the theory of spin-orbit tori which play an important role in beam dynamics studies of spin polarized beams. Here the emphasis is on blending given concepts and folklore into a full fledged theory of spin-orbit tori allowing to cast established as well as new results into the rigorous form of mathematical theorems. However the practical relevance of these concepts for spin polarized beams is covered in considerable detail as well. For my recent publications on spin polarized beams, see [BEH04, EH].

Chapter 2

Introduction to the Vlasov-Maxwell system

I now begin with the first part of this thesis which consists of Chapters 2-4 and Appendix A. In the present chapter I make some general remarks for the orientation of the reader. Since the first part was developed in collaboration with Bassi, Ellison and Warnock, I here often use the term 'we' instead of 'I'.

The first part of this thesis is concerned with the electron beam in a bunch compressor in a free electron laser (FEL). A bunch compressor is designed to increase the peak current of the beam and it typically consists of four dipole magnets. Fig. 1 shows the first bunch compressor system in the FERMI@Elettra free electron laser at Trieste, Italy. Note that the electron beam in a FEL consists of a train of separate bunches and that in a bunch compressor one can neglect the interaction between the bunches. Thus we only have to study a single bunch.

The purpose of an FEL is to produce intense coherent synchrotron radiation, but this does not take place in its bunch compressors. Nevertheless the electron beam produces, due to the dipole magnets, coherent synchrotron radiation in the bunch compressors and this warrants the study of bunch compressors. In fact bunch compressors can lead to a microbunching instability with detrimental effects on the beam quality. This is a major concern for free electron lasers where very bright electron beams are required, i.e. beams with low emittance and energy spread. Thus I discuss in some detail an initial condition on the bunch which we also studied in great detail in [MICRO].

A basic theoretical framework for understanding a bunch compressor is the 6D+3D Vlasov-Maxwell system (6D phase space for the bunch and 3D space for the self field of the bunch). Note that part of the self field accounts for the above mentioned coherent synchrotron radiation produced by the bunch compressor. However, the numerical integration of this system is computationally too intensive at the moment. Our basic ansatz is therefore a 4D+2D Vlasov-Maxwell system (4D phase space for the bunch and 2D space for the self field of the bunch). More precisely, we treat the beam evolution through a bunch compressor using a Monte Carlo mean field self-consistent approximation. We pseudo-randomly generate $\mathcal{N}$ points from an initial phase-space density. Here we use $\mathcal{N}$ for the simulated points to distinguish it from N for the number of particles in the bunch. We then calculate the charge density using a smooth density estimation. The electric and magnetic fields which constitute the self field of the bunch are calculated from the smooth charge/current density using a field formula that avoids singularities by using the retarded time as a variable of integration. The sample points are then moved forward in small time steps using the equations of motion in the beam frame with the fields frozen during a time step. We try to choose $\mathcal{N}$ large enough so that the charge density is a good approximation to the charge density that would be obtained from solving the 2D Vlasov-Maxwell system exactly. We call this method the 'Monte Carlo Particle (MCP) method' and we developed a FORTRAN code based on this method. We believe we calculate the charge density accurately and that for $\mathcal{N}$ sufficiently large one could obtain an accurate approximation to the 4D Vlasov phase-space density. That is beyond our current computer capability, however, and it is likely that a better approach would be to use the method of local characteristics to integrate the Vlasov equation directly.

Our MCP solver has been tested against other codes on the Zeuthen benchmark bunch compressors. Our results for the mean energy loss are in good agreement with 2D and 3D codes confirming that 1D codes underestimate the effect of coherent synchrotron radiation on the mean energy loss by a factor of 2. For more details see [PAC07-2],[PAC07-1] and references therein.

The above mentioned initial condition on the bunch corresponds to the bunch compressor of Fig. 1 which consists of a 4-dipole chicane between rf cavities and quadrupoles. This initial condition is a smooth beam frame initial phase-space density $a_0(z, x, p_z, p_x)$ modulated by a factor $1 + A\cos(2\pi z/\lambda_0)$ where A is a small amplitude and λ_0 is the perturbation wave length. The function a_0 contains the energy chirp, the $z - p_z$ correlation that is necessary for bunch compression. The beam frame coordinates (z, p_z, x, p_x) are standard and are defined in Section 3.2. The 4D+2D Vlasov-Maxwell system is described in two frames: the lab frame, which is tied to the cartesian coordinates Z, X, P_Z, P_X and the beam frame, which is tied to the accelerator coordinates z, x, p_z, p_x.

To define clearly our Vlasov-Maxwell starting point we begin with exact equations, but for practical work we later make approximations based on the following assumptions:

(A) The maximum bunch size Δ is small compared to the minimum bending radius.

(B) In beam frame coordinates the bunch form (and also the form of the phase-space distribution) changes very little during a time Δ/c. Correspondingly, the field of the bunch at a co-moving point changes little on such a time interval.

Here Δ is the biggest extent of the bunch in any direction. Under typical conditions

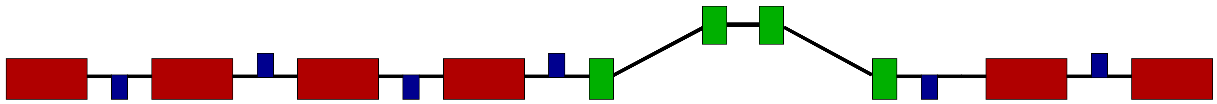


Figure 1: Proposed layout of FERMI@Elettra first bunch compressor system. Accelerating rf cavities in red, quadrupole magnets in blue, drift sections in black and dipoles in green. Parameters are given in (3.105).

(A) and (B) should be very well satisfied. We also assume that the beam is relativistic ($\gamma \gg 1$), as is true in the example studied, but that assumption could be removed without great cost.

The first part of this thesis is organized as follows. In Section 3.1 the 6D+3D Vlasov-Maxwell system is introduced via the 6D Vlasov equation and Maxwell's equations for the self field. By confining to a sheet bunch, the 6D+3D Vlasov-Maxwell system is then boiled down to the lab frame 4D+2D Vlasov-Maxwell system. Exact formulas for the self field of the 4D+2D Vlasov-Maxwell system are presented as well and the 4D Vlasov equation is derived from the 6D Vlasov equation. Section 3.2 is devoted to the definition of the beam frame and beam coordinates, the beam frame equations of motion, and the transformation of densities from beam frame to lab frame which is needed to determine the lab frame sources. Also the 4D Vlasov equation in the beam frame is derived from the 4D Vlasov equation in the lab frame and the above mentioned initial bunch condition is introduced. Some numerical results are presented in Section 3.3 where also the underlying initial condition on the bunch is introduced. In Section 3.4 we give the details of our MCP algorithm. In particular in Section 3.4.2 a causality issue is discussed and in Section 3.4.3 a convergence study of computation errors of our code is presented. Moreover Section 3.4.3 gives further insight into some density estimation techniques. Appendix A contains supplementary material needed for Chapter 3 and Chapter 4 gives an outlook on the Vlasov-Maxwell system.

My contributions to the Vlasov-Maxwell system are about as follows. I have been strongly involved in many aspects of our Vlasov-Maxwell work, part of which is documented in [EPAC06, PAC07-1, PAC07-2, EPAC08-1, EPAC08-2, MICRO, PAC09, ICAP09]. Among my main contributions are the development of the sheet beam ansatz (see Section 3.1), the parallelization of our code (see Section 3.4), the work on some aspects of the transformation steps in (3.54), the work on the 2D integral equation approach to the microbunching instability (see [MICRO]), the discovery of the causality issue of our code (see Section 3.4), the study of the kernel density estimation method (see Sections 3.4.3 and A.3) and the convergence studies (see Section 3.4.3).

Chapter 3

The Vlasov-Maxwell system

3.1 Generalities

Our basic starting point is the 6D+3D Vlasov-Maxwell system, i.e., we assume collisions can be ignored and that the $N-$particle bunch can be approximated by a continuum. Our final scheme for computation is less ambitious, but we think that it might be a reasonable approximation to the full system. We reduce the problem from $3D$ to $2D$, since we expect that most of the acceleration by a self field will be in the plane of the unperturbed orbit. We use a particle method that follows the spatial density rather than the phase-space density, but hope that with sufficient attention to smoothing the result approximates that defined by the Vlasov-Maxwell system.

We are studying the time evolution of an electron bunch and its Maxwell field (=self field) as the bunch moves through a chicane. In the model we use, the only force which acts on the bunch is the Lorentz force produced by the self field $\mathbf{E} = \mathbf{E}(\bar{\mathbf{R}}, u)$, $\mathbf{B} = \mathbf{B}(\bar{\mathbf{R}}, u)$ and the external magnetic field (the latter is produced by the magnets of the chicane) where $u = ct$ is the scaled time which we call 'time' and where the 3D position vector is written as

$$\bar{\mathbf{R}} = (Z, X, Y)^T \ . \tag{3.1}$$

There is no external electric field and the external magnetic field $\bar{\mathbf{B}}_{ext}$ is time independent and we write

$$(\bar{B}_{ext,Z}, \bar{B}_{ext,X}, \bar{B}_{ext,Y})^T = \bar{\mathbf{B}}_{ext} = \bar{\mathbf{B}}_{ext}(\bar{\mathbf{R}}) \ . \tag{3.2}$$

In the $Y = 0$ plane the external magnetic field $\bar{\mathbf{B}}_{ext}$ has the rather simple form

$$\bar{\mathbf{B}}_{ext}(Z, X, 0) = (0, 0, B_{ext}(Z))^T \ . \tag{3.3}$$

Clearly the total field is given by $\mathbf{E}$ and $\mathbf{B}+\bar{\mathbf{B}}_{ext}$. We use a Vlasov-Maxwell approach whereby the bunch is represented by a time-dependent 6D phase-space density $\bar{f} = \bar{f}(\bar{\mathbf{R}}, \bar{\mathbf{P}}; u)$ where $\dot{} = d/du$. Note that $\bar{\mathbf{P}}$ is the 3D momentum vector written as

$$\bar{\mathbf{P}} = (P_Z, P_X, P_Y)^T \ , \tag{3.4}$$

and that

$$\bar{\gamma} = \sqrt{1 + \frac{\bar{\mathbf{P}} \cdot \bar{\mathbf{P}}}{m^2 c^2}} \ , \qquad \dot{\bar{\mathbf{R}}} = \frac{\bar{\mathbf{P}}}{mc\bar{\gamma}} \ , \tag{3.5}$$

where m is the electron rest mass, c is the vacuum light velocity and $\bar{\gamma}$ is the 3D Lorentz factor. The phase-space variables $\bar{\mathbf{R}}, \bar{\mathbf{P}}$ characterize the 'lab frame'. Note that in the first part of this thesis the scalar product is denoted by $'\cdot'$ as in (3.5). The phase-space density evolves according to the 6D Vlasov-equation

$$\partial_u \bar{f} + \dot{\bar{\mathbf{R}}} \cdot \nabla_{\bar{\mathbf{R}}} \bar{f} + \dot{\bar{\mathbf{P}}} \cdot \nabla_{\bar{\mathbf{P}}} \bar{f} = 0 \ , \tag{3.6}$$

$$\bar{f}(\bar{\mathbf{R}}, \bar{\mathbf{P}}; u_0) = \bar{f}_0(\bar{\mathbf{R}}, \bar{\mathbf{P}}) \ , \tag{3.7}$$

where u_0 is the initial time. Note that the Lorentz force term, $\dot{\bar{\mathbf{P}}} \cdot \nabla_{\bar{\mathbf{P}}} \bar{f}$, of the 6D Vlasov-equation is determined by the Lorentz force of the total field whence we have the lab frame equations of motion

$$\dot{\bar{\mathbf{R}}} = \frac{\bar{\mathbf{P}}}{mc\bar{\gamma}} \,, \tag{3.8}$$

$$\dot{\bar{\mathbf{P}}} = \frac{q}{c}(\mathbf{E} + \frac{\bar{\mathbf{P}}}{m\bar{\gamma}} \times [\mathbf{B} + \bar{\mathbf{B}}_{ext}]) \,, \tag{3.9}$$

where q is the electron charge. We use SI units throughout. We used in (3.6) the fact that the vector field defined by the rhs of (3.8),(3.9) is divergence free. The self field satisfies Maxwell's equations

$$\begin{aligned} &\partial_u \mathbf{E} = c\nabla_{\bar{\mathbf{R}}} \times \mathbf{B} - \mu_0 c\bar{\mathbf{J}} \,, \qquad c\partial_u \mathbf{B} = -\nabla_{\bar{\mathbf{R}}} \times \mathbf{E} \,, \\ &\nabla_{\bar{\mathbf{R}}} \cdot \mathbf{E} = \frac{\bar{\rho}}{\varepsilon_0} \,, \qquad \nabla_{\bar{\mathbf{R}}} \cdot \mathbf{B} = 0 \,, \end{aligned} \tag{3.10}$$

where the 3D charge density $\bar{\rho}$ and the 3D current density $\bar{\mathbf{J}}$ of the bunch are determined by the 6D phase-space density $\bar{f}$ via

$$\bar{\rho}(\bar{\mathbf{R}}; u) := Q \int_{\mathbb{R}^3} d\bar{\mathbf{P}} \bar{f}(\bar{\mathbf{R}}, \bar{\mathbf{P}}; u) \,, \tag{3.11}$$

$$\bar{\mathbf{J}}(\bar{\mathbf{R}}; u) = (\bar{J}_Z(\bar{\mathbf{R}}; u), \bar{J}_X(\bar{\mathbf{R}}; u), \bar{J}_Y(\bar{\mathbf{R}}; u))^T := Q \int_{\mathbb{R}^3} d\bar{\mathbf{P}} \frac{\bar{\mathbf{P}}}{m\bar{\gamma}} \bar{f}(\bar{\mathbf{R}}, \bar{\mathbf{P}}; u) \,, \tag{3.12}$$

with Q being the charge of the bunch. Note that ε_0 is the vacuum electric permeability and μ_0 is the vacuum magnetic permeability whence $c^2 = 1/\mu_0\epsilon_0$. Maxwell's equations for the external magnetic field are homogeneous and read as

$$0 = \nabla_{\bar{\mathbf{R}}} \times \bar{\mathbf{B}}_{ext} \,, \qquad \nabla_{\bar{\mathbf{R}}} \cdot \bar{\mathbf{B}}_{ext} = 0 \,. \tag{3.13}$$

Clearly, Maxwell's equations for the total field are the same as for the self field. Since $\bar{f}$ is the 6D phase-space density it is normalized by

$$1 = \int_{\mathbb{R}^6} d\bar{\mathbf{R}} d\bar{\mathbf{P}} \bar{f}(\bar{\mathbf{R}}, \bar{\mathbf{P}}; u) \,, \tag{3.14}$$

whence, by (3.11), the 3D spatial density $(1/Q)\bar{\rho}$ is normalized, too:

$$Q = \int_{\mathbb{R}^3} d\bar{\mathbf{R}} \bar{\rho}(\bar{\mathbf{R}}; u) \,. \tag{3.15}$$

We assume that the initial self field vanishes, i.e.,

$$0 = \mathbf{E}(\bar{\mathbf{R}}, u_0) = \mathbf{B}(\bar{\mathbf{R}}, u_0) , \qquad 0 = \partial_u \mathbf{E}(\bar{\mathbf{R}}, u_0) = \partial_u \mathbf{B}(\bar{\mathbf{R}}, u_0) . \tag{3.16}$$

We abbreviate

$$\mathbf{E} := (E_Z, E_X, E_Y)^T , \qquad \mathbf{B} := (B_Z, B_X, B_Y)^T . \tag{3.17}$$

We consider two scenarios, the shielding resp. nonshielding one. In the shielding scenario we assume a perfect conductor at the planes $Y = \pm g$ modelling the vacuum chamber of the chicane where $2g$ is the distance between the two conductors which constitute the shielding. Thus in the shielding scenario we impose the following boundary conditions on the total field:

$$0 = E_Z(Z, X, \pm g, u) = E_X(Z, X, \pm g, u) = B_Y(Z, X, \pm g, u) + \bar{B}_{ext,Y}(Z, X, \pm g) , \tag{3.18}$$

$$\begin{aligned} 0 &= \partial_Y E_Y(Z, X, \pm g, u) = \partial_Y B_Z(Z, X, \pm g, u) + \partial_Y \bar{B}_{ext,Z}(Z, X, \pm g) \\ &= \partial_Y B_X(Z, X, \pm g, u) + \partial_Y \bar{B}_{ext,X}(Z, X, \pm g) . \end{aligned} \tag{3.19}$$

In fact (3.18) defines the perfect conductor [Ja] and (3.19) follows from (3.10),(3.13), (3.22). Note that in the shielding scenario we are only interested in the electromagnetic field between the two conductors, i.e., for $Y \in [-g, g]$. It follows from the initial conditions (3.16) for the self field and from the boundary conditions (3.18),(3.19) for the total field that the external field satisfies, in the shielding scenario,

$$0 = \bar{B}_{ext,Y}(Z, X, \pm g) , \tag{3.20}$$

$$0 = \partial_Y \bar{B}_{ext,Z}(Z, X, \pm g) = \partial_Y \bar{B}_{ext,X}(Z, X, \pm g) . \tag{3.21}$$

Of course, (3.18),(3.19),(3.20), (3.21) imply

$$0 = E_Z(Z, X, \pm g, u) = E_X(Z, X, \pm g, u) = B_Y(Z, X, \pm g, u) , \tag{3.22}$$

$$0 = \partial_Y E_Y(Z, X, \pm g, u) = \partial_Y B_Z(Z, X, \pm g, u) = \partial_Y B_X(Z, X, \pm g, u) . \tag{3.23}$$

We observe that, in the shielding scenario, the external field and the self field satisfy the same boundary conditions as the total field. It is clear by (3.22),(3.23) that, in the shielding scenario, E_Z, E_X, B_Y satisfy a Dirichlet condition and E_Y, B_Z, B_X satisfy a Neumann condition at $Y = \pm g$. Thus in the shielding scenario we have, for the self field, the initial boundary value problem consisting of eq.'s (3.6),(3.7),(3.10), (3.16),(3.22),(3.23) while in the nonshielding scenario we have the initial value problem consisting of eq.'s (3.6),(3.7),(3.10), (3.16). Fig. 2 shows our coordinate system (Z, X, Y) and the two conductors in the shielding scenario ($\mathbf{R}_r$ will be explained in Section 3.2).

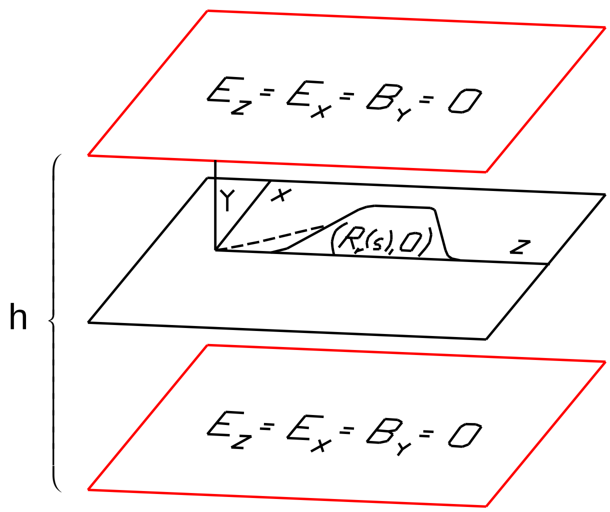


Figure 2: Basic Setup (h:=2g)

We assume that our 6D+3D Vlasov-Maxwell system is well-posed in both scenarios. Our problem is nonlinear since $\mathbf{E}, \mathbf{B}$ depend via (3.10), (3.11),(3.12) on $\bar{f}$ whence the term $\frac{q}{c}(\mathbf{E} + \frac{\bar{\mathbf{P}}}{m\bar{\gamma}} \times \mathbf{B}) \cdot \nabla_{\bar{\mathbf{P}}} \bar{f}$ in the 6D Vlasov equation (3.6) is nonlinear in $\bar{f}$. Thus we are faced with a complicated problem which warrants a numerical treatment. It is important to note that we need to compute the self field to the extent as it contributes to the 6D Vlasov equation (3.6). Thus for arbitrary initial values

(3.7) all six $3D$ self field components of $\mathbf{E}, \mathbf{B}$ are involved whence it is believed that for arbitrary initial values (3.7) our problem is numerically intractable. To arrive at a numerically tractable problem we confine to a sheet bunch, i.e., a situation where $\bar{f}$ is concentrated in the $(Y, P_Y) = 0$ plane:

$$\bar{f}(\bar{\mathbf{R}}, \bar{\mathbf{P}}; u) = \delta(Y)\delta(P_Y) f_L(\mathbf{R}, \mathbf{P}; u) \ , \tag{3.24}$$

with

$$\mathbf{R} = (Z, X)^T \ , \qquad \mathbf{P} = (P_Z, P_X)^T \ . \tag{3.25}$$

Thus from now in this chapter we assume that $\bar{f}$ has the form (3.24). It is shown in Section A.2 that $\bar{f}$ is of the form (3.24) if it is initially of this form. Moreover it is shown in Section A.1 that if $\bar{f}$ is of the form (3.24) then $E_Y(\bar{\mathbf{R}}, u), B_Z(\bar{\mathbf{R}}, u), B_X(\bar{\mathbf{R}}, u)$ are odd in Y and $E_Z(\bar{\mathbf{R}}, u), E_X(\bar{\mathbf{R}}, u), B_Y(\bar{\mathbf{R}}, u)$ are even in Y (see also the remarks after (3.45)). Furthermore it is shown in Section A.2 that if $\bar{f}$ is of the form (3.24) then f_L satisfies the 4D Vlasov equation

$$\partial_u f_L + \dot{\mathbf{R}} \cdot \nabla_{\mathbf{R}} f_L + \dot{\mathbf{P}} \cdot \nabla_{\mathbf{P}} f_L = 0 \ , \tag{3.26}$$

where

$$\dot{\mathbf{R}} = \frac{\mathbf{P}}{mc\gamma} \ , \tag{3.27}$$

$$\dot{\mathbf{P}} = \frac{q}{c}\left(\mathbf{E}_{\parallel}(\mathbf{R}, u) + \frac{1}{m\gamma}(B_{ext}(Z) + B_{\perp}(\mathbf{R}, u))(P_X, -P_Z)^T\right) \ , \tag{3.28}$$

and where

$$\gamma = \gamma(\mathbf{P}) = \sqrt{1 + \frac{\mathbf{P} \cdot \mathbf{P}}{m^2 c^2}} \ , \tag{3.29}$$

$$\mathbf{E}_{\parallel}(\mathbf{R}, u) = (E_{L,Z}(\mathbf{R}, u), E_{L,X}(\mathbf{R}, u))^T := (E_Z(\mathbf{R}, 0, u), E_X(\mathbf{R}, 0, u))^T \ , \tag{3.30}$$

$$B_{\perp}(\mathbf{R}, u) := B_Y(\mathbf{R}, 0, u) \ . \tag{3.31}$$

Note that the vector field defined by the rhs of (3.27),(3.28) is divergence free. Writing the initial form of f_L as

$$f_L(\mathbf{R}, \mathbf{P}; u_0) = f_{L,0}(\mathbf{R}, \mathbf{P}) \ , \tag{3.32}$$

we have, by (3.7), $\bar{f}_0(\bar{\mathbf{R}}, \bar{\mathbf{P}}) = \delta(Y)\delta(P_Y)f_{L,0}(\mathbf{R}, \mathbf{P})$. Only the three components E_Z, E_X, B_Y of the self field contribute to the 4D Vlasov equation (3.26) (and they only contribute in the $Y = 0$ plane). As explained in Section A.2, the reason for this is that $E_Y(\bar{\mathbf{R}}, u), B_Z(\bar{\mathbf{R}}, u), B_X(\bar{\mathbf{R}}, u)$ are odd in Y. Thus $\mathbf{E}_\parallel, B_\perp$ are the only parts of the self field which have to be computed whence we only have to deal with a $4D$ phase-space density and three $2D$ self field components, i.e., the 6D+3D Vlasov-Maxwell system boils down to a 4D+2D Vlasov-Maxwell system. We believe that the 4D+2D Vlasov-Maxwell problem is numerically tractable. We define

$$\mathcal{F} := (E_Z, E_X, B_Y)^T \ , \tag{3.33}$$

$$\mathcal{F}_L(\mathbf{R}, u) := \mathcal{F}(\mathbf{R}, 0, u) = (\mathbf{E}_\parallel(\mathbf{R}, u), B_\perp(\mathbf{R}, u))^T \ . \tag{3.34}$$

Computing f_L and $\mathcal{F}_L$ solves the 4D+2D Vlasov-Maxwell problem. Maxwell's equations (3.10) and the initial conditions (3.16) give us

$$\Box\mathcal{F}(\mathbf{R}, Y, u) = \delta(Y)\mathbf{S}(\mathbf{R}, u) \ , \tag{3.35}$$

$$0 = \mathcal{F}(\mathbf{R}, Y, u_0) = \partial_u\mathcal{F}(\mathbf{R}, Y, u_0) \ , \tag{3.36}$$

where

$$\mathbf{S} = Z_0 \begin{pmatrix} c\partial_Z\rho_L + \partial_u J_{L,Z} \\ c\partial_X\rho_L + \partial_u J_{L,X} \\ \frac{1}{c}[\partial_X J_{L,Z} - \partial_Z J_{L,X}] \end{pmatrix} \ , \tag{3.37}$$

$$\rho_L(\mathbf{R}; u) = Q\int_{\mathbb{R}^2} d\mathbf{P} f_L(\mathbf{R}, \mathbf{P}; u) \ , \tag{3.38}$$

$$\mathbf{J}_L(\mathbf{R}; u) = (J_{L,Z}(\mathbf{R}; u), J_{L,X}(\mathbf{R}; u))^T = Q\int_{\mathbb{R}^2} d\mathbf{P}\frac{\mathbf{P}}{m\gamma} f_L(\mathbf{R}, \mathbf{P}; u) \ , \tag{3.39}$$

and where $\Box = \partial_Z^2 + \partial_X^2 + \partial_Y^2 - \partial_u^2$ with $Z_0 := \sqrt{\frac{\mu_0}{\epsilon_0}}$ being the free space impedance. Note that

$$\bar{\rho}(\bar{\mathbf{R}}; u) = \delta(Y)\rho_L(\mathbf{R}; u) \ , \qquad \bar{\mathbf{J}}(\bar{\mathbf{R}}; u) = \begin{pmatrix} \delta(Y)\mathbf{J}_L(\mathbf{R}; u) \\ 0 \end{pmatrix} \ , \tag{3.40}$$

and that by (3.14)

$$1 = \frac{1}{Q}\int_{\mathbb{R}^2} d\mathbf{R}\rho_L(\mathbf{R};u) = \int_{\mathbb{R}^4} d\mathbf{R}d\mathbf{P}f_L(\mathbf{R},\mathbf{P};u)\ . \tag{3.41}$$

We refer to ρ_L as the 2D charge density, $\mathbf{J}_L$ as the 2D current density and ρ_L/Q as the 2D spatial density. In the nonshielding scenario we write $\mathcal{F} = \mathcal{F}^{nsh}$ and in the shielding scenario we write $\mathcal{F} = \mathcal{F}^{sh}$ and we abbreviate

$$\mathcal{F}_L^{nsh}(\mathbf{R},u) := \mathcal{F}^{nsh}(\mathbf{R},0,u)\ , \qquad \mathcal{F}_L^{sh}(\mathbf{R},u) := \mathcal{F}^{sh}(\mathbf{R},0,u)\ . \tag{3.42}$$

Thus by (3.22)

$$0 = \mathcal{F}^{sh}(Z,X,\pm g,u)\ . \tag{3.43}$$

It is shown in Section A.1 that (3.35),(3.36),(3.43) imply

$$\mathcal{F}^{nsh}(\mathbf{R},Y,u) = -\frac{1}{4\pi}\int_{\mathbb{R}^2} d\mathbf{R}' 1_{[u_0,\infty)}(u-\sqrt{|\mathbf{R}-\mathbf{R}'|^2+Y^2}) \cdot\frac{\mathbf{S}(\mathbf{R}',u-\sqrt{|\mathbf{R}-\mathbf{R}'|^2+Y^2})}{\sqrt{|\mathbf{R}-\mathbf{R}'|^2+Y^2}}\ , \tag{3.44}$$

$$\mathcal{F}^{sh}(\mathbf{R},Y,u) = \sum_{k\in\mathbb{Z}}(-1)^k\mathcal{F}^{nsh}(\mathbf{R},Y-2kg,u)\ , \tag{3.45}$$

where $1_{[u_0,\infty)}$ is the indicator function of the set $[u_0,\infty)$. Clearly $\mathcal{F}^{nsh}(\mathbf{R},Y,u)$ and $\mathcal{F}^{sh}(\mathbf{R},Y,u)$ are even in Y and only those values $\mathbf{S}(\bar{\mathbf{R}},u)$ contribute to $\mathcal{F}^{nsh}$ and $\mathcal{F}^{sh}$ for which $u \geq u_0$. It follows from (3.42), (3.44),(3.45) that

$$\mathcal{F}_L^{nsh}(\mathbf{R},u) = -\frac{1}{4\pi}\int_{\mathbb{R}^2} d\mathbf{R}' 1_{[u_0,\infty)}(u-|\mathbf{R}-\mathbf{R}'|)\frac{\mathbf{S}(\mathbf{R}',u-|\mathbf{R}-\mathbf{R}'|)}{|\mathbf{R}-\mathbf{R}'|}\ , \tag{3.46}$$

$$\begin{aligned}\mathcal{F}_L^{sh}(\mathbf{R},u) &= \sum_{k\in\mathbb{Z}}(-1)^k\mathcal{F}^{nsh}(\mathbf{R},2kg,u)\\ &= -\frac{1}{4\pi}\sum_{k\in\mathbb{Z}}(-1)^k\int_{\mathbb{R}^2} d\mathbf{R}' 1_{[u_0,\infty)}(u-\sqrt{|\mathbf{R}-\mathbf{R}'|^2+(2kg)^2})\\ &\quad\cdot\frac{\mathbf{S}(\mathbf{R}',u-\sqrt{|\mathbf{R}-\mathbf{R}'|^2+(2kg)^2})}{\sqrt{|\mathbf{R}-\mathbf{R}'|^2+(2kg)^2}}\ .\end{aligned} \tag{3.47}$$

Clearly $\mathcal{F}_L^{nsh}$ equals the $k=0$ term in the expression (3.47) of $\mathcal{F}^{sh}$. The integration in (3.46),(3.47) is restricted to a very small part of $\mathbb{R}^2$, because of the small size of the bunch, but it is awkward to locate this region owing to the fact that spatial and temporal arguments of the source both depend on $\mathbf{R}'$. The task of integration is greatly simplified if we take the temporal argument to be a new variable of integration. We first transform the integrand in (3.44) to polar coordinates (χ, θ), then take the temporal argument v in place of the radial coordinate χ. That is,

$$\mathbf{R}' - \mathbf{R} = \chi \mathbf{e}(\theta) \ , \qquad \mathbf{e}(\theta) = (\cos\theta, \sin\theta)^T \ , \qquad v = u - \sqrt{\chi^2 + Y^2} \ . \quad (3.48)$$

This conveniently gets rid of the potentially small divisor in (3.44) giving the self field simply as an integral over the source. In fact it is shown in Section A.1 that $\mathcal{F}^{nsh}$ can be written as

$$\mathcal{F}^{nsh}(\mathbf{R}, Y, u) = -\frac{1}{4\pi}\int_{-\infty}^{u-|Y|} dv 1_{[u_0,\infty)}(v) \int_{-\pi}^{\pi} d\theta \mathbf{S}(\mathbf{R} + \sqrt{(u-v)^2 - Y^2}\mathbf{e}(\theta), v) \ , \quad (3.49)$$

whence by (3.47)

$$\begin{aligned}\mathcal{F}_L^{sh}(\mathbf{R}, u) &= \sum_{k\in\mathbb{Z}} (-1)^k \mathcal{F}^{nsh}(\mathbf{R}, 2kg, u) \\ &= -\frac{1}{4\pi}\sum_{k\in\mathbb{Z}}(-1)^k \int_{-\infty}^{u-|2kg|} dv 1_{[u_0,\infty)}(v) \int_{-\pi}^{\pi} d\theta \mathbf{S}(\mathbf{R} + \sqrt{(u-v)^2 - (2kg)^2}\mathbf{e}(\theta), v) \ ,\end{aligned}$$

i.e.

$$\mathcal{F}_L^{sh}(\mathbf{R}, u) = -\frac{1}{2\pi}\sum_{k=0}^{\infty}(-1)^k(1 - \delta_{k0}/2) \int_{u_0}^{u-2kg} dv 1_{[u_0,\infty)}(v) \int_{-\pi}^{\pi} d\theta \mathbf{S}(\tilde{\mathbf{R}}(\theta, v; u), v) \ , \quad (3.50)$$

where $\tilde{\mathbf{R}}(\theta, v; u) = \mathbf{R} + \sqrt{(u-v)^2 - (2kg)^2}\mathbf{e}(\theta)$. Of course $\mathcal{F}_L^{nsh}$ is the $k = 0$ term in (3.50), i.e.,

$$\mathcal{F}_L^{nsh}(\mathbf{R}, u) = -\frac{1}{4\pi}\int_{-\infty}^{u} dv 1_{[u_0,\infty)}(v) \int_{-\pi}^{\pi} d\theta \mathbf{S}(\mathbf{R} + (u-v)\mathbf{e}(\theta), v) \ . \quad (3.51)$$

To estimate the effective region of the θ integration in (3.50), note that the source in (3.50) has significant values only for $\tilde{\mathbf{R}}(\theta, v; u)$ restricted to a bunch-sized neighborhood of $\mathbf{R}_r(\beta_r v)$; i.e., the bunch is close to the reference particle (see Section 3.2 for the definition of $\mathbf{R}_r$) where β_r is the constant speed of the reference particle. For $\mathcal{F}_L^{sh}$ at time u we are interested only in $\mathbf{R}$ in a bunch-sized neighborhood of $\mathbf{R}_r(\beta_r u)$. Thus for $\mathbf{R}$ in a small neighborhood of $\mathbf{R}_r(\beta_r u)$ the integrand is appreciable only when

$$\left|\tilde{\mathbf{R}}(\theta, v; u) - \mathbf{R}_r(\beta_r v)\right| \approx \left|\mathbf{R}_r(\beta_r u) - \mathbf{R}_r(\beta_r v) + \sqrt{(u-v)^2 - (2kg)^2}\,\mathbf{e}(\theta)\right| = O(\Delta)\ , \tag{3.52}$$

where Δ was introduced in Chapter 2. For $k = 0$ and $u - v$ large compared to Δ, this cannot be satisfied unless $\mathbf{e}(\theta)$ has nearly the same direction as $\mathbf{R}_r(\beta_r u) - \mathbf{R}_r(\beta_r v)$, which is to say that the domain of θ integration is tiny (and close to $\theta = 0$ for a chicane with small bending angle). When $u - v$ gets close to Δ the domain expands precipitously to the full $[-\pi, \pi]$. For $k \neq 0$ the condition (3.52) cannot be met unless $u - v \gg 2kg$, so for image charges there are no contributions to the v-integral close to its upper limit.

The θ integration is over an arc centered at the observation point $\mathbf{R}$ at time u with radius $\sqrt{(u-v)^2 - (2kg)^2}$, its extent being its intersection with the bunch at time v. This is illustrated in Fig. 3 for $k = 0$. When v is close to u the source bunch and the observation region (the region of the bunch at time u) overlap and the $\theta-$support of the source is large. However, for most v the $\theta-$support is small and it is important to determine the approximate support as shown in the figure. Currently the θ integration is done with the trapezoidal rule, which is superconvergent. The remaining $v-$integrand varies with v, $\mathbf{R}$ and u in ways we have not yet quantified and so we use an adaptive integrator (Gauss-Kronrod).

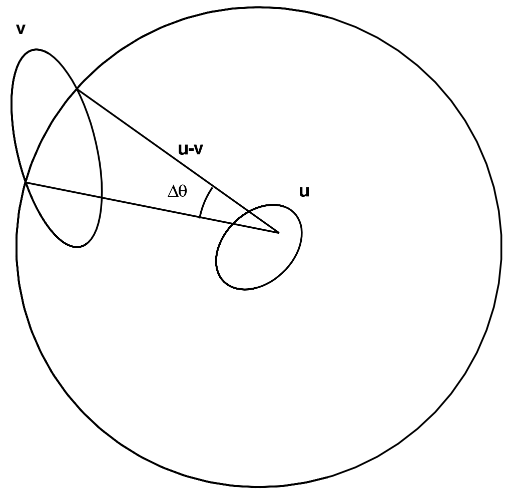


Figure 3: Plan for θ integration

3.2 Beam frame

3.2.1 Exact formulas

In our approach the Maxwell equations are solved in the lab frame (recall that $\mathcal{F}_L$ depends on $\mathbf{R}$) but the equations of motion are integrated in the beam frame (to be defined in this section). Here we discuss the beam frame coordinates and the transformation of the densities between the two frames. The beam frame is defined in terms of the reference curve $\mathbf{R}_r(s) = (Z_r(s), X_r(s))^T$ which in turn is defined by the Lorentz force without self field. We take $s = 0$ at the entrance of the chicane, i.e., $\mathbf{R}_r(0)$ is the entry point of the reference curve into the chicane. We also write $s = s_f$

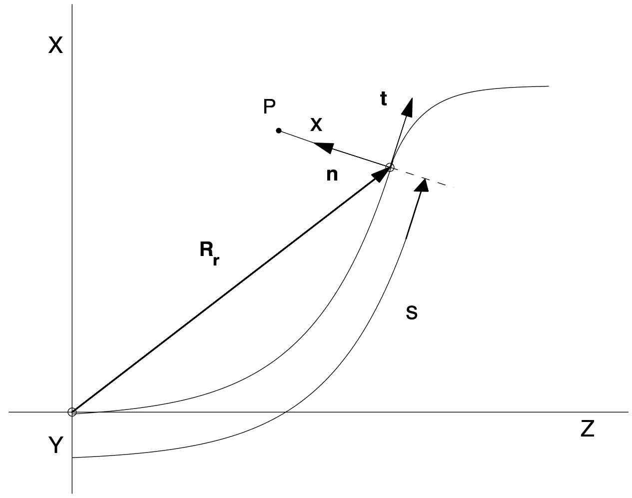


Figure 4: Beam Frame Coordinates ($P \equiv \mathbf{R}_r(s) + x\mathbf{n}(s)$)

for the end of the chicane. The unit tangent vector, $\mathbf{t}$, to the reference curve is just $\mathbf{t}(s) = \mathbf{R}_r'(s)$ and we define the unit normal vector, $\mathbf{n}$, by $\mathbf{n}(s) = (-X_r'(s), Z_r'(s))^T$ so that $\mathbf{n}$ is a $\pi/2$ counterclockwise rotation from $\mathbf{t}$ as shown in Fig. 4. It follows from the equations of motion (3.27),(3.28) that $\mathbf{t}'(s) = -qB_{ext}(Z_r(s))\mathbf{n}(s)/P_r$ where $P_r = m\gamma_r\beta_r c$ is the momentum of the reference particle and $\gamma_r = (1-\beta_r^2)^{-1/2}$. This determines the curvature κ up to a sign and we choose $\kappa(s) = qB_{ext}(Z_r(s))/P_r$. Thus $\mathbf{t}'(s) = -\kappa(s)\mathbf{n}(s)$ and $\mathbf{n}'(s) = \kappa(s)\mathbf{t}(s)$. In terms of Fig. 2 this makes κ negative in the first magnet, positive in the second magnet and so on.

The beam frame Frenet-Serret coordinates are s, x, where s is the arc length along the reference curve and x is the perpendicular distance along $\mathbf{n}$. Thus the transformation from (s, x) to (Z, X) is

$$\mathbf{R} \;=\; \mathbf{R}_r(s) + x\mathbf{n}(s). \tag{3.53}$$

Based on (3.53) our lab to beam transformation can be performed in three steps:

$$(Z, X, P_Z, P_X; u) \rightarrow (s, x, p_s, p_x; u) \rightarrow (u, x, p_s, p_x; s) \rightarrow (z, x, p_z, p_x; s)\;. \tag{3.54}$$

The phase-space variables z, x, p_z, p_x characterize the 'beam frame'. The first step in (3.54) is defined by (3.53) and by

$$\mathbf{P} = P_r(p_s\mathbf{t}(s) + p_x\mathbf{n}(s))\;. \tag{3.55}$$

In the second step the variables s and u are interchanged making s the new independent variable. In the final step $z = s - \beta_r u$ replaces u as a dependent variable and $p_z = (\gamma - \gamma_r)/\gamma_r$ replaces p_s where γ depends on p_s, p_x via (3.29),(3.55) as $\gamma = \sqrt{1 + (P_r^2/m^2c^2)(p_s^2 + p_x^2)}$. The variables z, x, p_z, p_x are small near the reference curve which corresponds to $z = x = p_z = p_x = 0$.

To summarize, the coordinate transformation is written as

$$\begin{aligned}
&\mathbf{R} = \mathbf{R}_r(s) + x\mathbf{n}(s),\\
&\mathbf{P} = P_r[p_s(p_z, p_x)\mathbf{t}(s) + p_x\mathbf{n}(s)],\\
&u = (s - z)/\beta_r\;,
\end{aligned} \tag{3.56}$$

with inverse

$$\begin{aligned}
&s = \hat{s}(\mathbf{R})\;,\; z = \hat{z}(\mathbf{R}, u) = \hat{s}(\mathbf{R}) - \beta_r u\;,\; x = \hat{x}(\mathbf{R})\;,\\
&p_z = \hat{p}_z(\mathbf{P}) := -1 + (1 + \mathbf{P}\cdot\mathbf{P}/m^2c^2)^{1/2}/\gamma_r\;,\\
&p_x = \hat{p}_x(\mathbf{R}, \mathbf{P}) := \mathbf{P}\cdot\mathbf{n}(\hat{s}(\mathbf{R}))/P_r\;,
\end{aligned} \tag{3.57}$$

where

$$p_s(p_z, p_x) = \left[(\frac{1}{\beta_r})^2(1+p_z)^2 - p_x^2 - \frac{1}{\gamma_r^2\beta_r^2}\right]^{1/2}$$
$$= [1 + (2p_z + p_z^2)/\beta_r^2 - p_x^2]^{1/2} \ . \tag{3.58}$$

The transformation $(z, x, p_z, p_x; s) \to (Z, X, P_Z, P_X; u)$ in (3.56) is only considered in a neighborhood of the reference curve $\mathbf{R}_r$, i.e., for small x so that it is one-one. Furthermore (3.56) is restricted to $p_s > 0$ in order to have $ds/du > 0$ which allows to use s as the independent variable. Under the transformation (3.57), the beam frame equations of motion become

$$z' = 1 - \frac{[1+\kappa(s)x](1+p_z)}{p_s(p_z, p_x)}, \tag{3.59}$$
$$x' = \frac{[1+\kappa(s)x]p_x}{p_s(p_z, p_x)}, \tag{3.60}$$
$$p_z' = \frac{q[1+x\kappa(s)]}{m\gamma_r c^2}[\mathbf{t}(s) + \frac{p_x}{p_s(p_z, p_x)}\mathbf{n}(s)] \cdot \mathbf{E}_\parallel(\mathbf{R}_r(s) + x\mathbf{n}(s); \frac{s-z}{\beta_r}), \tag{3.61}$$
$$p_x' = p_s(p_z, p_x)\kappa(s) - \frac{q}{P_r}[1+\kappa(s)x]B_{\text{ext}}[Z_r(s) - xX_r'(s)]$$
$$+ \frac{q(1+p_z)}{P_r\beta_r c p_s(p_z, p_x)}[1+\kappa(s)x]\mathbf{n}(s) \cdot \mathbf{E}_\parallel(\mathbf{R}_r(s) + x\mathbf{n}(s); \frac{s-z}{\beta_r})$$
$$- \frac{q}{P_r}[1+\kappa(s)x]B_\perp(\mathbf{R}_r(s) + x\mathbf{n}(s), \frac{s-z}{\beta_r}) \ , \tag{3.62}$$

where $' = d/ds$. The beam frame equations of motion (3.59-3.62) can be written compactly as

$$\zeta' = B(s, \zeta) \ , \tag{3.63}$$

where $\zeta = (z, x, p_z, p_x)^T$. Thus the beam frame Vlasov equation is

$$\partial_s f_B + B(s, \zeta) \cdot \nabla_\zeta f_B = 0, \tag{3.64}$$

where f_B is the beam frame phase space density and where we have made use of the fact that the vector field $B(s, \cdot)$ is divergence free.

Our field formula is in the lab frame so the lab charge and current densities must be determined from the beam frame phase-space density. The relation between the lab frame phase-space density, f_L, and the beam frame phase-space density, f_B, is

$$f_L(Z,X,P_Z,P_X;u) = \frac{\beta_r^2}{P_r^2} f_B(\hat{z}(\mathbf{R};u),\hat{x}(\mathbf{R}),\hat{p}_z(\mathbf{P}),\hat{p}_x(\mathbf{R},\mathbf{P});\hat{s}(\mathbf{R}))\ . \tag{3.65}$$

Here f_B is normalized, i.e.,

$$1 = \int_{\mathbb{R}^4} dz dp_z dx dp_x f_B(z,x,p_z,p_x;s)\ , \tag{3.66}$$

as is f_L in (3.41). Even though the derivation of (3.65) is somewhat subtle (see, e.g.,[StoT]) the end result is quite simple. To determine the charge density in terms of the beam frame phase space density we use (3.38) and (3.65) to obtain

$$\begin{aligned}\rho_L(\mathbf{R};u) &= Q\int_{\mathbb{R}^2} f_L(\mathbf{R},\mathbf{P};u)d\mathbf{P} \\ &= Q\int_{\mathbb{R}^2} \frac{(1+p_z)}{p_s(p_z,p_x)} f_B\big(\hat{z}(\mathbf{R},u),\hat{x}(\mathbf{R}),p_z,p_x;\hat{s}(\mathbf{R})\big)dp_z dp_x\ . \end{aligned}\tag{3.67}$$

To determine the current density in terms the beam frame phase space density we use (3.39) and (3.65) to obtain

$$\begin{aligned}\mathbf{J}_L(\mathbf{R};u) &= Q\int_{\mathbb{R}^2} \frac{\mathbf{P}}{m\gamma(\mathbf{P})} f_L(\mathbf{R},\mathbf{P};u)d\mathbf{P} \\ &= Q\beta_r c\int_{\mathbb{R}^2} \left[\mathbf{t}(\hat{s}(\mathbf{R})) + \frac{p_x}{p_s(p_z,p_x)}\mathbf{n}(\hat{s}(\mathbf{R}))\right] \\ &\qquad \cdot f_B\big(\hat{z}(\mathbf{R},u),\hat{x}(\mathbf{R}),p_z,p_x;\hat{s}(\mathbf{R})\big)dp_z dp_x\ . \end{aligned}\tag{3.68}$$

The formulas (3.67) and (3.68) are derived by substituting (3.65) and changing the variables of integration from P_Z, P_X to p_z, p_x. where the Jacobian is

$$\begin{aligned}\det\left[\frac{\partial\mathbf{P}}{\partial p_z},\frac{\partial\mathbf{P}}{\partial p_x}\right] &= P_r^2\det\left[\frac{\partial p_s}{\partial p_z}\mathbf{t},\frac{\partial p_s}{\partial p_x}\mathbf{t}+\mathbf{n}\right] \\ &= P_r^2\det\left[\frac{\partial p_s}{\partial p_z}\mathbf{t},\mathbf{n}\right] = P_r^2\frac{\partial p_s}{\partial p_z} = \frac{P_r^2}{\beta_r^2}\frac{1+p_z}{p_s(p_z,p_x)}\ . \end{aligned}\tag{3.69}$$

We define the beam frame spatial density ρ_B by

$$\rho_B(z,x;s) = \int_{\mathbb{R}^2} dp_z dp_x f_B(z,x,p_z,p_x;s)\ , \tag{3.70}$$

and the longitudinal beam frame spatial density ρ by

$$\rho(z;s) = \int_{\mathbb{R}} dx \rho_B(z,x;s) = \int_{\mathbb{R}^3} dx dp_z dp_x f_B(\zeta;s) \ . \tag{3.71}$$

Note that $\int_{\mathbb{R}^2} \rho_B(z,x;s) dz dx = 1$ and that $Q\rho_B$ is the beam frame charge density. For more background material on this section, see [MICRO, StoT].

3.2.2 Approximations

To approximate the inverse functions $\hat{z}, \hat{x}$ in a neighborhood of $\mathbf{R} = \mathbf{R}_r(s)$ we compute by Taylor expansion

$$\begin{pmatrix} \hat{s}(\mathbf{R}) - s \\ \hat{x}(\mathbf{R}) \end{pmatrix} = M^T(s)(\mathbf{R} - \mathbf{R}_r(s)) + \mathcal{O}(\kappa(s)\|\mathbf{R} - \mathbf{R}_r(s)\|^2) \ , \tag{3.72}$$

where

$$M(s) := [\mathbf{t}(s), \mathbf{n}(s)] \ . \tag{3.73}$$

To approximate the beam frame equations of motion (3.63) we linearize B w.r.t. z, x, p_z, p_x and use that $\gamma_r \ll 1$. Using also (see Assumption A of Chapter 2) that $\kappa(s)x \ll 1$ we obtain

$$\begin{aligned}
z' &= -\kappa(s)x \ , \quad x' = p_x \ , \\
p_z' &= \frac{q}{P_r c}[\mathbf{t}(s) + p_x \mathbf{n}(s)] \cdot \mathbf{E}_{\|}(\mathbf{R}_r(s) + x\mathbf{n}(s), (s-z)/\beta_r) \ , \\
p_x' &= \kappa(s)p_z + \frac{q}{P_r c}\big[\mathbf{n}(s) \cdot \mathbf{E}_{\|}(\mathbf{R}_r(s) + x\mathbf{n}(s), (s-z)/\beta_r) \\
&\qquad - cB_{\perp}(\mathbf{R}_r(s) + x\mathbf{n}(s), (s-z)/\beta_r)\big] \ .
\end{aligned} \tag{3.74}$$

The approximate equations of motion (3.74) have $\mathcal{F}_L(\mathbf{R}, u)$ evaluated at $\mathbf{R} = \mathbf{R}_r(s) + x\mathbf{n}(s)$ and $u = (s-z)/\beta_r$. Since it is inconvenient for numerical computations to compute $\mathcal{F}_L(\mathbf{R}_r(s) + x\mathbf{n}(s), (s-z)/\beta_r$ for different values of z, we perform the following additional approximations:

$$\begin{aligned}\mathcal{F}_L(\mathbf{R}_r(s) + x\mathbf{n}(s), (s-z)/\beta_r) &\approx \mathcal{F}_L(\mathbf{R}_r(s+z) + x\mathbf{n}(s+z), s)\\ &\approx \mathcal{F}_L(\mathbf{R}_r(s) + M(s)(z,x)^T, s) \ . \end{aligned} \tag{3.75}$$

At the first approximation we use the fact that the self field is slowly varying in s for fixed z, x (see Assumption B of Chapter 2) and that $\beta_r \approx 1$. The second approximation uses the fact that we are only interested in the self field in the bunch for z, x small, which again uses Assumption A of Chapter 2 and drops the O term in (3.72) giving us

$$\begin{aligned}\begin{pmatrix} s+z \\ x \end{pmatrix} &= \begin{pmatrix} \hat{s}(\mathbf{R}_r(s+z) + x\mathbf{n}(s+z)) \\ \hat{x}(\mathbf{R}_r(s+z) + x\mathbf{n}(s+z)) \end{pmatrix}\\ &\approx M^T(s)(\mathbf{R}_r(s+z) + x\mathbf{n}(s+z) - \mathbf{R}_r(s)) + \begin{pmatrix} s \\ 0 \end{pmatrix} . \end{aligned} \tag{3.76}$$

From (3.74) and (3.75) we obtain the approximate equations of motion

$$\begin{aligned} &z' = -\kappa(s)x \ , \qquad x' = p_x \ , \\ &p_z' = F_{z1}(\hat{\mathbf{R}}, s) + p_x F_{z2}(\hat{\mathbf{R}}, s) \ , \qquad p_x' = \kappa(s)p_z + F_x(\hat{\mathbf{R}}, s), \end{aligned} \tag{3.77}$$

where $\hat{\mathbf{R}} = \hat{\mathbf{R}}(z, x, s) = \mathbf{R}_r(s) + M(s)(z,x)^T$ and

$$\begin{aligned} &F_{z1}(\hat{\mathbf{R}}, s) = \frac{q}{P_r c}\mathbf{E}_\parallel(\hat{\mathbf{R}}, s)\cdot\mathbf{t}(s), \qquad F_{z2}(\hat{\mathbf{R}}, s) = \frac{q}{P_r c}\mathbf{E}_\parallel(\hat{\mathbf{R}}, s)\cdot\mathbf{n}(s) \ , \\ &F_x(\hat{\mathbf{R}}, s) = \frac{q}{P_r c}\left[\mathbf{E}_\parallel(\hat{\mathbf{R}}, s)\cdot\mathbf{n}(s) - cB_\perp(\hat{\mathbf{R}}, s)\right] . \end{aligned} \tag{3.78}$$

Including the self field we write the initial value problem for (3.77) as

$$\zeta' = A(s)\zeta + G(\zeta, s; \mathcal{F}_L), \tag{3.79}$$

$$\zeta(0) = \zeta_0 \ , \tag{3.80}$$

where

$$A(s) = \begin{pmatrix} 0 & -\kappa(s) & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \kappa(s) & 0 \end{pmatrix} . \tag{3.81}$$

The vector field defined by the rhs of (3.79) is divergence free, thus the beam frame Vlasov equation is

$$\partial_s f_B(\zeta; s) + (\nabla_\zeta f_B(\zeta; s)) \cdot [A(s)\zeta + G(\zeta, s; \mathcal{F}_L)] = 0 . \tag{3.82}$$

The equations of motion (3.79), without the self field, represent the Lorentz force in linearized form and they can be written as

$$\zeta' = A(s)\zeta , \qquad \zeta(0) = \zeta_0 . \tag{3.83}$$

Eq. (3.83) can be solved and the solution reads as $\zeta = \Phi(s, 0)\zeta_0$ where the principal solution matrix $\Phi(s, \tau)$ can be written in terms of the dispersion function,

$$D(s, \tau) = \int_\tau^s ds' \int_\tau^{s'} ds'' \kappa(s'') , \tag{3.84}$$

and the momentum compaction function

$$R_{56}(s, \tau) = - \int_\tau^s ds' \kappa(s') D(s', \tau) , \tag{3.85}$$

as

$$\Phi(s, \tau) = \begin{pmatrix} 1 & -D'(s, \tau) & R_{56}(s, \tau) & D(s, \tau) - (s - \tau)D'(s, \tau) \\ 0 & 1 & D(s, \tau) & s - \tau \\ 0 & 0 & 1 & 0 \\ 0 & 0 & D'(s, \tau) & 1 \end{pmatrix} , \tag{3.86}$$

where $D'(s, \tau) = \partial_s D(s, \tau)$. Note that without self field we have

$$f_B(\zeta; s) = f_B(\Phi(0, s)\zeta; 0) . \tag{3.87}$$

The approximate equations of motion in the interaction picture become

$$\zeta_0' = \Phi(0,s)G(\Phi(s,0)\zeta_0, s; \mathcal{F}_L). \tag{3.88}$$

We have found that it is numerically more efficient to integrate (3.88) than to integrate (3.77). Equations (3.67) and (3.68) lead by Taylor expansion of (3.58) in p_z, p_x to

$$\rho_L(\mathbf{R}; u) \approx Q\rho_B(\hat{z}(\mathbf{R}, u), \hat{x}(\mathbf{R}); \hat{s}(\mathbf{R})) \;, \tag{3.89}$$

$$\begin{aligned}\mathbf{J}_L(\mathbf{R}; u) \approx Q\beta_r c[&\rho_B(\hat{z}(\mathbf{R}, u), \hat{x}(\mathbf{R}); \hat{s}(\mathbf{R}))\mathbf{t}(\hat{s}(\mathbf{R})) \\ &+\tau_B(\hat{z}(\mathbf{R}, u), \hat{x}(\mathbf{R}); \hat{s}(\mathbf{R}))\mathbf{n}(\hat{s}(\mathbf{R}))] \;,\end{aligned} \tag{3.90}$$

where $\tau_B(z, x; s) = \int_{\mathbb{R}^2} p_x f_B(z, x, p_z, p_x; s) dp_z dp_z$. Using (3.72) gives us the approximation

$$\begin{pmatrix} \hat{z}(\mathbf{R}, u) \\ \hat{x}(\mathbf{R}) \end{pmatrix} = \begin{pmatrix} \hat{s}(\mathbf{R}) - \beta_r u \\ \hat{x}(\mathbf{R}) \end{pmatrix} \approx M^T(\beta_r u)(\mathbf{R} - \mathbf{R}_r(\beta_r u)) \;, \tag{3.91}$$

and using the fact that $\rho_B(z, x; s)$ has its support for z, x small, we have
$\rho_B(\hat{z}(\mathbf{R}, u), \hat{x}(\mathbf{R}); \hat{s}(\mathbf{R})) \approx \rho_B(M^T(\beta_r u)(\mathbf{R} - \mathbf{R}_r(\beta_r u)); \hat{s}(\mathbf{R}))$. Using also the fact that $f_B(z, x, p_z, p_x; s)$ is slowly varying in s we have
$\rho_B(\hat{z}(\mathbf{R}, u), \hat{x}(\mathbf{R}); \hat{s}(\mathbf{R})) \approx \rho_B(M^T(\beta_r u)(\mathbf{R} - \mathbf{R}_r(\beta_r u)); \beta_r u)$. With similar approximations of the current density we thus arrive at

$$\rho_L(\mathbf{R}; u) \approx Q\rho_B(M^T(\beta_r u)(\mathbf{R} - \mathbf{R}_r(\beta_r u)); \beta_r u) \;, \tag{3.92}$$

$$\begin{aligned}\mathbf{J}_L(\mathbf{R}; u) \approx Q\beta_r c\big[&\rho_B(M^T(\beta_r u)(\mathbf{R} - \mathbf{R}_r(\beta_r u)); \beta_r u)\mathbf{t}(\beta_r u) \\ &+\tau_B(M^T(\beta_r u)(\mathbf{R} - \mathbf{R}_r(\beta_r u)); \beta_r u)\mathbf{n}(\beta_r u)\big] \;.\end{aligned} \tag{3.93}$$

For more background material on this section, see [MICRO, StoT].

3.3 The FERMI@Elettra first bunch compressor system

We studied the microbunching instability in great detail for the FERMI@Elettra first bunch compressor system [MICRO]. This bunch compressor system was proposed as a benchmark for testing codes. The complete layout of the system is shown in Fig. 1. The system consists of a 4-dipole chicane between rf cavities and quadrupoles. The initial beam frame phase-space density to be

$$f_B(z, x, p_z, p_x; 0) = (1 + \varepsilon(z))a_0(z, x, p_z, p_x) \ , \tag{3.94}$$

where

$$a_0(z, x, p_z, p_x) = \mu(z)\rho_c(p_z - hz)\rho_t(x, p_x) \ , \tag{3.95}$$

$$\mu(z) = \frac{\alpha}{4a}[\tanh((z + a)/b) - \tanh((z - a)/b)] \ , \tag{3.96}$$

$$\rho_c(p_z) = \exp[-p_z^2/2\sigma_u^2]/\sqrt{2\pi}\sigma_u \ , \tag{3.97}$$

$$\rho_t(x, p_x) = \exp[-(x^2 + (\alpha_0 x + \beta_0 p_x)^2)/2\epsilon_0\beta_0]/2\pi\epsilon_0 \ , \tag{3.98}$$

$$\varepsilon(z) = A\cos(2\pi z/\lambda_0) = A\cos(k_0 z) \ . \tag{3.99}$$

Note that the 'linear energy chirp' parameter h in (3.95) is unrelated to the shielding parameter $h = 2g$ in Fig. 2. The smooth a_0 is perturbed by a modulation, ε, with wave length λ_0 and small amplitude A. The purpose of α is to normalize f_B, as demanded by (3.66). However, since it is a good approximation, we use $\alpha = 1$ in our computations. Taking the limit $b \to 0+$ in (3.96) we get $\mu \to \mu_0$ where $\mu_0(z) = (\alpha/2a)I_{(-a,a)}(z)$. The function μ_0 is a rough pointwise approximation to μ, so that the bunch length is $\approx 2a$. We use the smooth μ instead of μ_0 because the discontinuous μ_0 gives rise to a Gibbs phenomenon which causes problems in our numerics. Due to (3.71),(3.94),(3.95),(3.97), (3.98) the initial longitudinal spatial density is

$$\rho(z; 0) = (1 + \varepsilon(z))\mu(z) \ , \tag{3.100}$$

whence

$$f_B(z,x,p_z,p_x;0) = \rho(z;0)\rho_c(p_z - hz)\rho_t(x,p_x)\ . \tag{3.101}$$

The density $\rho_c(p_z - hz)$ contains the linear energy chirp which is created by 'off-crest rf acceleration' such that particles in front of the reference particle gain less energy than particles behind the reference particle. This creates the correlation needed for bunch compression. To discuss bunch compression we define

$$C(s) := [1 + hR_{56}(s,0)]^{-1}, \quad C_f := C(s_f)\ , \quad R_f := R_{56}(s_f,0)\ , \tag{3.102}$$

where R_{56} is defined by (3.85). Note that $C(s) > 0$ for $s \in [0, s_f]$ and that, by (3.85), $C(0) = 1$. Recalling that s_f is the s-value at the end of the chicane, we conclude from (3.71),(3.87),(3.94) that

$$\begin{aligned}
\rho(z;s_f) &= \int_{\mathbb{R}^3} dp_z dx dp_x f_B(\Phi(0,s_f)\zeta;0) \\
&= \int_{\mathbb{R}^3} dp_z dx dp_x f_B(z - R_f p_z, x - s_f p_x, p_z, p_x; 0) \\
&= \int_{\mathbb{R}} dp_z \rho(z - R_f p_z; 0)\rho_c(p_z/C_f - hz) \\
&= C_f \int_{\mathbb{R}} dy \rho\Big(C_f(z - R_f y); 0\Big)\rho_c(y)\ ,
\end{aligned} \tag{3.103}$$

where in the second equality we used the fact that $D(0,s_f) = 0 = D'(0,s_f)$. It is easy to check that $\rho(\cdot;s_f)$ in (3.103) is even and so its first moment is zero. A short calculation shows that the second moment of $\rho(\cdot;s_f)$ is equal to $1/C_f^2$ times the second moment of $\rho(\cdot;0)$ plus the term $R_f^2\sigma_u^2$. For our parameters (see below) σ_u is so small that we have, to very good approximation,

$$\rho(z;s_f) = C_f\rho(C_f z;0)\ . \tag{3.104}$$

This is just (3.103) with ρ_c replaced by the delta function. The approximation (3.104) of (3.103) clearly shows the compression and the meaning of the term 'compression factor $C(s)$'. We limited our study in [MICRO] to the chicane with the following parameter values:

$$\begin{aligned}
&\text{Energy of reference particle} : E_r = 233MeV \\
&\text{Peak current} : I = 120A \\
&\text{Bunch charge} : Q = 1nC \\
&\text{Normalized transverse emittance} : \gamma\epsilon_0 = 10^{-6}m \\
&\text{Alpha function} : \alpha_0 = 0 \\
&\text{Beta function} : \beta_0 = 10m \\
&\text{Linear energy chirp} : h = -12.6m^{-1} \\
&\text{Uncorrelated energy spread} : \sigma_E = 2KeV \\
&\text{Momentum compaction} : R_{56}(s_f, 0) = 0.057m \\
&\text{Radius of curvature} : r_0 = 5m \\
&\text{Magnetic length} : L_b = 0.5m \\
&\text{Distance 1st} - \text{2nd}, \text{3rd} - \text{4th bend} : L_1 = 2.5m \\
&\text{Distance 2rd} - \text{3nd bend} : L_2 = 1m
\end{aligned} \tag{3.105}$$

The external magnetic field $\bar{\mathbf{B}}_{ext}$ is approximated by a hard edge model whence $B_{ext}(Z)$ in (3.3) is approximated by a step function of Z. Thus the lengths L_1, L_2 and L_b in (3.105) are in terms of the lab frame variable Z and the total length of the chicane is 8m. The total arc length traversed by the reference particle is $s_f = 8.029$m and the compression factor at s_f is $C(s_f) = (1+hR_{56}(s_f, 0))^{-1} = 3.545$. The uncorrelated energy spread $\sigma_E = 2KeV$ gives $\sigma_u = \sigma_E/E_r = 8.6 \cdot 10^{-6}$ whence (3.104) is a good approximation in the case without self field. In the calculations we vary λ_0 and take $A = .05$, $a = 1180\mu$m and $b = 150\mu$m.

Next we present a typical density plot (with self field) computed by our code. In

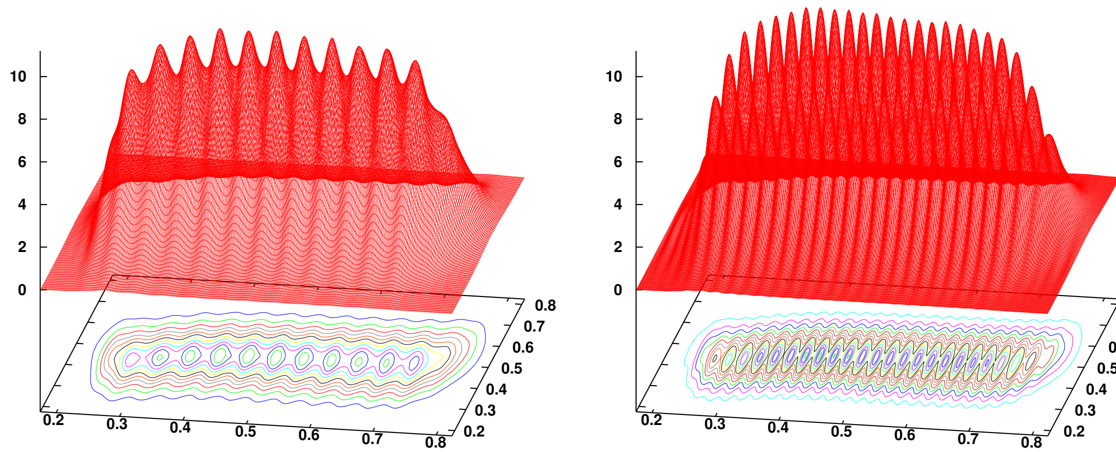
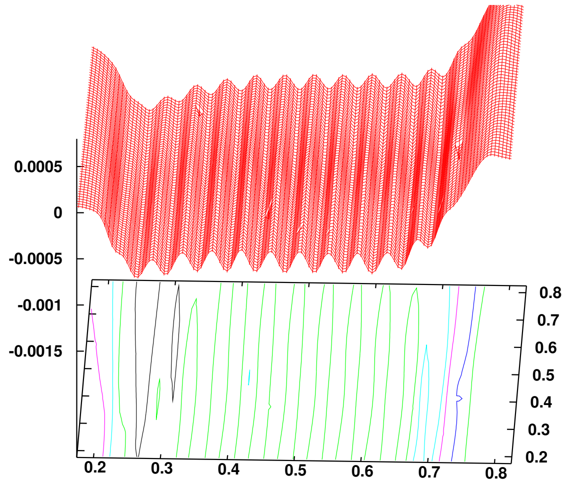


Figure 5: Spatial density in grid coordinates (x_1, x_2) at $s = s_f$ for $\lambda_0 = 200\mu$m (left) and $\lambda_0 = 100\mu$m (right).

fact, Fig. 5 shows the spatial density in grid coordinates (x_1, x_2) for $\lambda_0 = 200\mu$m (left) and $\lambda_0 = 100\mu$m (right) at $s = s_f$ (the grid coordinates are explained in Section 3.4). Here we simply state that we are able to calculate accurately this 2D spatial density, the basic quantity in our 4D+2D Vlasov-Maxwell system. In Fig. 6 we show the longitudinal force F_{z1} from (3.78), proportional to $\mathbf{E}_{\parallel}(\cdot, s) \cdot \mathbf{t}(s)$, at $s = s_f$ for $\lambda_0 = 200\mu$m (left) and 100μm (right). Notice that the maximum intensity of F_{z1} increases as λ_0 decreases.

The results are obtained in the free space case; i.e., neglecting shielding effects from the vacuum chamber. In our simulations of the FERMI@Elettra first bunch compressor system we noticed that τ_B in (3.93) has a negligible effect therefore we ignored its contribution.



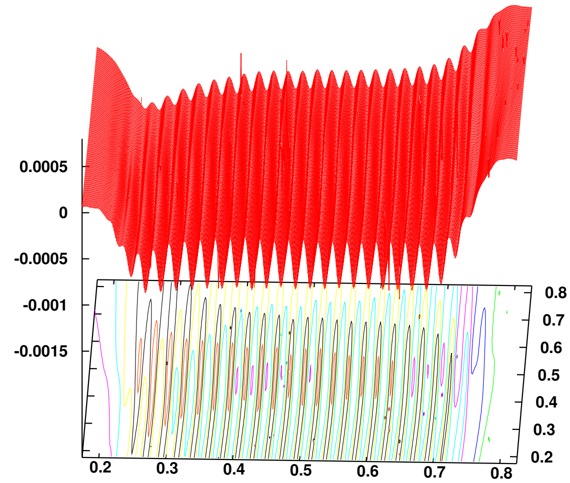


Figure 6: Longitudinal force in grid coordinates (x_1, x_2) at $s = s_f$ for $\lambda_0 = 200\mu$m (left) and $\lambda_0 = 100\mu$m (right).

3.4 The Monte Carlo particle method

3.4.1 Generalities

We have discussed our method for calculating the self field in the lab frame and the determination of the lab frame charge and current densities from the beam frame phase-space density. Here we discuss a method of solution of the coupled Vlasov-Maxwell system similar to traditional particle methods, variously called 'particle-in-cell (PIC)' or 'macro-particle methods'. We call it the 'Monte Carlo particle (MCP) method', because it uses a Monte Carlo method to determine a smooth charge distribution from an ensemble of particles. The MCP is self-consistent in the sense that it takes into account the interaction between the bunch and its self field.

Before we developed the MCP method we considered solving the Vlasov equation using the method of local characteristics (or 'semi-Lagrangian method'), which has been extremely effective in problems with a 2D phase space. This deals with the

Vlasov equation in a very direct way, defining the phase-space density by its values on a grid with interpolation to off-grid points. The density is updated by integrating backward from grid points, with the collective force regarded as constant during a time step. Since the backward orbits land at off-grid points, this update requires interpolation. In comparison with usual particle methods, this method offers much lower noise and the possibility of a relatively direct control of accuracy by monitoring interpolation error. On the other hand, it is relatively expensive in computation time and memory, and in the case of a chicane it is technically complicated because the density is concentrated in a narrow region of phase space that evolves in time in a manner that is not known *a priori* [Li]. We are studying ways to deal with this evolving support, since it would be inefficient to use many grid points where the density is negligibly small. Possible techniques include changes of variable [VWZ], an evolving selection of fiducial grid points, and the use of forward characteristics rather than backward [EPAC08-2]. Although we have high hopes for success in this direction, at present we stick to the more modest goal of improving the particle method, in which it is much easier to deal with the support question since one has to work only with the charge density in 2D rather than the phase-space density in 4D. In particle methods the connection to the Vlasov equation is unfortunately indirect, and the control of accuracy relies entirely on the experiment of increasing the number of particles. Even if one believes that a solution of the Vlasov equation is obtained in the limit, it is usually too expensive to make a convincing empirical demonstration of convergence.

An essential ingredient of the MCP method is the technique of density estimation since, when marching forward in time, for every update of the sample of points the charge/current density has to be updated. We consider three different methods of density estimation.

One approach ('Method 1') to density estimation is based on orthogonal series and we have studied the Fourier series case in some detail following [Ef]. Here ρ_L and $\mathbf{J}_L$ are obtained at every s step by computing the Fourier coefficients of the truncated Fourier series via Monte Carlo integration of the sample of phase-space points. Details are also given in [MICRO] and [PAC07-2]. The computational effort is $O(\mathcal{N} J_1 J_2) + O(N_1 N_2 J_1 J_2)$, where $\mathcal{N}$ is the number of simulated points, J_1, J_2 are the number of Fourier coefficients, and N_1, N_2 are the number of grid points in x_1, x_2 respectively (the grid coordinates x_1, x_2 are explained further below). Typical values in our microbunching simulations are $\mathcal{N} = 5 \times 10^8$, $J_1 = 150$ and $J_2 = 50$. Therefore the computational effort is $O(10^{12})$ and is of the same order as the computational effort for the polar coordinate field calculation discussed in item 2 below. Method 1 is done in parallel as outlined in item 1 below.

A second approach ('Method 2') employs cloud in cell charge deposition where at every s step the sample is placed on our fixed grid (see [BPT] and Section 3.5 of [Si]). Here ρ_L and $\mathbf{J}_L$ are obtained by computing the Fourier coefficients of the truncated Fourier series by a simple quadrature. The computational effort in this case is $O(\mathcal{N}) + O(N_1 N_2 J_1 J_2)$. We have found that using $\mathcal{N}, J_1, J_2$ as above, $N_1 = 1000$ and $N_2 = 128$, gives the same approximation as for Method 1. This computational effort of $O(10^9)$ is much smaller than for the orthogonal series method and negligible with respect to the computational effort for the polar coordinate field calculation (for the latter, see item 2, below). In Method 2 the Fourier coefficients are computed in parallel by partitioning the $\mathcal{N}$ scattered phase space points into N_p groups where N_p is the number of processors and the quadrature is done in parallel by partitioning the grid into N_p groups.

A third approach ('Method 3') applies the kernel density estimation technique to the sample. This approach is still in the testing phase where we are investigating standard kernels like bivariate Gaussians or bivariate Epanechnikov kernels (all with a uniform bandwith, H). The computational effort for the bivariate product

Epanechnikov kernel is $O(\mathcal{N}\tilde{N}_1\tilde{N}_2)$, where $\mathcal{N}$ is as before but now $\tilde{N}_1\tilde{N}_2$ is the number of grid points inside the square of diameter $2H$ centered at the scattered particle position (x_1, x_2). For $\mathcal{N} = 5 \times 10^8, N_1 = 1000, N_2 = 128$ we approximately get $\tilde{N}_1 = 24, \tilde{N}_2 = 3$, $O(\mathcal{N}\tilde{N}_1\tilde{N}_2) = O(10^{10})$. Thus this method is comparable in speed to Method 2 and is worthy of further investigation (for further details on the kernel density estimation technique, see Sections 3.4.3 and A.3). In the kernel density estimator method the densities are computed in parallel by partitioning the $\mathcal{N}$ scattered phase space points into N_p groups where N_p is the number of processors.

For all three density estimation methods, the initial sample is generated from pseudo-random numbers [Ca, Ni] by using the Acceptance-Rejection method [Ros], assuming particles are independent identically distributed according to the initial phase-space density.

In density estimation Methods 1 and 2 we represent the charge/current density in the beam frame as a truncated Fourier series, thus giving ourselves a density that is smooth, of class C^∞. Ideally one would use the resulting Fourier series and its gradient to evaluate the source in the field formula. That is too expensive, however, since it involves multiple summations of the Fourier series, at points not amenable to the fast Fourier transform. Instead, we use the Fourier series to put the density and its gradient on a grid, and then use low order polynomial interpolation for evaluations at off-grid points. Thus we accomplish something similar to particle-in-cell codes, but by a different route, and get the gradient as well as the density itself at grid points. Our method gives low noise, but is costly at high levels of resolution. We have not yet carried out a careful comparison with more usual methods at similar levels of cost and resolution.

It is cost effective to make an s-dependent coordinate transformation so that the 2D spatial density can be accurately represented in a grid which does not depend on s. Since in our studies the uncorrelated energy spread σ_u and the spread in the transverse momentum σ_{px_0} at entrance of the chicane are small, we found that the coordinate transformation $(z,x) \leftrightarrow (\tilde{z},\tilde{x})$ via

$$z = (1 + hR_{56}(s,0))\tilde{z} - D'(s,0)\tilde{x}\ , \qquad x = hD(s,0)\tilde{z} + \tilde{x} \tag{3.106}$$

gives an almost stationary situation, where particles are at rest in the limit of no self field, and $\sigma_u = 0$ and $\sigma_{px_0} = 0$. The chirp parameter h and D, R_{56} were introduced in Section 3.2. The transformation (3.106) is obtained solving (3.79) without self field, i.e., by solving (3.83) and with initial conditions $z = \tilde{z}$, $p_z = h\tilde{z}$, $x = \tilde{x}$, and $p_x = 0$. Since we do density estimation in the unit square $[0,1] \times [0,1]$, our final grid transformation $(\tilde{z},\tilde{x}) \leftrightarrow (x_1,x_2)$ is obtained by a simple scaling and translation.

We now describe our algorithm more concretely and to be specific we choose Method 1 of the density estimation. Since the reference particle corresponds to $z = 0$ and since $z = s - \beta_r u$, the reference particle arrives at the chicane entrance at $u = 0$. At $s = 0$ our bunch effectively has z supported in $(-a,a)$ where the longitudinal size parameter a was explained in Section 3.2. Thus the particle at the head of the bunch arrives at $s = 0$ at the time $-a/\beta_r$ and we take the latter to be u_0 whence at $u = u_0$ the particles have s coordinates in the interval $(-2a,0)$. The field formulas (3.50),(3.51) can now be applied (we here confine to (3.51)).

For a small step $s \rightarrow s + \Delta s$ we proceed as follows:

1. Denoting ρ_B, τ_B in the grid coordinates (x_1,x_2) by ρ_g, τ_g respectively, we expand $\rho_g(x_1,x_2;s)$ and $\tau_g(x_1,x_2;s)$ in a finite Fourier series

$$\rho_g(x_1,x_2;s) = \sum_{i=0}^{J_1}\sum_{j=0}^{J_2} \theta_{ij}(s)\phi_i(x_1)\phi_j(x_2), \tag{3.107}$$

$$\tau_g(x_1,x_2;s) = \sum_{i=0}^{J_1}\sum_{j=0}^{J_2} \Theta_{ij}(s)\phi_i(x_1)\phi_j(x_2), \tag{3.108}$$

where

$$\theta_{ij}(s) = \int_A dx_1 dx_2 \phi_i(x_1)\phi_j(x_2)\rho_g(x_1,x_2;s), \tag{3.109}$$

$$\Theta_{ij}(s) = \int_A dx_1 dx_2 \phi_i(x_1)\phi_j(x_2)\tau_g(x_1,x_2;s). \tag{3.110}$$

Here $\{\phi_i\}$ is the orthonormal basis $\phi_0(x) = 1$ and $\phi_i(x) = \sqrt{2}\cos(i\pi x)$ for $i \geq 1$, $x \in [0,1]$. Note that ρ_g is now the actual spatial density, in the coordinates x_1, x_2, with nonzero σ_u and $\sigma_{p_{x0}}$ and with self field.

Since ρ_g is a probability density the Fourier coefficients θ_{ij} may be written as the expected value E of $\phi_i(X_1)\phi_j(X_2)$ with respect to $\rho_g(\cdot;s)$

$$\begin{aligned}\theta_{ij}(s) &= E\{\phi_i(X_1)\phi_j(X_2)\}\\ &= \int_A dx_1 dx_2 \phi_i(x_1)\phi_j(x_2)\rho_g(x_1,x_2;s),\end{aligned} \tag{3.111}$$

where $X = (X_1, X_2)$ is the random variable with probability density ρ_g. To estimate τ_g, which is not a probability density, we notice that the Fourier coefficients Θ_{ij} may be written as the expected value E of $\phi_i(X_1)\phi_j(X_2)P_X$ with respect to $f_g(\cdot;s)$

$$\begin{aligned}\Theta_{ij}(s) &= E\{\phi_i(X_1)\phi_j(X_2)P_X\}\\ &= \int_A dx_1 dx_2 \int_{\mathbb{R}^2} dp_z dp_x \phi_i(x_1)\phi_j(x_2)p_x\\ &\times f_g(x_1,x_2,p_z,p_x;s),\end{aligned} \tag{3.112}$$

where $X = (X_1, X_2, P_Z, P_X)$ is the random variable with probability density $f_g(\cdot;s)$.

It follows that the natural estimate of E is the sample mean, i.e., we have the following two Monte Carlo formulas:

$$\theta_{ij}(s) \approx \frac{1}{\mathcal{N}}\sum_{n=1}^{\mathcal{N}} \phi_i(X_{1n})\phi_j(X_{2n}), \tag{3.113}$$

$$\Theta_{ij}(s) \approx \frac{1}{\mathcal{N}}\sum_{n=1}^{\mathcal{N}} \phi_i(X_{1n})\phi_j(X_{2n})P_{Xn}, \tag{3.114}$$

where a realization of the random variable $X = (X_1, X_2, P_Z, P_X)$ is obtained from beam frame scattered phase-space points z_i, x_i, p_{zi}, p_{xi} at s, i=1,..,$\mathcal{N}$ (via the transformation: $(z_i, x_i, p_{z_i}, p_{x_i}) \to (x_{1i}, x_{2i}, p_{z_i}, p_{x_i})$). The Monte Carlo computation is done in parallel, i.e., the sums in (3.113),(3.114) are each split into N_p pieces where N_p is the number of processors. In other words, each processor only computes the sum over $\mathcal{N}/N_p$ terms in (3.113),(3.114).

2. The force fields $\mathbf{E}_\parallel(\cdot, s)\cdot\mathbf{t}(s), \mathbf{E}_\parallel(\cdot, s)\cdot\mathbf{n}(s), F_x(\cdot, s)$, which are needed in (3.78), are computed by using the s-independent grid defined above. That is, given a grid point (x_1, x_2), we compute the associated beam frame values z and x, then compute $\mathbf{R} = \mathbf{R}_r(s) + M(s)(z, x)^T$. The force fields can then be determined at these $\mathbf{R}$-values from $\mathcal{F}_L(\mathbf{R}, s)$. Using (3.51) we have

$$\mathcal{F}_L(\mathbf{R}, s) = -\frac{1}{4\pi}\int_{u_0}^{s} dv 1_{[u_0,\infty)}(v) \int_{-\pi}^{\pi} d\theta \mathbf{S}(\tilde{\mathbf{R}}(\theta, v; s), v) \ , \tag{3.115}$$

where

$$\tilde{\mathbf{R}}(\theta, v; s) = \mathbf{R} + (s - v)\mathbf{e}(\theta) \ . \tag{3.116}$$

Here we have considered the nonshielding scenario since often the shielding effect is not important. For some designs shielding could well play a role, so our code allows it to be included.

To do the double integral in (3.115) we apply a Gauss-Kronrad adaptive algorithm to the outer integral. Gauss-Kronrod picks a v and then we determine the θ support, $(\theta_{min}, \theta_{max})$. The inner θ integral is then done with the trapezoidal rule on a uniform mesh. For each point $(\tilde{\mathbf{R}}(\theta, v; s), v)$ of demand the source value $\mathbf{S}(\tilde{\mathbf{R}}(\theta, v; s), v)$ is determined by a tri-quadratic interpolation of $\mathbf{S}$-values. We notice that the Fourier method of item 1 not only gives an analytical representation at s of ρ_g and τ_g but of $\nabla\rho_g$ and $\nabla\tau_g$ as well. A representation of $\partial\rho_g/\partial s$ and $\partial\tau_g/\partial s$ is obtained by differentiating the Fourier coefficients with a finite difference scheme. Even though it is possible to construct the source term

S by storing the 'history' of the Fourier coefficients, i.e. θ_{ij} and Θ_{ij}, $d\theta_{ij}/ds$ and $d\Theta_{ij}/ds$ on a grid in s, we found it is more efficient to store ρ_g, $\nabla\rho_g$ and $\partial\rho_g/\partial s$ (the same for τ_g) on a 3D grid in (x_1, x_2, s). We use a uniform grid in (x_1, x_2, s) with $N_1 N_2$ grid points in (x_1, x_2).

The computational effort for the calculation of one component of the self field is $O(N_1 N_2 N_v N_\theta)$, where N_v is the number of evaluations for the v integration, and N_θ is the number of evaluations for the θ integration. Typical values for our simulations in [MICRO] are $N_1 = 1000$, $N_2 = 128$, $N_v = N_\theta = 1000$, therefore $O(N_1 N_2 N_v N_\theta) = O(10^{12})$. Note that the field computation is done in parallel by letting each processor compute $\mathbf{E}_\parallel(\mathbf{R}, s) \cdot \mathbf{t}(s), \mathbf{E}_\parallel(\mathbf{R}, s) \cdot \mathbf{n}(s), F_x(\mathbf{R}, s)$ for only $N_1 N_2 / N_p$ points $\mathbf{R}$ where N_p is the number of processors.

3. We use item 2 to push the particles in the interaction picture of (3.88). This allows us to use an Euler scheme where the integration step Δs is determined by the strength and smoothness of the self field. The force fields have been calculated by using a grid in (x_1, x_2) as outlined in item 2 above. To calculate the fields at particle positions needed in (3.78) we use a bi-quadratic interpolation. The particle pushing is done in parallel, i.e., each processor only pushes $\mathcal{N}/N_p$ particles.

4. The procedure is iterated going back to item 1.

3.4.2 Causality issue

Because the code marches forward in s there is a causality issue as follows. First of all one notes by (3.115) that $\mathcal{F}_L(\mathbf{R}, s)$ is, as one also expects from relativity, only affected by source values $\mathbf{S}(\tilde{\mathbf{R}}, v)$ for which $(\tilde{\mathbf{R}}, v)$ lie on the backward lightcone, $L(\mathbf{R}, s)$, of $(\mathbf{R}, s)$ which is defined by

$$L(\mathbf{R}, s) := \{(\mathbf{R}', s') \in \mathbb{R}^3 : |\mathbf{R} - \mathbf{R}'| = s - s'\} . \tag{3.117}$$

In particular according to the discussion in item 2 of Section 3.4.1, the self field values, which are needed in (3.78) when the algorithm is at s, are affected only by source values $\mathbf{S}(\tilde{\mathbf{R}}, v)$ for which $(\tilde{\mathbf{R}}, v)$ lie on $L(\mathbf{R}_r(s) + M(s)(z,x)^T, s)$. It is easy to see that $L(\mathbf{R}_r(s) + M(s)(z,x)^T, s)$ contains points $(\mathbf{R}', s')$ for which $\hat{s}(\mathbf{R}') < s$ and $\mathbf{S}(\mathbf{R}', s') \neq 0$ and points $(\mathbf{R}', s')$ for which $\hat{s}(\mathbf{R}') > s$ and $\mathbf{S}(\mathbf{R}', s') \neq 0$. Obviously the points $(\mathbf{R}', s')$ in $L(\mathbf{R}_r(s) + M(s)(z,x)^T, s)$ for which $\hat{s}(\mathbf{R}') > s$ and $\mathbf{S}(\mathbf{R}', s') \neq 0$ raise a causality issue. Nevertheless we believe that in general the causality issue is not serious since, as one can easily see, for points $(\mathbf{R}', s')$ in $L(\mathbf{R}_r(s) + M(s)(z,x)^T, s)$ for which $\hat{s}(\mathbf{R}') > s$ and $\mathbf{S}(\mathbf{R}', s') \neq 0$ we have that $\hat{s}(\mathbf{R}') - s$ is less or equal to the z-size of the bunch. For more details on the causality issue, see [ICAP09, StoT].

Another aspect of the backward light cone is the fact that for points $(\mathbf{R}', s')$ in $L(\mathbf{R}_r(s) + M(s)(z,x)^T, s)$ for which $\hat{s}(\mathbf{R}') < s$ and $\mathbf{S}(\mathbf{R}', s') \neq 0$ the difference $s - \hat{s}(\mathbf{R}')$ can be much bigger than the z-size of the bunch. For example for the FERMI@Elettra first bunch compressor system and with the parameter values (3.105) we have the situation that the reference particle at the end of a dipole is subjected to a collective force which is even affected by the bunch at the *entrance* into that dipole. Thus indeed the code has to store a lot of history of the bunch in order to compute the collective force, see also item 2 of Section 3.4.1.

3.4.3 Convergence study

I now discuss a technique which allows a convergence study of the error of various quantities, computed by the code. We here concentrate on a convergence study w.r.t. the parameter $\mathcal{N}$, i.e., the particle number. I also present two applications of this technique to the spatial density, i.e., I present some results we got by density estimation via Method 3 (=kernel density estimation) resp. Method 2 (=cloud in cell charge deposition). For more details on density estimation, see Sections 3.4.1 and A.3.

I now outline the technique (for more details, see Section A.4). Let Ψ be a normed space and let $\psi \in \Psi$ be an unknown element approximated by the elements $\psi(\mathcal{N}) \in \Psi$ where $\psi(\mathcal{N})$ denotes the approximant of ψ computed with $\mathcal{N}$ particles. Underlying the technique is the assumption that, for $\mathcal{N} \to \infty$, the error $||\psi - \psi(\mathcal{N})||$ satisfies

$$||\psi - \psi(\mathcal{N})|| = \mathcal{O}(\mathcal{N}^{-d}) \ , \tag{3.118}$$

where $d > 0$ is called the 'consistency order' of the approximant $\psi(\mathcal{N})$. Thus, by assumption, a $c > 0$ exists such that for large $\mathcal{N}$ we have

$$||\psi - \psi(\mathcal{N})|| \approx c\mathcal{N}^{-d} \ . \tag{3.119}$$

In fact the technique we outline here allows to approximate d by $\tilde{d}$ where

$$\tilde{d} := \frac{1}{\ln(\mathcal{N}_2/\mathcal{N}_1)} \ln(\frac{||\psi(\mathcal{N}_1) - \psi(\mathcal{N}_3)||}{||\psi(\mathcal{N}_2) - \psi(\mathcal{N}_4)||}) \ , \tag{3.120}$$

and where the particle numbers $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4$ are supposed to be sufficiently large such that (3.119) is a good approximation for $\mathcal{N} = \mathcal{N}_i$ $(i = 1, 2, 3, 4)$. Choosing, in addition, $\mathcal{N}_3/\mathcal{N}_1$ and $\mathcal{N}_4/\mathcal{N}_2$ sufficiently large, we obtain for the relative error, $|1 - \tilde{d}/d|$, of $\tilde{d}$ that

$$|1 - \frac{\tilde{d}}{d}| \lessapprox \frac{(\mathcal{N}_3/\mathcal{N}_1)^{-d} + (\mathcal{N}_4/\mathcal{N}_2)^{-d}}{d \ln(\mathcal{N}_2/\mathcal{N}_1)} \ . \tag{3.121}$$

Thus $\tilde{d}$ is a good approximation of d if $\mathcal{N}_3/\mathcal{N}_1$ and $\mathcal{N}_4/\mathcal{N}_2$ are sufficiently large. This rule may be followed in practice and it does not involve aprori knowledge of d. In fact, if one does not know d apriori, then one may apply (3.120) for different sets of $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4$ and may stop when $\tilde{d}$ begins to converge to some fixed value, d. Of course, due to (3.121), choosing also $\mathcal{N}_2/\mathcal{N}_1$ large, may further improve $\tilde{d}$. To discuss in more detail the quality of $\tilde{d}$, we note that if one imposes, for some $\varepsilon > 0$, the condition:

$$\frac{(\mathcal{N}_3/\mathcal{N}_1)^{-d} + (\mathcal{N}_4/\mathcal{N}_2)^{-d}}{d \ln(\mathcal{N}_2/\mathcal{N}_1)} \leq \varepsilon \ , \tag{3.122}$$

then by (3.121) one obtains

$$|1 - \frac{\tilde{d}}{d}| \lessapprox \varepsilon \ . \tag{3.123}$$

It is convenient to restrict the choice of $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4$ to

$$\mathcal{N}_2 = k_1 \mathcal{N}_1 \ , \qquad \mathcal{N}_3 = k_2 \mathcal{N}_1 \ , \qquad \mathcal{N}_4 = k_1 k_2 \mathcal{N}_1 \ , \qquad k_2 \geq k_1 > 1 \ , \tag{3.124}$$

which leaves $\mathcal{N}_1, k_1, k_2$ as the only free parameters. Note that k_1, k_2 are not necessarily integers and that (3.124) gives us the ordering

$$\mathcal{N}_4 > \mathcal{N}_3 \geq \mathcal{N}_2 > \mathcal{N}_1 \ . \tag{3.125}$$

Thus for (3.124) the particle numbers $\mathcal{N}_1, \mathcal{N}_4$ are the smallest resp. largest whence the minimization of $\mathcal{N}_4/\mathcal{N}_1$ under the condition (3.122) is an important issue here. With (3.124) the condition (3.122) reads as

$$\frac{2k_2^{-d}}{d \ln(k_1)} \leq \varepsilon \ , \tag{3.126}$$

whence, for (3.124), the condition (3.126) leads to (3.123).

We now need more detail for two special cases of (3.124), Choices 1 and 2. Choice 1 is that special case of (3.124) which minimizes $\mathcal{N}_4/\mathcal{N}_1$ under the condition (3.126) and Choice 2 is that special case of (3.124) which minimizes $\mathcal{N}_4/\mathcal{N}_1$ under the conditions (3.126) and $k_1 = 2$. We begin with Choice 1. Since $\mathcal{N}_4/\mathcal{N}_1 = k_1k_2$ we have to minimize k_1k_2 whence we define

$$(k_1k_2)_{opt} := \inf\{k_1k_2 : k_2 \geq k_1 > 1, \frac{2k_2^{-d}}{d\ln(k_1)} \leq \varepsilon\} \ . \tag{3.127}$$

Choice 1 branches into Choice 1a where

$$\varepsilon < \frac{2}{e} \ , \tag{3.128}$$

and Choice 1b where

$$\varepsilon \geq 2/e \ . \tag{3.129}$$

For Choice 1a we obtain from (3.127)

$$(k_1k_2)_{opt} = (k_1)_{opt}(k_2)_{opt} = (\frac{2e}{\varepsilon})^{1/d} \ , \tag{3.130}$$

$$(k_1)_{opt} := \exp(1/d) \ , \qquad (k_2)_{opt} := (\frac{2}{\varepsilon})^{1/d} \ . \tag{3.131}$$

Note that, for Choice 1a,

$$\mathcal{N}_4 > \mathcal{N}_3 > \mathcal{N}_2 > \mathcal{N}_1 \ . \tag{3.132}$$

For Choice 1b we obtain from (3.127)

$$(k_1k_2)_{opt} = (k_1)_{opt}(k_2)_{opt} \ , \tag{3.133}$$

$$(k_1)_{opt} = (k_2)_{opt} \ , \tag{3.134}$$

where $(k_1)_{opt}$ is the unique solution of

$$\frac{2(k_1)_{opt}^{-d}}{d\ln((k_1)_{opt})} = \varepsilon \ , \qquad (k_1)_{opt} > 1 \ . \tag{3.135}$$

Note that, for Choice 1b,

$$\mathcal{N}_4 > \mathcal{N}_3 = \mathcal{N}_2 > \mathcal{N}_1 \ . \tag{3.136}$$

We now consider Choice 2. We thus define

$$(k_1k_2)_{opt,k_1=2} := \inf\{2k_2 : k_2 \geq 2, \frac{2k_2^{-d}}{d\ln(2)} \leq \varepsilon\} . \tag{3.137}$$

Choice 2 branches into Choice 2a where

$$\varepsilon < \frac{2^{1-d}}{d\ln(2)} , \tag{3.138}$$

and Choice 2b where

$$\varepsilon \geq \frac{2^{1-d}}{d\ln(2)} . \tag{3.139}$$

Clearly, for Choice 2a,

$$(k_1k_2)_{opt,k_1=2} = 2(k_2)_{opt,k_1=2} = 2(\frac{2}{\varepsilon d\ln(2)})^{1/d} , \tag{3.140}$$

$$(k_2)_{opt,k_1=2} := (\frac{2}{\varepsilon d\ln(2)})^{1/d} . \tag{3.141}$$

Also, by (3.124),(3.138),(3.141), we have, for Choice 2a, that (3.132) holds. Moreover, for Choice 2b,

$$(k_1k_2)_{opt,k_1=2} = (k_2)^2_{opt,k_1=2} = 4 , \tag{3.142}$$

$$(k_2)_{opt,k_1=2} := 2 . \tag{3.143}$$

By (3.124),(3.143) we have, for Choice 2b, that (3.136) holds. Note also that, except for the rather uninteresting Choice 2b, we observe that $(k_1k_2)_{opt}$ and $(k_1k_2)_{opt,k_1=2}$ are strictly decreasing functions of d whence the computational cost of $\tilde{d}$ increases with decreasing d (we will see this confirmed in our applications below).

I now present two applications of the above technique of approximating d by $\tilde{d}$ to the spatial density ρ_g. In both situations $\Psi = L^2(\mathbb{R}^2)$ and ψ is the spatial density in grid coordinates at some fixed s, i.e.,

$$\psi = \rho_g(\cdot; s) . \tag{3.144}$$

Recall that the relation of ρ_g with the beam frame spatial density ρ_B is discussed in Section 3.4.1. Note also that we choose the same initial condition for f_B as in Section 3.3 with the additional restriction that the initial modulation is zero, i.e., that $A = 0$ in (3.99). We use, for the approximant $\psi(\mathcal{N})$ of ψ, the abbreviation

$$\psi(\mathcal{N}) = \rho_{g,\mathcal{N}}(\cdot; s) \ , \tag{3.145}$$

where $\rho_{g,\mathcal{N}}$ denotes the density estimate of ρ_g for $\mathcal{N}$ particles. Note that the explicit form of $\rho_{g,\mathcal{N}}$ depends on the choice of the density estimator. Since $\Psi = L^2(\mathbb{R}^2)$, we have for arbitrary particle numbers $\mathcal{N}, \mathcal{N}'$, by (3.144),(3.145),

$$||\psi - \psi(\mathcal{N})||^2 = \int_{\mathbb{R}^2} dx_1 dx_2 \Big(\rho_g(x_1, x_2; s) - \rho_{g,\mathcal{N}}(x_1, x_2; s) \Big)^2 \ , \tag{3.146}$$

$$||\psi(\mathcal{N}) - \psi(\mathcal{N}')||^2 = \int_{\mathbb{R}^2} dx_1 dx_2 \Big(\rho_{g,\mathcal{N}}(x_1, x_2; s) - \rho_{g,\mathcal{N}'}(x_1, x_2; s) \Big)^2 \ . \tag{3.147}$$

In the first application $\rho_{g,\mathcal{N}}$ is computed by the kernel density estimation method (referred to 'Method 3' in Section 3.4.1) and in the second application $\rho_{g,\mathcal{N}}$ is computed by the cloud in cell charge deposition method (referred to 'Method 2' in Section 3.4.1). Since for the first application we want a situation where we know d, we will restrict our first application to the case $s = 0$. For both applications we compute the integral on the rhs of (3.147) by the midpoint rule whence

$$||\psi(\mathcal{N}) - \psi(\mathcal{N}')||^2 \approx \frac{1}{N_1 N_2} \sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} \Big(\rho_{g,\mathcal{N}}(\frac{i_1}{N_1}, \frac{i_2}{N_2}; s) - \rho_{g,\mathcal{N}'}(\frac{i_1}{N_1}, \frac{i_2}{N_2}; s) \Big)^2 \ , \tag{3.148}$$

where the grid on $[0, 1] \times [0, 1]$ has $N_1 N_2$ points and where we also used the fact that $\rho_g(\cdot; s)$ is supported in $[0, 1] \times [0, 1]$. Note that for both applications we choose $N_1 = N_2 = 128$. Since for the first application we have $s = 0$, we here even know ψ whence, for $s = 0$, we compute the integral on the rhs of (3.146) by the midpoint rule, i.e.,

$$||\psi - \psi(\mathcal{N})||^2 \approx \frac{1}{N_1 N_2} \sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} \Big(\rho_g(\frac{i_1}{N_1}, \frac{i_2}{N_2}; 0) - \rho_{g,\mathcal{N}}(\frac{i_1}{N_1}, \frac{i_2}{N_2}; 0) \Big)^2 \ . \tag{3.149}$$

I now consider the first application in more detail which, at the same time, illustrates the kernel density estimation method. Thus here $\rho_{g,\mathcal{N}}(\cdot;0)$ is to be computed by the kernel density estimation method. First I have to show that $d = 1/3$ and then I present some results about $\tilde{d}$. Thus the emphasis in the first application is on analyzing $\tilde{d}$ in a situation where d is known apriori. For details on the kernel density estimation method, see Section A.3. We begin with

$$\rho_{g,\mathcal{N}}(x_1, x_2; 0) := \frac{1}{H^2\mathcal{N}} \sum_{j=1}^{\mathcal{N}} K_{C1,2D,P}(\frac{x_1 - x_1^{(j)}}{H}, \frac{x_2 - x_2^{(j)}}{H}) , \qquad (3.150)$$

where the sample $(x_1^{(1)}, x_2^{(1)})^T, ..., (x_1^{(\mathcal{N})}, x_2^{(\mathcal{N})})^T$, which is generated from pseudo-random numbers by using the Acceptance-Rejection method, is distributed according to the initial spatial density $\rho_g(\cdot;0)$ and where $H > 0$ is called the 'bandwith' and the 'kernel' $K_{C1,2D,P}$ is given by

$$K_{C1,2D,P}(x_1, x_2) := \frac{225}{256}(1 - (x_1)^2)^2(1 - (x_2)^2)^2 1_{[-1,1]}(x_1)1_{[-1,1]}(x_2) , \quad (3.151)$$

with $1_{[-1,1]}$ being the indicator function of the interval $[-1, 1]$. Note that $\int_{\mathbb{R}^2} dx_1 dx_2 K_{C1,2D,P}(x_1, x_2) = 1$ whence $\int_{\mathbb{R}^2} dx_1 dx_2 \rho_{g,\mathcal{N}}(x_1, x_2; 0) = 1$. For kernels different from $K_{C1,2D,P}(x_1, x_2)$, see Section A.3. Note also that $K_{C1,2D,P}$ is continuously differentiable whence $\rho_{g,\mathcal{N}}(\cdot;0)$ is continuously differentiable which is an important property for being effective in our code. Moreover $K_{C1,2D,P}$ is essentially 'optimal' among those kernels which are continuously differentiable, but the 'optimality' is a topic which is beyond the scope of this thesis (see however the textbooks on density estimation in the reference list). To come to a situation where $d = 1/3$, the bandwith H in (3.150) must not be arbitrary since it has to be optimized to the value H_{MISE}, as follows. We first have to discuss $MISE(H)$. Let $(\tilde{x}_1^{(1)}, \tilde{x}_2^{(1)})^T, ..., (\tilde{x}_1^{(\mathcal{N})}, \tilde{x}_2^{(\mathcal{N})})^T$ be $\mathbb{R}^2$-valued random vectors which are independent identically distributed with probability density $\rho_g(\cdot;0)$. We define

$$\tilde{\rho}_{g,\mathcal{N}}(x_1, x_2; 0) := \frac{1}{H^2\mathcal{N}} \sum_{j=1}^{\mathcal{N}} K_{C1,2D,P}(\frac{x_1 - \tilde{x}_1^{(j)}}{H}, \frac{x_2 - \tilde{x}_2^{(j)}}{H}) , \qquad (3.152)$$

$$\tilde{\psi}(\mathcal{N}) := \tilde{\rho}_{g,\mathcal{N}}(\cdot;0) , \qquad (3.153)$$

whence, since pseudo-random numbers approximate random numbers, we obtain the approximate equality:

$$\psi(\mathcal{N}) = \rho_{g,\mathcal{N}}(\cdot;0) \approx \tilde{\rho}_{g,\mathcal{N}}(\cdot;0) = \tilde{\psi}(\mathcal{N}) \ , \tag{3.154}$$

where we also used (3.145),(3.153). Since $s = 0$ we have, by (3.144), $\psi = \rho_g(\cdot;0)$ whence, by (3.146),(3.154),

$$\begin{aligned} ISE(H) &:= ||\psi - \tilde{\psi}(\mathcal{N})||^2 \approx ||\psi - \psi(\mathcal{N})||^2 \\ &= \int_{\mathbb{R}^2} dx_1 dx_2 \Big(\rho_g(x_1,x_2;0) - \rho_{g,\mathcal{N}}(x_1,x_2;0) \Big)^2 \ , \end{aligned} \tag{3.155}$$

where $ISE(H)$ depends on H since $\tilde{\psi}(\mathcal{N})$ depends on H via (3.152),(3.153). One approximates $ISE(H)$ by its expectation value, $MISE(H)$, i.e.,

$$ISE(H) \approx E(ISE(H)) =: MISE(H) \ , \tag{3.156}$$

and approximates $MISE(H)$ by its large-$\mathcal{N}$-asymptote $AMISE(H)$, i.e.,

$$\begin{aligned} MISE(H) \approx AMISE(H) := \frac{H^4}{4}\mu^2(K_{C1,2D,P}) \int_{\mathbb{R}^2} dx_1 dx_2 \Big(\Delta\rho_g(x_1,x_2;0) \Big)^2 \\ + \frac{1}{\mathcal{N}H^2} \int_{\mathbb{R}^2} dx_1 dx_2 K^2_{C1,2D,P}(x_1,x_2) \ , \end{aligned} \tag{3.157}$$

where the positive constant $\mu(K_{C1,2D,P})$ is determined by $\mu(K_{C1,2D,P}) = \int_{\mathbb{R}^2} dx_1 dx_2 (x_1)^2 K_{C1,2D,P}(x_1,x_2)$ and where $\Delta\rho_g(\cdot;0)$ is the Laplacian of $\rho_g(\cdot;0)$. One defines

$$H_{MISE} := argmin_{H>0}(MISE(H)) \ , \tag{3.158}$$

and approximates

$$\begin{aligned} H_{MISE} &\approx H_{AMISE} := argmin_{H>0}(AMISE(H)) \\ &= \left(\frac{2\int_{\mathbb{R}^2} dx_1 dx_2 K^2_{C1,2D,P}(x_1,x_2)}{\mathcal{N}\mu^2(K_{C1,2D,P}) \int_{\mathbb{R}^2} dx_1 dx_2 [\Delta\rho_g(x_1,x_2;0)]^2} \right)^{1/6} \ , \end{aligned} \tag{3.159}$$

leading to

$$\begin{aligned} MISE(H_{MISE}) &\approx AMISE(H_{MISE}) \\ &\approx \frac{3}{4}\mathcal{N}^{-2/3}\Big(16\mu^4(K_{C1,2D,P})[\int_{\mathbb{R}^2} dx_1 dx_2 K^2_{C1,2D,P}(x_1,x_2)]^4 \\ &\quad \cdot[\int_{\mathbb{R}^2} dx_1 dx_2 (\Delta\rho_g(x_1,x_2;0))^2]^2\Big)^{1/6} . \end{aligned} \tag{3.160}$$

Note that, by (3.159), $H_{MISE} = \mathcal{O}(\mathcal{N}^{-1/6})$. We conclude from (3.155),(3.156),(3.160) that

$$\begin{aligned} ||\psi - \psi(\mathcal{N})||^2 &\approx ISE(H_{MISE}) \approx MISE(H_{MISE}) \\ &\approx \frac{3}{4}\mathcal{N}^{-2/3}\Big(16\mu^4(K_{C1,2D,P})[\int_{\mathbb{R}^2} dx_1 dx_2 K^2_{C1,2D,P}(x_1,x_2)]^4 \\ &\quad \cdot[\int_{\mathbb{R}^2} dx_1 dx_2 [\Delta\rho_g(x_1,x_2;0)]^2]^2\Big)^{1/6} . \end{aligned} \tag{3.161}$$

Note that, by (3.161), $||\psi - \psi(\mathcal{N})|| = \mathcal{O}(\mathcal{N}^{-1/3})$ so that, by (3.118), we have shown, as promised, that $d = 1/3$. To finish off the first application, we now present some results on $\tilde{d}$. Of course when we compute the $\rho_{g,\mathcal{N}_i}(\cdot;0)$ in $\tilde{d}$, we have to use the bandwith H_{MISE}. We compute H_{MISE} by using (3.149),(3.155),(3.156), (3.158) and by using the fact that we know $\rho_g(\cdot;0)$:

$$\begin{aligned} H_{MISE} &= argmin_{H>0}(MISE(H)) \approx argmin_{H>0}(ISE(H)) \\ &\approx argmin_{H>0}(||\psi - \psi(\mathcal{N})||^2) \\ &\approx argmin_{H>0}\Bigg(\frac{1}{N_1 N_2}\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2}\Big(\rho_g(\frac{i_1}{N_1},\frac{i_2}{N_2};0) - \rho_{g,\mathcal{N}}(\frac{i_1}{N_1},\frac{i_2}{N_2};0)\Big)^2\Bigg) . \end{aligned} \tag{3.162}$$

Note that, if $s > 0$, then we could not use the generalization of (3.162) from $s = 0$ to $s > 0$ since do not know $\rho_g(\cdot;s)$ for $s > 0$. One wayout would be to approximate H_{MISE} without the knowledge of $\rho_g(\cdot;s)$ by using the technique of least squares cross validation which is outlined in Section A.3. However, here I stick to $s = 0$ and I did computations with the following particle numbers:

$$\mathcal{N} = 8\cdot 10^5, 4\cdot 10^6, 8\cdot 10^6, 16\cdot 10^6, 32\cdot 10^6, 64\cdot 10^6, 80\cdot 10^6, 128\cdot 10^6 . \tag{3.163}$$

First of all, before discussing $\tilde{d}$, it is good to have numerical evidence for (3.159),(3.161). Using (3.149),(3.162) we obtain

$$\begin{aligned}
&\mathcal{N} = 8\cdot 10^5 \Longrightarrow (H_{MISE} = 0.025\ , ||\psi - \psi(\mathcal{N})|| = 0.0382)\\
&\mathcal{N} = 4\cdot 10^6 \Longrightarrow (H_{MISE} = 0.02\ , ||\psi - \psi(\mathcal{N})|| = 0.0219)\\
&\mathcal{N} = 8\cdot 10^6 \Longrightarrow (H_{MISE} = 0.018\ , ||\psi - \psi(\mathcal{N})|| = 0.017)\\
&\mathcal{N} = 16\cdot 10^6 \Longrightarrow (H_{MISE} = 0.016\ , ||\psi - \psi(\mathcal{N})|| = 0.0145)\\
&\mathcal{N} = 32\cdot 10^6 \Longrightarrow (H_{MISE} = 0.014\ , ||\psi - \psi(\mathcal{N})|| = 0.0115)\\
&\mathcal{N} = 64\cdot 10^6 \Longrightarrow (H_{MISE} = 0.012\ , ||\psi - \psi(\mathcal{N})|| = 0.00926)\\
&\mathcal{N} = 80\cdot 10^6 \Longrightarrow (H_{MISE} = 0.012\ , ||\psi - \psi(\mathcal{N})|| = 0.00871)\\
&\mathcal{N} = 128\cdot 10^6 \Longrightarrow (H_{MISE} = 0.011\ , ||\psi - \psi(\mathcal{N})|| = 0.00761)\ ,
\end{aligned} \tag{3.164}$$

where $\psi = \rho_g(\cdot;0)$ and $\psi(\mathcal{N}) = \rho_{g,\mathcal{N}}(\cdot;0)$. Fig.7 plots $\ln(H_{MISE})$ versus $\ln(\mathcal{N})$, confirming that $H_{MISE} = \mathcal{O}(\mathcal{N}^{-1/6})$. In fact the eight circles in Fig.7 are data from (3.164) and the dashed line in Fig.7 is the curve: $H_{MISE} = c\mathcal{N}^{-1/6}$ where c is fitted by the data point $(\mathcal{N}, H_{MISE}) = (8\cdot 10^5, 0.025)$, i.e., $c = 0.025\cdot(8\cdot 10^5)^{1/6}$. Fig.8 plots $\ln(||\psi - \psi(\mathcal{N})||)$ versus $\ln(\mathcal{N})$, confirming that the L^2-error of ψ satisfies $||\psi - \psi(\mathcal{N})|| = \mathcal{O}(\mathcal{N}^{-1/3})$. In fact the eight circles in Fig.8 are data from (3.164) and the dashed line in Fig.8 is the curve: $||\psi - \psi(\mathcal{N})|| = c\mathcal{N}^{-1/3}$ where c is fitted by the data point $(\mathcal{N}, ||\psi - \psi(\mathcal{N})||) = (8\cdot 10^5, 0.0382)$, i.e., $c = 0.0382\cdot(8\cdot 10^5)^{1/3}$.

We are now ready to discuss $\tilde{d}$ and we do that in the same situation as Fig.'s 7 and 8, i.e., the situation when $\psi = \rho_g(\cdot;0)$ and $\psi(\mathcal{N}) = \rho_{g,\mathcal{N}}(\cdot;0)$ with $\rho_{g,\mathcal{N}}(\cdot;0)$ given by (3.150) and where $\mathcal{N}$ is from (3.163). Using (3.120),(3.148) we approximate

$\tilde{d}$ by using the midpoint rule:

$$\tilde{d} \approx \frac{1}{\ln(\mathcal{N}_2/\mathcal{N}_1)} \ln\left(\sqrt{\frac{\sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} \Big(\rho_{g,\mathcal{N}_1}(\frac{i_1}{N_1}, \frac{i_2}{N_2}; 0) - \rho_{g,\mathcal{N}_3}(\frac{i_1}{N_1}, \frac{i_2}{N_2}; 0) \Big)^2}{\sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} \Big(\rho_{g,\mathcal{N}_2}(\frac{i_1}{N_1}, \frac{i_2}{N_2}; 0) - \rho_{g,\mathcal{N}_4}(\frac{i_1}{N_1}, \frac{i_2}{N_2}; 0) \Big)^2}} \right) . \tag{3.165}$$

To choose the proper size of the particle numbers $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4$ in (3.165) we first apply the theory outlined at the beginning of this section, i.e., we discuss Choices 1 and 2. Note that with the particle numbers in (3.163), the maximum possible value of $\mathcal{N}_4/\mathcal{N}_1$ is merely 160. If $d = 1/3$ and $\varepsilon = 0.1$ then (3.130), (3.131) give us, for Choice 1a,

$$(k_1)_{opt} \approx 20.1 \,, \qquad (k_2)_{opt} = 8000 \,, \qquad (k_1 k_2)_{opt} \approx 160700 \,, \tag{3.166}$$

whence $\mathcal{N}_4/\mathcal{N}_1 \approx 160700$. If $d = 1/3$ and $\varepsilon = 0.1$ then (3.140), (3.141) give us, for Choice 2a,

$$(k_2)_{opt,k_1=2} \approx 648600 \,, \qquad (k_1 k_2)_{opt,k_1=2} \approx 1297200 \,, \tag{3.167}$$

whence $\mathcal{N}_4/\mathcal{N}_1 \approx 1297200$. If $d = 1/3$ and $\varepsilon = 0.3$ then (3.130), (3.131) give us, for Choice 1a,

$$(k_1)_{opt} \approx 20.1 \,, \qquad (k_2)_{opt} \approx 296 \,, \qquad (k_1 k_2)_{opt} \approx 5950 \,, \tag{3.168}$$

whence $\mathcal{N}_4/\mathcal{N}_1 \approx 5950$. If $d = 1/3$ and $\varepsilon = 0.3$ then (3.140), (3.141) give us, for Choice 2a,

$$(k_2)_{opt,k_1=2} \approx 24020 \,, \qquad (k_1 k_2)_{opt,k_1=2} \approx 48040 \,, \tag{3.169}$$

whence $\mathcal{N}_4/\mathcal{N}_1 \approx 48040$. We see that even for the modest choice $\varepsilon = 0.3$ the theory demands $\mathcal{N}_4/\mathcal{N}_1 \approx 5950$ which is considerably larger than 160. In other words, $d = 1/3$ is so small that rather large particle numbers are suggested. However the values of $\mathcal{N}_4/\mathcal{N}_1$, suggested by our theory, are merely *sufficient* for the validity of (3.123), *not necessary* as we will see now. In fact, computing $\tilde{d}$ for the modest particle numbers (3.163) we obtain, by using (3.165),

$$(\mathcal{N}_1 = 8 \cdot 10^5 \ , \ \mathcal{N}_2 = 8 \cdot 10^6 \ , \ \mathcal{N}_3 = 8 \cdot 10^6 \ , \ \mathcal{N}_4 = 80 \cdot 10^6) \Longrightarrow \tilde{d} = 0.369 \ ,$$
$$(\mathcal{N}_1 = 4 \cdot 10^6 \ , \ \mathcal{N}_2 = 8 \cdot 10^6 \ , \ \mathcal{N}_3 = 64 \cdot 10^6 \ , \ \mathcal{N}_4 = 128 \cdot 10^6) \Longrightarrow \tilde{d} = 0.351 \ . \tag{3.170}$$

Note that $\tilde{d} = 0.369$ gives $|1 - \tilde{d}/d| = 0.11$ and $\tilde{d} = 0.351$ gives $|1 - \tilde{d}/d| = 0.05$. Thus indeed the modest particle numbers (3.163) give already rather good results for $\tilde{d}$. This indeed gives evidence that the values of $\mathcal{N}_4/\mathcal{N}_1$, demanded by our theory, are sufficient but not necessary for the validity of (3.123). Note that the particle numbers in (3.170) are selected from (3.163) via Choices 1,2, as follows. For the first example in (3.170) we have $10 = k_1 = \mathcal{N}_2/\mathcal{N}_1 = \mathcal{N}_3/\mathcal{N}_1 = k_2$ and

$$\frac{2k_1^{-d}}{d\ln(k_1)} = \frac{2(\mathcal{N}_2/\mathcal{N}_1)^{-d}}{d\ln(\mathcal{N}_2/\mathcal{N}_1)} = \frac{6(10)^{-1/3}}{\ln(10)} \approx 1.2 \ ,$$

whence, by (3.135), $\varepsilon \approx 1.2$, so that (3.129) holds which implies that the first example in (3.170) belongs to Choice 1b. For the second example in (3.170) we have $k_1 = \mathcal{N}_2/\mathcal{N}_1 = 2, k_2 = \mathcal{N}_3/\mathcal{N}_1 = 16$ and, by (3.141),

$$\varepsilon = \frac{2k_2^{-d}}{d\ln(2)} = \frac{2(\mathcal{N}_3/\mathcal{N}_1)^{-d}}{d\ln(2)} = \frac{6(16)^{-1/3}}{\ln(2)} \approx 3.44 \ ,$$

whence (3.138) holds so that the second example in (3.170) belongs to Choice 2a. This concludes our first application. In retrospective it is clear why for the first application we restricted ourselves to the case $s = 0$. In fact, having $s = 0$ guarantees (i) that the sample $(x_1^{(1)}, x_2^{(1)})^T, ..., (x_1^{(\mathcal{N})}, x_2^{(\mathcal{N})})^T$ approximates a sequence of random vectors which are independent identically distributed with probability density $\rho_g(\cdot; 0)$ and (ii) that H_{MISE} can be computed.

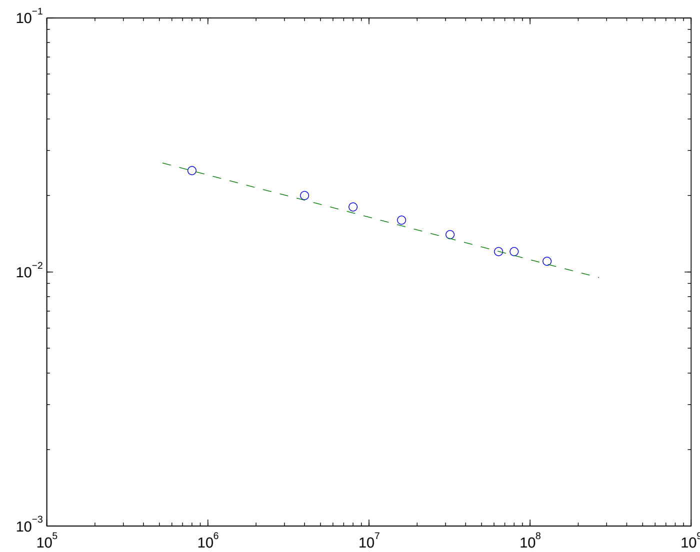


Figure 7: Loglog plot of the bandwith H_{MISE} versus particle number $\mathcal{N}$

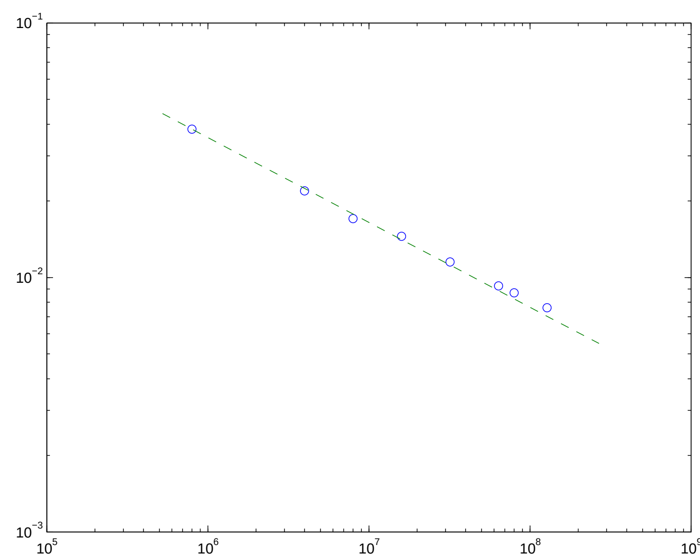


Figure 8: Loglog plot of the error $||\psi - \psi(\mathcal{N})||$ versus particle number $\mathcal{N}$

I now consider the second application where $\rho_{g,\mathcal{N}}$ is computed by the cloud in cell charge deposition method (referred to 'Method 2' in Section 3.4.1) for the parameter values $J_1 = J_2 = 40$. Here we deal with a situation where we neither know d nor where we know if a meaningful d exists at all. Note that we choose the same initial condition for f_B as for the first application. We choose the particle numbers $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4$ in (3.120) via Choice 2 with $\mathcal{N}_1 = 8 \cdot 10^6$ and we vary k_2 from 2 to 32 with the aim to see if $\tilde{d}$ converges to some d when k_2 grows. Thus we use the particle numbers $8 \cdot 10^6, 16 \cdot 10^6, 32 \cdot 10^6, 64 \cdot 10^6, 128 \cdot 10^6, 256 \cdot 10^6, 512 \cdot 10^6$. Fig.9 plots $\tilde{d}$ versus k_2 when $s = 0$ and Fig.10 plots $\tilde{d}$ versus k_2 when $s = s_f$. Fig.9 indicates $d \approx 0.5$ while Fig.10 indicates that d is around 0.35. Since the theory of the cloud in cell charge deposition method is beyond the scope of this thesis and since the purpose of Fig.'s 9 and 10 is to illustrate the application of (3.120), we leave the d-values $0.5, 0.35$ uncommented.

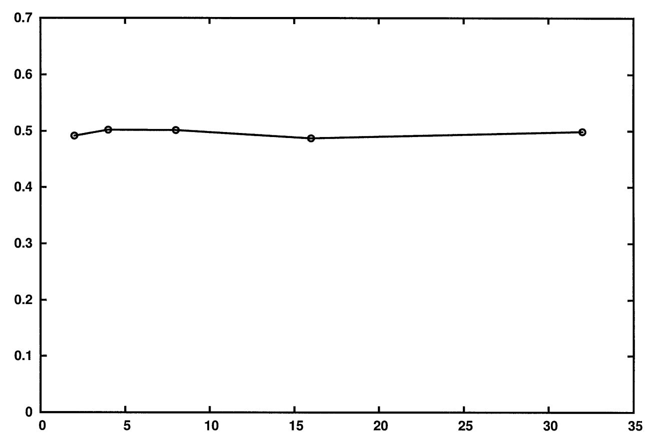


Figure 9: $\tilde{d}$ versus k_2 when $k_1 = 2$ and $s = 0$

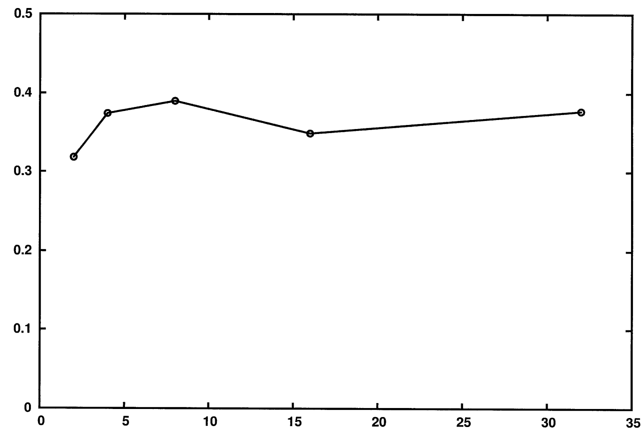


Figure 10: $\tilde{d}$ versus k_2 when $k_1 = 2$ and $s = s_f$

Chapter 4

Summary of Vlasov-Maxwell system and outlook

We have demonstrated a procedure with some new features for self-consistent simulation with application to a bunch compressor. Although it is based on tracking an ensemble of particles, as in usual macro-particle or PIC codes, the method of smoothing the charge distribution is quite different, using density estimation methods. The resulting smooth distribution is used in an accurate solution of the field equations by applying exact field formulas. We hope that the resulting time evolution of the spatial density approximates that which would be obtained from a solution of the Vlasov-Maxwell system on the $4D$ phase space, but there is no direct check on accuracy of such an approximation. However, the evident lack of noise in the simulation is encouraging.

We anticipate improvements in the code regarding treatment of the spatial density, but at present the most costly part is the field calculation. We intend to review the choice of integration variables and the integration algorithms to see if the field evaluation can be speeded up. Parts of the integration, for large retarded time intervals (see (3.50),(3.51)) may have been done more accurately than necessary.

We mentioned that the MCP method can be time consuming. We are attempting to improve the Monte Carlo integrations by trying variance reduction techniques, which build on the Central Limit Theorem [Ca, Ros], and also by trying quasi-random sequences (also called 'low-discrepancy sequences') in place of pseudo-random sequences [Ca, Ni]. Quasi-random sequences allow one to break the 'curse of dimensionality' in grid-based multi-dimensional integration, giving a true error bound (i.e., not probabilistic) of order $(\log \mathcal{N})^{k-1}/\mathcal{N}$, with only logarithmic dependence on the dimension k of the space. We are also attempting to improve the Monte Carlo integrations by considering a FFT for nonequidistant points offered by the subroutine library NFFT [NFFT]. Moreover we want to extend the convergence studies of Section 3.4.3.

Furthermore we intend to take further advantage of the exact field formulas. For example (3.44),(3.45) will allow us to study the energy balance between the bunch and the self field by applying Poynting's theorem. Moreover one of Maxwell's eight equations (3.10), i.e., the equation

$$c\partial_u B_Y = -\partial_Z E_X + \partial_X E_Z \ ,$$

carries useful information about $\mathbf{E}_\parallel$ and $B_\perp$ since, at $Y = 0$, it yields to

$$c\partial_u B_\perp = -\partial_Z E_{L,X} + \partial_X E_{L,Z} \ ,$$

which can be used to reduce the computational cost of the field computation and/or as a double check since the field formulas (3.50),(3.51) for $E_{L,X}, E_{L,Z}, B_\perp$ are exact.

Chapter 5

Introduction to spin-orbit tori

I now come to the second part of this thesis which consists of Chapters 5-10 and Appendices B-G. It presents the topic of spin-orbit tori as a mathematical theory and it is based on the map formalism equations of motion (6.1),(6.2).

5.1 Physical context and mathematical approach

I begin with some brief general remarks on the physical context for the orientation of the reader. More details can be found in [BEH04, Hof, MSY, Vo].

Spin is of central importance for the understanding of the behavior of fundamental particles and their interactions. This is made clear, for example, in [SPIN09] where up-to-date accounts of experimental and theoretical work are given. In particular, the differential cross sections for particle-particle interactions depend on the spin states of the particles. These interactions are typically studied by colliding a beam of spin-1/2 particles (e.g. electrons or protons) either with another beam of spin-1/2 particles or with nuclei located at a fixed 'target'. Various considerations, such as the need for high energies, often dictate that the particles circulate in a beam consisting of a train of separate bunches in a so-called storage ring. Typically the motion of a bunch for 10^9 turns around the ring is of interest. The particle interactions to be studied in such a storage ring take place at the centers of detectors mounted at specially configured interaction points. The task of Accelerator Physics is to provide and describe the transport of the bunches through the interaction points and it requires mathematical tools which are different from those needed to describe the collision processes in the interaction points (the latter tools are from Quantum Field Theory). This thesis deals exclusively with the Accelerator Physics aspects and its tools are from Dynamical Systems Theory. Descriptions of storage rings can be found in standard text books. See for example [CT, Wi]. However, to summarize, the common feature of a storage ring is that the electrically charged particles are confined by combinations of electric and magnetic fields to move in bunches on approximately circular orbits in a vacuum tube. The dimensions of a bunch are millimeters whence they are very small compared to the average radius of the ring which can be kilometers. A bunch typically contains around $N = 10^{11}$ particles. Accelerator Physics involves various levels of description depending on how accurately one wants to study the bunches. So I now have to characterize the level needed for this thesis. At this level a phase-space variable $\tilde{u}$ and a spin variable $\tilde{S}$ provide a classical description of a particle located at $\tilde{u} \in \mathbb{R}^6$ with spin value $\tilde{S} \in \mathbb{R}^3$. Experiments aimed at exploiting the influence of spin on particle-particle collisions usually require that the bunches be spin polarized. This means that the polarization $\tilde{P} := (1/N)\sum_{i=1}^{N} \tilde{S}_i$, namely the average over the spin vectors $\tilde{S}_1, ..., \tilde{S}_N$ of the bunch be non-zero. Thus the task of Polarized Beam Physics is to provide and describe the transport of bunches through the interaction points such that $|\tilde{P}|$ is 'sufficiently' large. Note that in the definition of $\tilde{P}$ the spin vectors have to be normalized, i.e., $|\tilde{S}_i| = 1$. Nevertheless for the purposes of this work there is no need to assume that the spin vectors are normalized. For the purposes of this thesis I ignore all interactions between the particles, the emission of electromagnetic radiation by the particles and the effects of the electric and magnetic fields set up in the vacuum pipe by the particles themselves. This leads to a classical Hamiltonian description (for a derivation of the Hamiltonian from Quantum Physics, see [BH98]). Furthermore I shall neglect the extremely small Stern-Gerlach force acting from $\tilde{S}$ onto $\tilde{u}$ [BEH04] (for details on the relativistic Stern-Gerlach force in Accelerator Physics, see e.g. [He96]). Then the particle motion is described by the equation for the Lorentz force and the spin motion by the Thomas-Bargmann-Michel-Telegdi (T-BMT) equation [Ja]. Thus the equations of motion for the combined $\tilde{u}$,$\tilde{S}$ system are no longer Hamiltonian (albeit the equations of motion for $\tilde{u}$ are still Hamiltonian).

Although dynamical systems are usually analyzed by taking time as the independent variable, this is usually not convenient for storage rings since there, the vacuum tube and the electric and magnetic guide fields have a fixed, 1-turn periodic, approximately circular spatial layout. It is then common practice to define the angular distance, $\theta = 2\pi s/L$, around the ring where s is the distance around the ring and L is the circumference. The equations of motion for $\tilde{u}$ and $\tilde{S}$ are then transformed into forms in which θ is the independent variable. The one-turn periodicity of the positions of the electric and magnetic guide fields then becomes a 2π–periodicity in θ of the equations of motion for $\tilde{u}$ and $\tilde{S}$. As a next step one constructs the curvilinear closed orbit, i.e., the orbit along which the particle motion is one-turn periodic and one defines coordinates with respect to this orbit. Then $\tilde{u}$ consists of three pairs of canonical variables. For example, two of the pairs can describe transverse motion and one pair can describe longitudinal (synchrotron) motion within a bunch. One of this latter pair quantifies the deviation of the particle energy from the energy of a 'reference particle' fixed at the center of a bunch and the other describes the time delay w.r.t. the reference particle [BHR]. With respect to the average radius of the closed orbit and the nominal particle energy, the canonical position variable and the energy variable are very small.

Spin and particle motion in storage rings is usually described using either the 'flow formalism' or the 'map formalism'. In the flow formalism $\tilde{u}$ and $\tilde{S}$ are functions of θ: $\tilde{u} = \tilde{u}(\theta), \tilde{S} = \tilde{S}(\theta)$ and in the map formalism one samples $\tilde{S}$ and $\tilde{u}$ at a fixed θ turn by turn.

In this thesis I focus on the map formalism which I now derive from the flow formalism. The magnetic and electric fields in storage rings are usually set up so that the motion of the particles is close to integrable. In the following I shall assume that it is exactly integrable. Once the spin motion has been classified on this basis, the effect of non-integrability can be included as a perturbation. I therefore choose $\tilde{u}$ to consist of d pairs of action–angle variables, i.e., $\tilde{u} = \begin{pmatrix} \tilde{\phi} \\ \tilde{J} \end{pmatrix}$, where $\tilde{\phi}, \tilde{J} \in \mathbb{R}^d$ and where $d = 3$ is the case of main interest. Then in the flow formalism one writes

$$\frac{d\tilde{\phi}}{d\theta} = \tilde{\omega}(\tilde{J}) , \qquad \tilde{\phi}(\theta_0) = \phi_0 , \tag{5.1}$$

$$\frac{d\tilde{J}}{d\theta} = 0 , \qquad \tilde{J}(\theta_0) = J_0 , \tag{5.2}$$

$$\frac{d\tilde{S}}{d\theta} = \mathcal{A}(\theta, \tilde{\phi}, \tilde{J})\tilde{S} , \qquad \tilde{S}(\theta_0) = S_0 , \tag{5.3}$$

where the d components of $\tilde{\omega}(\tilde{J})$ are called the 'orbital tunes' and $\mathcal{A}$ is a real skew–symmetric 3×3 matrix, i.e., $\mathcal{A}_{12} = -\mathcal{A}_{21}, \mathcal{A}_{13} = -\mathcal{A}_{31}$ and $\mathcal{A}_{23} = -\mathcal{A}_{32}$. The function $\mathcal{A}$ is derived from the rotation rate vector of the T-BMT equation [BEH04] and it is 2π-periodic in θ and in the d components of $\tilde{\phi}$. Of course, (5.3) is an incarnation of the T–BMT equation. Analogously (5.1),(5.2) are an incarnation of the Lorentz force law. One can call the pair $(\tilde{\omega}, \mathcal{A})$ the 'spin-orbit system' in the flow formalism and it was studied in [BEH04].

To proceed from the flow formalism to the map formalism I write the solution of

(5.1),(5.2),(5.3) as

$$\tilde{\phi}(\theta) = \phi_0 + (\theta - \theta_0)\tilde{\omega}(J_0) \ , \tag{5.4}$$

$$\tilde{J}(\theta) = J_0 \ , \tag{5.5}$$

$$\tilde{S}(\theta) = \tilde{\Psi}(\theta, \theta_0; \phi_0, J_0) S_0 \ , \tag{5.6}$$

where $\tilde{\Psi}$ is the principal solution matrix for $d\tilde{S}/d\theta = \mathcal{A}(\theta, \phi_0 + (\theta - \theta_0)\tilde{\omega}(J_0), J_0)\tilde{S}$ and where $\tilde{\Psi}(\theta, \theta_0; \phi_0, J_0)$ is 2π-periodic in the d components of ϕ_0 and $\tilde{\Psi}$ is $SO(3)$-valued. For the definition of $SO(3)$, see after (6.2). It follows from (5.4),(5.5),(5.6) that

$$\tilde{\Psi}(\theta_2, \theta_0; \phi_0, J_0) = \tilde{\Psi}(\theta_2, \theta_1; \phi_0 + (\theta_1 - \theta_0)\tilde{\omega}(J_0), J_0)\tilde{\Psi}(\theta_1, \theta_0; \phi_0, J_0) \ ,$$

whence, for integers m, n,

$$\begin{aligned} &\tilde{\Psi}(\theta_0 + 2\pi(n+m), \theta_0; \phi_0, J_0) \\ &\quad = \tilde{\Psi}(\theta_0 + 2\pi n, \theta_0; \phi_0 + 2\pi m\tilde{\omega}(J_0), J_0)\tilde{\Psi}(\theta_0 + 2\pi m, \theta_0; \phi_0, J_0) \ , \end{aligned} \tag{5.7}$$

where I used the fact that, due to the 2π-periodicity of $\mathcal{A}(\theta, \cdot, \cdot)$ in θ,

$$\tilde{\Psi}(\theta + 2\pi m, \theta_0 + 2\pi m; \phi_0, J_0) = \tilde{\Psi}(\theta, \theta_0; \phi_0, J_0) \ . \tag{5.8}$$

Without loss of generality one can take $\theta_0 = 0$ and so, by letting

$$\phi(n) := \tilde{\phi}(2\pi n) \ , \tag{5.9}$$

$$J(n) := \tilde{J}(2\pi n) \ , \tag{5.10}$$

$$S(n) := \tilde{S}(2\pi n) \ , \tag{5.11}$$

I obtain from (5.4),(5.5),(5.6)

$$\phi(n+1) = \phi(n) + 2\pi\tilde{\omega}(J(n)) \ , \qquad \phi(0) = \phi_0 \ , \tag{5.12}$$

$$J(n+1) = J(n) \ , \qquad J(0) = J_0 \ , \tag{5.13}$$

$$S(n+1) = \tilde{\Psi}(2\pi, 0; \phi(n), J(n))S(n) \ , \qquad S(0) = S_0 \ . \tag{5.14}$$

The initial value problem (5.12),(5.13), (5.14) characterizes the 'spin-orbit system' $(\tilde{\omega}, \tilde{\Psi}(2\pi, 0; \cdot, \cdot))$ taken in the map formalism. Letting

$$\omega := \tilde{\omega}(J_0) \ , \tag{5.15}$$

$$\Psi(n; x) := \tilde{\Psi}(2\pi n, 0; x, J_0) \ , \tag{5.16}$$

I obtain from (5.12),(5.13), (5.14)

$$\phi(n+1) = \phi(n) + 2\pi\omega \ , \qquad \phi(0) = \phi_0 \ , \tag{5.17}$$

$$S(n+1) = A(\phi(n))S(n) \ , \qquad S(0) = S_0 \ , \tag{5.18}$$

where

$$A(\cdot) := \tilde{\Psi}(2\pi, 0; \cdot, J_0) \ , \tag{5.19}$$

and from (5.7) the 'cocycle condition'

$$\Psi(n+m; \phi) = \Psi(n; \phi + 2\pi m\omega)\Psi(m; \phi) \ . \tag{5.20}$$

Note that $A(\cdot) = \Psi(1; \cdot)$. The initial value problem (5.17),(5.18) characterizes the 'spin-orbit torus' (ω, A) taken in the map formalism. Thus (5.17),(5.18) are the basic equations for this second part of the thesis. We will see in Section 6.1 that Ψ is uniquely determined by ω and A, whence I will use for Ψ the notation $\Psi_{\omega,A}$. In this work I will assume that A is continuous and accordingly continuity is assumed in many other definitions as well. For example, the generators of the invariant spin fields (see Definition 6.2) and the transfer fields (see Definition 7.2) between spin-orbit tori are continuous functions. In contrast, in [BEH04] A is of class C^1 since $\tilde{\Psi}(\cdot, J_0)$ is of class C^1 (as well as the invariant spin fields and the transfer fields). Note that assuming mere continuity in the present work is fruitful since I here deal with the map formalism (in contrast, in the flow formlism of [BEH04] it is natural to impose the C^1-property since one has to deal with differential equations).

Although accelerator physicists tend to concentrate on studying spin motion in real storage rings, many of the issues surrounding the so-called invariant spin field (introduced in Section 6.3) and the spin-orbit resonance (introduced in Sections 7.4 and 8.4) depend just on the structure of the initial value problem (5.17),(5.18) and can be treated in isolation from the original physical system. This is the strategy to be adopted here and it clears the way for the focus on purely mathematical matters, in particular for the exploitation of theorems from Topology and Fourier Analysis. For example, the Homotopy Lifting Theorem (see also Section 6.4) facilitates the study of continuous functions (in particular it allows to apply the so-called quaternion formalism to functions like $\Psi(n;\cdot)$ in (5.16)). Another example is Fejér's multivariate theorem which facilitates the study of so-called quasiperiodic functions (in particular it allows, via Theorem 8.6, to characterize the set of the so-called spin tunes of second kind).

Now that the background to this work has been presented as well as an introduction to the map formalism, I finish this chapter with an outline of the structure of the following chapters. For thorough overviews of the importance of the invariant spin field and the so-called amplitude-dependent spin tune for classifying spin motion in storage rings see [BEH04, Hof, Vo]. Note that the spin tunes of first kind introduced in Section 7.4 are the amplitude-dependent spin tunes at a fixed, but arbitrary value of the 'amplitude' J_0.

5.2 Synopsis

Chapters 5-10 and Appendices B-G are structured as follows.

In Chapter 6 I introduce the most basic concepts. In particular, in Section 6.1 I introduce the spin-orbit torus (ω, A) where ω is the orbital tune vector and A is a 1-turn spin transfer matrix which is modeled after the situation of (5.19). I also introduce in Section 6.1 the symbol $\mathcal{SOT}(d,\omega)$ for the set of all spin-orbit tori which have the orbital tune vector $\omega \in \mathbb{R}^d$ and the symbol $\mathcal{SOT}(d)$ for the set of all spin-orbit tori which have an orbital tune vector in $\mathbb{R}^d$. I then derive the n-turn spin transfer matrix $\Psi_{\omega,A}$ from ω and A and establish some basic relations between the $\Psi_{\omega,A}(n;\cdot)$ for different values of the integer n. This leads naturally in Section 6.2 to the definition of the $\mathbb{Z}$-action, $L_{\omega,A}$, on $\mathbb{R}^{d+3}$ which is a function associated with every spin-orbit torus $(\omega, A) \in \mathcal{SOT}(d,\omega)$ encoding the information about the spin-orbit torus in a very useful form. Some group theoretical properties of $L_{\omega,A}$ are discussed too. Also the $\mathbb{Z}$-action L_ω on $\mathbb{R}^d$ is introduced which formalizes the orbital translations on $\mathbb{R}^d$ associated with each $(\omega, A) \in \mathcal{SOT}(d,\omega)$. In Section 6.3 I consider a distribution or field of spins constructed by attaching a spin to each $\phi_0 \in \mathbb{R}^d$ at $n = 0$ and thereby introduce the polarization fields (and, as a special subclass, the spin fields) associated with every (ω, A). I also define the $\mathbb{Z}$-action $L^{(PF)}_{\omega,A}$ which governs the evolution of the polarization fields. Polarization fields are important tools to study the polarization of a bunch (see also Section 5.1), however this aspect of polarization fields plays no role in this work. Chapter 6 is closed with Section 6.4 where the impact of Homotopy Theory on the present work is outlined and where some related concepts and facts are mentioned which are needed in this work. In particular I show how to exploit the 2π-periodicity of some functions and I point out how Homotopy Theory is related with the $SO(3)$-index. The $SO(3)$-index is based on the quaternion formalism of $\mathbb{S}^3$ which is employed in this work to deal with continuous $SO(3)$-valued functions.

One is particularly interested in spin-orbit tori for which spin precesses around a fixed axis and perhaps even at a fixed rate. Such a fixed rate leads to the definition of spin tune of first kind. Moreover to fully exploit those spin-orbit tori one needs a transformation group which allows to transform the spin motion from one spin-orbit torus to another. Thus in Chapter 7 I introduce the transformation group (=group action), $R_{d,\omega}$, on $\mathcal{SOT}(d,\omega)$. The group action $R_{d,\omega}$ is motivated by some observations made at the beginning of Section 7.1 of how spin-orbit tori should be transformed into each other in an efficient way. This leads to the notion of the $R_{d,\omega}$-

orbit. Roughly speaking, an $R_{d,\omega}$-orbit of a spin-orbit torus, (ω, A), is the set of spin-orbit tori which can be reached from (ω, A) by varying the parameters of $R_{d,\omega}$ over the underlying group, $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. Thus with Chapter 7 I begin to consider the set $\mathcal{SOT}(d, \omega)$ as a whole and we will see that spin-orbit tori, which belong to the same $R_{d,\omega}$-orbit, share many of their properties. The way in which spin-orbit trajectories and polarization fields transform with $R_{d,\omega}$ from one spin-orbit torus to another is stated in Theorem 7.3 of Section 7.1. The aim of studying reference frames in which spins precess around a fixed axis, possibly at a fixed rate, prompts the definition in Section 7.2 of trivial, almost trivial and weakly trivial spin-orbit tori to embrace these cases. Section 7.2 also shows how Homotopy Theory impacts on weakly trivial spin-orbit tori via the $SO_3(2)$-index. Then in Section 7.3 I use $R_{d,\omega}$ acting on trivial, almost trivial and weakly trivial spin-orbit tori to classify spin-orbit tori into so-called coboundaries, almost coboundaries, weak coboundaries, and those which are not weak coboundaries. Thus I deal with four major subsets of $\mathcal{SOT}(d, \omega)$ (where some of them overlap - see the inclusions (7.18)). The terminology of 'coboundary' and 'almost coboundary' is borrowed from Dynamical Systems Theory since, given a spin-orbit torus (ω, A) in $\mathcal{SOT}(d, \omega)$, the function $\Psi_{\omega,A}$ is a $SO(3)$-cocycle over the topological $\mathbb{Z}$-space $(\mathbb{R}^d, L_\omega)$. Section 7.3 displays the close connection between the concepts of weak coboundary and invariant spin field (ISF) and the impact of Homotopy Theory on weak coboundaries. In Section 7.4 I define for every spin-orbit torus a (possibly empty) set of spin tunes of first kind (and the associated spin-orbit resonances) which are reincarnations of the spin tunes introduced by Yokoya [Yo1] and show that this set is nonempty iff the spin-orbit torus is an almost coboundary. Spin tunes of the first kind are always associated with almost coboundaries so that they are always associated with invariant spin fields. In Section 7.5 I present the celebrated uniqueness theorem of Yokoya [Yo1], which relates the uniqueness issue of the invariant spin field with the condition of spin-orbit resonance of first kind. In Section 7.6 I put the present work, and weak coboundaries in particular, into the context of Polarized Beam Physics. Thus I relate the present work with other work of Polarized Beam Physics. In Section 7.7 I address the question of whether two weakly trivial spin-orbit tori belong to the same $R_{d,\omega}$-orbit. In particular the relevance of the small divisor problem and Diophantine sets of orbital tunes is pointed out.

In Chapter 8 I widen and deepen the study of spin-orbit tori by using the tool of quasiperiodic functions. In particular I show that, off orbital resonance, the existence of just one quasiperiodic spin trajectory ensures the existence of an ISF. Then in Section 8.2 I consider reference frames, called 'simple precession frames', in which spins precess around an axis which can be any spin trajectory and I define a phase advance for spin motion in such a frame. In Section 8.3 I introduce special simple precession frames, called 'uniform precession frames', for which the phase advance is the same from turn to turn and show their connection with the so-called generalized Floquet Theorem. Armed with the concept of the uniform precession frame I define, in Section 8.4, for every spin-orbit torus a (possibly empty) set of spin tunes of second kind (and the associated spin-orbit resonances) and show that the spin tunes of second kind are identical with the spin tunes of first kind in most situations. In this work the spin tunes of second kind mainly serve to analyze the spin tunes of first kind. In Section 8.5 I resume the theme of Section 7.7 and, on the basis of Corollary 8.12, I am able to outline an algorithm employed in the code SPRINT for computing spin tunes of first and second kind. In Section 8.6 I show how Homotopy Theory has an impact on the individual values of the spin tunes of first kind, i.e., how it affects the structure of the sets $\Xi_1(\omega, A)$. Section 8.7 returns to the question, already addressed in Section 7.3, of whether the existence of an ISF implies that a spin-orbit torus can be transformed to become a weakly trivial one.

Chapter 9 reconsiders the basic $\mathbb{Z}$-actions $L_{\omega,A}$ and $L^{(PF)}_{\omega,A}$ used in Chapters 6,7,8 and introduces further associated $\mathbb{Z}$-actions. In particular, in Section 9.1 it is shown how the peculiar structure of the cocycle condition (see (5.20) and (6.6)) follows from the fact that $L_{\omega,A}$ is a skew-product of the orbital $\mathbb{Z}$-action L_ω. In Section 9.2 I show that the $\mathbb{Z}$-action $L_{\omega,A}$ is an extension of the $\mathbb{Z}$-action $L^{(T)}_{\omega,A}$. I thereby relate the orbital translations on $\mathbb{R}^d$ to the corresponding orbital translations on the d-torus $\mathbb{T}^d$. Thus Section 9.2 gives a brief glimpse into the $\mathbb{T}^d$-treatment of spin-orbit tori. In Section 9.3 I widen the perspective by showing how a single principal $SO(3)$-bundle, $\lambda_{\mathcal{SOT}(d)}$, underlies $\mathcal{SOT}(d)$. It leads in Section 9.3.5 to Theorem 9.5a, which is a special case of Zimmer's Reduction Theorem. As an application of this I obtain Theorem 9.5b which shows the concept of the invariant spin field in a new light.

The appendices, B-F, provide material needed in Chapters 6-9. While most of the material of Appendices B-E is standard, these appendices provide sufficient precision and make this part of the thesis essentially self contained. Appendix F contains those proofs which are not given elsewhere. Appendix G contains a guide which will help the reader with some subjects appearing in this part of the thesis.

5.3 Scope and limitations

I now mention the possible merits and shortcomings of this part of the thesis.

The intention and flavor of this work is to present a piece of Mathematical Physics. In fact an abundance of mathematical definitions is introduced, which transfigure the topic of spin-orbit tori into a mathematical theory. Accordingly, an abundance of lemmas, propositions, theorems, corollaries is stated and the proofs are, without exception, intended to be rigorous.

Three important issues related with this work, but not covered by it at all, are the spinor formalism, the synthesis of families of spin-orbit tori into spin-orbit systems and the use of Borel algebras. Note that the spinor formalism deals with spinor valued functions which are associated with the spin trajectories and spinor valued functions which are associated with the polarization fields (in contrast, the present work uses the 3D formalism where the spin lives in $\mathbb{R}^3$). Note also that both associations can be performed via liftings w.r.t. the so-called complex Hopf bundle whose projection has domain $\mathbb{S}^3$ and range $\mathbb{S}^2$. It turns out that that the spinor formalism can be pursued along similar lines as the quaternion formalism in Sections C.2,C.3 (the latter is based on the Hurewicz fibration $(\mathbb{S}^3, p_2, SO(3))$). In fact if in the quaternion formalism one replaces the Hurewicz fibration $(\mathbb{S}^3, p_2, SO(3))$ by the complex Hopf bundle (the latter is a Hurewicz fibration, too) then one obtains the spinor formalism [He] (for Hurewicz fibrations, see Appendix C). In contrast, the issue of the synthesis of families of spin-orbit tori into spin-orbit systems seems to have a less geometrical and more analytical flavor. While in this work the emphasis is on continuous functions, large parts of spin-orbit theory can be formulated by using Borel measurable functions [He]. Such an approach is feasible for the statistical description of spin-orbit tori (e.g., the study of the polarization) and it allows to apply more tools from Ergodic Theory, e.g., Birkhoff's Ergodic Theorem [EH].

This work puts some effort into the taxonomy of spin-orbit tori, in particular, due to their importance, some effort into the taxonomy of weak coboundaries. A minor shortcoming is that many results focus on the generic case where $(1, \omega)$ is nonresonant. However since the nongeneric case can be reduced to the generic case, it would be easy to modify and prove many of my results for the nongeneric case [He]. The following conjecture, which I call the 'ISF-conjecture', plays a fruitful role in Polarized Beam Physics. The ISF-conjecture, which, at least to my knowledge (see also Section 7.6), is unsettled, goes as follows: "If a spin-orbit torus (ω, A) is off orbital resonance, then it has an invariant spin field". Albeit no attempt is made in this work to settle the ISF-conjecture, the present work presents some conditions which transform the ISF-conjecture into equivalent conjectures. For example, by Theorems 7.9,7.10, a $(\omega, A) \in \mathcal{SOT}(d, \omega)$ with $d = 1$ is a weak coboundary iff it has an ISF. Note finally that numerical procedures exist which 'solve' the ISF problem numerically (see Section 7.6).

Chapter 6

The spin-orbit tori

In this section I introduce the most basic concepts and facts needed for this work.

6.1 Introducing the spin-orbit tori (ω, A)

The main purpose of this section is to state Definition 6.1 which introduces the basic entity of this work, the 'spin-orbit torus'. The orbital motion underlying the definition of (ω, A) is a translational motion in $\mathbb{R}^d$, where d is the number of degrees of freedom (whenever I write $\mathbb{R}^k$, this implies that k is a positive integer).

As pointed out in Chapter 5, the orbital motion in the present work is assumed to be integrable. So its simplest formulation is by choosing the orbital variables as angles $\phi_1, ..., \phi_d$ which are the components of $\phi \in \mathbb{R}^d$. Accordingly the orbital motion is a constant translation of ϕ per turn. In contrast, the spin motion is modeled by A after the situation of (5.19), i.e., after the T—BMT equation so that the spin variable S is $\mathbb{R}^3$-valued and its motion is a rotation which is affected by the orbital motion and can therefore be very complicated. For more details on the T—BMT aspect see the remarks after Definition 6.1.

In this work the spin-orbit trajectories $\begin{pmatrix} \phi \\ S \end{pmatrix} : \mathbb{Z} \to \mathbb{R}^{d+3}$ are required to satisfy the following map formalism equations of motion

$$\phi(n+1) = \phi(n) + 2\pi\omega \ , \tag{6.1}$$

$$S(n+1) = A(\phi(n))S(n) \ , \tag{6.2}$$

where $n \in \mathbb{Z}$ and $\omega \in \mathbb{R}^d, A \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. It is clear from Section 5.1 that arbitrary initial values $\phi(0) \in \mathbb{R}^d, S(0) \in \mathbb{R}^3$ are allowed.

Here $\mathbb{Z}$ denotes the set of integers and $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ denotes the set of 2π–periodic and continuous functions from $\mathbb{R}^d$ into $SO(3)$ (for the general definition of $\mathcal{C}_{per}(\mathbb{R}^d, X)$ with topological space X, see Section C.1). Note that a function on $\mathbb{R}^d$ is called '2π–periodic' if it is 2π–periodic in each of its d arguments. The set $SO(3)$ consists of those real 3×3–matrices R with $\det(R) = 1$ for which $R^T R = I_{3\times3}$ where R^T denotes the transpose of R and $I_{3\times3}$ denotes the 3×3 unit matrix. As is common, the topology of $SO(3)$ is defined as the relative topology from $\mathbb{R}^{3\times3}$ whence each of the nine components of A are continuous functions from $\mathbb{R}^d$ into $\mathbb{R}$. Thus these components are functions in $\mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$ where $\mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^k)$ denotes the set of 2π–periodic and continuous functions from $\mathbb{R}^d$ into $\mathbb{R}^k$. That the 2π–periodicity of A has to be imposed follows from (5.19). Loosely speaking, A is 2π–periodic since $\phi_1, ..., \phi_d$ are angle variables.

The terminology 'orbital motion' is common in Polarized Beam Physics and it should not be confused with the mathematical meaning of 'orbital' in the context of group actions where one deals with orbit spaces (see Appendix B). For the present work $\mathbb{R}^d$ is the appropriate carrier of the orbital motion but if one would go deeper into the matter of spin-orbit tori then the d-torus $\mathbb{T}^d$ is an important alternative. To give a brief glimpse into this matter see Section 9.2 where I employ the orbital motion on $\mathbb{T}^d$. While for the most part of this work $\mathbb{R}^d$ is the arena of the orbital motion, the d-torus $\mathbb{T}^d$ plays an ubiquitous role in this work in the study of the sets

$\mathcal{C}_{per}(\mathbb{R}^d, X)$ as is outlined in Section 6.4.

The system (6.1),(6.2) is autonomous because its r.h.s. does not *explicitly* depend on n (it depends on n only via $\phi(n)$ and $S(n)$!). I summarize the three basic facts about the system (6.1),(6.2): it is autonomous and nonlinear, it is uniquely determined by ω and A, and the 'orbital trajectories' $\phi(\cdot)$ are unaffected by the 'spin trajectories' $S(\cdot)$.

By induction in n one obtains from (6.1),(6.2) that every spin-orbit trajectory $\begin{pmatrix} \phi \\ S \end{pmatrix}$ satisfies, for $n \in \mathbb{Z}$,

$$\begin{pmatrix} \phi(n) \\ S(n) \end{pmatrix} = \begin{pmatrix} \phi(0) + 2\pi n\omega \\ \Psi_{\omega,A}(n;\phi(0))S(0) \end{pmatrix} , \tag{6.3}$$

where, for $\phi \in \mathbb{R}^d$,

$$\begin{aligned} &\Psi_{\omega,A}(0;\phi) := I_{3\times3} , \\ &\Psi_{\omega,A}(n;\phi) := A(\phi + 2\pi(n-1)\omega)\cdots A(\phi + 2\pi\omega)A(\phi) , \qquad (n = 1, 2, ...) , \\ &\Psi_{\omega,A}(n;\phi) = A^T(\phi + 2\pi n\omega)\cdots A^T(\phi - 4\pi\omega)A^T(\phi - 2\pi\omega) , \quad (n = -1, -2, ...) . \end{aligned} \tag{6.4}$$

The function $\Psi_{\omega,A} : \mathbb{Z} \times \mathbb{R}^d \to SO(3)$ defined by (6.4) is uniquely determined by ω and A. Clearly $A(\cdot) = \Psi_{\omega,A}(1;\cdot)$ and $\Psi_{\omega,A}(n;\cdot) \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. For every $\phi(0) \in \mathbb{R}^d, S(0) \in \mathbb{R}^3$ the initial value problem of (6.1),(6.2) has the unique solution (6.3). It also follows easily from (6.4) that a function $\Psi : \mathbb{Z} \times \mathbb{R}^d \to SO(3)$, which satisfies for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d$ the initial value problem

$$\Psi(n+1;\phi) = A(\phi + 2\pi n\omega)\Psi(n;\phi) , \qquad \Psi(0;\phi) = I_{3\times3} , \tag{6.5}$$

satisfies $\Psi = \Psi_{\omega,A}$. Note also that, by (6.4), for $m, n \in \mathbb{Z}, \phi \in \mathbb{R}^d$,

$$\Psi_{\omega,A}(n+m;\phi) = \Psi_{\omega,A}(n;\phi + 2\pi m\omega)\Psi_{\omega,A}(m;\phi) . \tag{6.6}$$

I call S in (6.3) the 'spin trajectory over $\phi(0)$'. We are led to:

Definition 6.1 *(Spin-orbit torus) Given a* $\omega \in \mathbb{R}^d$*, a pair* (ω, A) *is called a 'd-dimensional spin-orbit torus' if* $A \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$*. I call* ω *the 'orbital tune vector' of the spin-orbit torus. The function* $\Psi_{\omega,A} : \mathbb{Z} \times \mathbb{R}^d \to SO(3)$ *is defined by (6.4) and* $\Psi_{\omega,A}(n; \cdot)$ *is called the 'n-turn spin transfer matrix of* (ω, A)*'. I denote, for* $\omega \in \mathbb{R}^d$*, the set of those spin-orbit tori, whose orbital tune vector is* ω*, by* $\mathcal{SOT}(d, \omega)$*. The set of all d-dimensional spin-orbit tori I denote by* $\mathcal{SOT}(d)$ *and the set of all spin-orbit tori by* $\mathcal{SOT}$*. A function* $\begin{pmatrix} \phi \\ S \end{pmatrix} : \mathbb{Z} \to \mathbb{R}^{d+3}$ *is called a 'spin-orbit trajectory of* (ω, A)*' if it satisfies (6.1),(6.2). Accordingly* ϕ *is called an 'orbital trajectory of* (ω, A)*' and* S *is called a 'spin trajectory of* (ω, A) *over* $\phi(0)$*'.* □

In the remaining parts of this section I give some comments on Definition 6.1.

Clearly, for a given $\omega \in \mathbb{R}^d$, there are as many elements in the set $\mathcal{SOT}(d, \omega)$ and as many equations of moton (6.1),(6.2) as there are elements in $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. To put this into perspective one has to recall that the spin-orbit tori are modeled after the situation of (5.19), i.e., after the T—BMT equation. Therefore the spin-orbit tori obtained from (5.19) constitute only a small subset of $\mathcal{SOT}$. Thus in effect the present work demonstrates that important features of the spin-orbit tori can be studied without using (5.19), i.e., without referring to the actual T—BMT equation at all. For example, while the uniqueness theorem of Yokoya (see Section 7.5) holds for a vast set of spin-orbit tori, only a small (but, of course very important) subset of those spin-orbit tori is connected with (5.19) and the T—BMT equation.

Since $\Psi_{\omega,A}(n; \phi) \in SO(3)$, the angle between two spin trajectories over the same $\phi(0)$ is a constant of motion. Of course the Euclidean norm $|S(n)|$ of $S(n)$ is a constant of motion, too.

It follows from (6.6) that, for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d$, we have the useful formula

$$\Psi^T_{\omega,A}(n; \phi) = \Psi_{\omega,A}(-n; \phi + 2\pi n\omega) . \tag{6.7}$$

Picking, for $(\omega, A) \in \mathcal{SOT}(d, \omega)$, a $\phi_0 \in \mathbb{R}^d$, then the equation of spin motion (6.2) for the corresponding orbital trajectory $\phi(n) = \phi_0 + 2\pi n\omega$ reads as

$$S(n+1) = A(\phi_0 + 2\pi n\omega)S(n) \,. \tag{6.8}$$

Of course, every function $S : \mathbb{Z} \to \mathbb{R}^3$, which satisfies (6.8), is a spin trajectory over ϕ_0 of (ω, A) (and vice versa). Moreover if $S : \mathbb{Z} \to \mathbb{R}^3$ satisfies (6.8), then the function $\begin{pmatrix} \phi \\ S \end{pmatrix}$, with $\phi(n) = \phi_0 + 2\pi n\omega$, is a spin-orbit trajectory of (ω, A).

While the system of equations of motion (6.1),(6.2) for $\begin{pmatrix} \phi \\ S \end{pmatrix}$ is autonomous and nonlinear, the equation of motion (6.8) for S is linear and non-autonomous.

Furthermore, if $(\omega, A) \in \mathcal{SOT}(d, \omega)$ and if $\omega, \omega' \in \mathbb{R}^d$ differ only by an element of $\mathbb{Z}^d$ then, due to the 2π–periodicity of A, the spin-orbit tori $(\omega, A), (\omega', A)$ are essentially the same since the associated equation of motion (6.8) is the same for both.

To interpret Definition 6.1 along the lines of Section 5.1 in the context of the map formalism for polarized beams in storage rings, the reader should view $\phi(n)$ as the value of the orbital angle variable and $S(n)$ as the value of the spin variable after n 'turns' around the storage ring. This means that n can be as large as 10^9 whence the present section is definitely not the last word to be said about spin-orbit trajectories. In particular the numerical calculation of $\Psi_{\omega,A}(n; \cdot)$ for large n is a challenging task. Furthermore this calculation can be hampered by the circumstance that A is only approximately known. These circumstances warrant the more involved discussion of spin-orbit tori in this work.

6.2 Introducing the $\mathbb{Z}$-action $L_{\omega,A}$ associated with every spin-orbit torus (ω, A)

Since the equations of motion (6.1),(6.2) are autonomous, each spin-orbit torus (ω, A) is associated with a $\mathbb{Z}$-action $L_{\omega,A}$ which determines the evolution of the spin-orbit trajectories as follows (for details on group actions in general and $\mathbb{Z}$-actions in particular, see Appendix B). Defining the function $L_{\omega,A} : \mathbb{Z} \times \mathbb{R}^{d+3} \to \mathbb{R}^{d+3}$, for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d, S \in \mathbb{R}^3$, by

$$L_{\omega,A}(n; \phi, S) := \begin{pmatrix} \phi + 2\pi n\omega \\ \Psi_{\omega,A}(n; \phi)S \end{pmatrix} , \tag{6.9}$$

I obtain from (6.3) that, for every spin-orbit trajectory $\begin{pmatrix} \phi \\ S \end{pmatrix}$ of (ω, A) and every $n \in \mathbb{Z}$,

$$\begin{pmatrix} \phi(n) \\ S(n) \end{pmatrix} = L_{\omega,A}(n; \phi(0), S(0)) . \tag{6.10}$$

Clearly, by (6.4),(6.6),(6.9), we have, for $m, n \in \mathbb{Z}, \phi \in \mathbb{R}^d, S \in \mathbb{R}^3$,

$$L_{\omega,A}(0; \phi, S) = \begin{pmatrix} \phi \\ S \end{pmatrix} , \tag{6.11}$$

$$L_{\omega,A}(m + n; \phi, S) = L_{\omega,A}(m; L_{\omega,A}(n; \phi, S)) , \tag{6.12}$$

$$L_{\omega,A}(m + n; \phi, S) = L_{\omega,A}(n; L_{\omega,A}(m; \phi, S)) . \tag{6.13}$$

One concludes from (6.11),(6.12) that $L_{\omega,A}$ is a left $\mathbb{Z}$-action on $\mathbb{R}^{d+3}$ and from (6.11),(6.13) that $L_{\omega,A}$ is a right $\mathbb{Z}$-action on $\mathbb{R}^{d+3}$. In fact since the group $\mathbb{Z}$ is Abelian, every left $\mathbb{Z}$-action is a right $\mathbb{Z}$-action and every right $\mathbb{Z}$-action is a left $\mathbb{Z}$-action. Left actions are also called ‘actions’. Since $L_{\omega,A}$ is a $\mathbb{Z}$-action on $\mathbb{R}^{d+3}$, one calls $(\mathbb{R}^{d+3}, L_{\omega,A})$ a ‘$\mathbb{Z}$-space’. In a more loose sense, $L_{\omega,A}$ would be called the 'general solution map' of (6.1),(6.2). Note that $L_{\omega,A}(n;\cdot)$ is continuous whence $(\mathbb{R}^{d+3}, L_{\omega,A})$ is a topological $\mathbb{Z}$-space. Note also that, because $L_{\omega,A}$ is a $\mathbb{Z}$-action, we have, for $n = 1, 2, ...$, that $L_{\omega,A}(n;\cdot)$ is the n-fold composition of $L_{\omega,A}(1;\cdot)$ and, for $n = -1, -2, ...$, that $L_{\omega,A}(n;\cdot)$ is the $|n|$-fold composition of $L_{\omega,A}(-1;\cdot)$. While all these details on $L_{\omega,A}$ are trivial, they are meant for setting the stage for later chapters where I have to study more group actions.

If $\omega \in \mathbb{R}^d$ then I define the function $L_\omega : \mathbb{Z} \times \mathbb{R}^d \to \mathbb{R}^d$, for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d$, by

$$L_\omega(n;\phi) := \phi + 2\pi n\omega \ . \tag{6.14}$$

Clearly L_ω is a $\mathbb{Z}$-action on $\mathbb{R}^d$ and moreover $(\mathbb{R}^d, L_\omega)$ is a topological $\mathbb{Z}$-space.

In Section 9.1 it will be shown how the peculiar structure of (6.6) follows from the fact that $L_{\omega,A}$ is a so-called skew-product of the orbital $\mathbb{Z}$-action L_ω.

Given a spin-orbit torus (ω, A) in $\mathcal{SOT}(d,\omega)$, it follows from (6.6) and Appendix B that $\Psi_{\omega,A}$ is a $SO(3)$-cocycle over the topological $\mathbb{Z}$-space $(\mathbb{R}^d, L_\omega)$ whence $(L_\omega, \Psi_{\omega,A}) \in COC(\mathbb{R}^d, \mathbb{Z}, SO(3))$. I thus have a natural injection $\rho_{\mathcal{SOT}(d)} : \mathcal{SOT}(d) \to COC(\mathbb{R}^d, \mathbb{Z}, SO(3))$, defined for $(\omega, A) \in \mathcal{SOT}(d)$ by

$$\rho_{\mathcal{SOT}(d)}(\omega, A) := (L_\omega, \Psi_{\omega,A}) \ . \tag{6.15}$$

6.3 Introducing the polarization fields of every spin-orbit torus (ω, A) and the associated $\mathbb{Z}$-action $L^{(PF)}_{\omega,A}$

Each spin-orbit torus is associated with a set of functions, called 'polarization fields', which are introduced in this section. The evolution of the polarization fields of a spin-orbit torus (ω, A) is determined by the $\mathbb{Z}$-action $L^{(PF)}_{\omega,A}$ introduced below.

In this work the main impact of polarization fields is that invariant spin fields (which are special polarization fields) show up in Theorem 7.9, i.e., polarization fields impact the group action $R_{d;\omega}$ on $\mathcal{SOT}(d,\omega)$. This group action, to be introduced in Section 7.1, allows to study $\mathcal{SOT}(d,\omega)$ as a whole and exploits some fundamental symmetry properties of $\mathcal{SOT}(d,\omega)$ leading in particular to a definition of spin tune (see Definition 7.11). Not pursued in this work (and only briefly mentioned in Sections 5.1 and 7.6) is a second purpose of polarization fields being an important tool in the statistical treatment of the polarization [EH]. The statistical treatment is needed for coping with the fact that a storage ring bunch contains many particles (typically 10^{11}).

To motivate the concept of polarization field, consider an initial assignment of spins $G:\mathbb{R}^d\to\mathbb{R}^3$, i.e., a spin attached to every point $\phi_0\in\mathbb{R}^d$. Under the $\mathbb{Z}$-action $L_{\omega,A}$ the point $\begin{pmatrix}\phi_0\\ G(\phi_0)\end{pmatrix}$ evolves to $\begin{pmatrix}\phi_0+2\pi n\omega\\ \Psi_{\omega,A}(n;\phi_0)G(\phi_0)\end{pmatrix}$ at n-th turn. Denoting $\phi_0+2\pi n\omega$ by ϕ and $\Psi_{\omega,A}(n;\phi_0)G(\phi_0)$ by $\mathcal{S}_G(n,\phi)$ one obtains

$$\mathcal{S}_G(n,\phi)=\Psi_{\omega,A}(n;\phi-2\pi n\omega)G(\phi-2\pi n\omega)\ . \tag{6.16}$$

The 2π–periodicity of G has to be imposed for the same reason as mentioned in Section 6.1, namely because the components of ϕ are angle variables. One is thus led to:

Definition 6.2 *(Polarization field, spin field) Let (ω,A) be a spin-orbit torus. I call a function $\mathcal{S}_G:\mathbb{Z}\times\mathbb{R}^d\to\mathbb{R}^3$ a 'polarization field of (ω,A)', if it satisfies (6.16) for all ϕ,n and if $G\in\mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3)$. The function G will be called the 'generator of $\mathcal{S}_G$'.*

I call a polarization field $\mathcal{S}_G$ 'invariant' if $\mathcal{S}_G(n,\cdot)$ is independent of n. A polarization field $\mathcal{S}_G$ with $|\mathcal{S}_G(n,\phi)|=1$ is called a 'spin field'. An invariant polarization field which is a spin field is called an 'invariant spin field (ISF)'. □

Remark:

(1) It follows from Definition 6.1 and (6.16) that if $\mathcal{S}_G$ is an invariant polarization field then, for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d$,

$$G(\phi) = \Psi_{\omega,A}(n;\phi - 2\pi n\omega)G(\phi - 2\pi n\omega) . \tag{6.17}$$

This has an interesting implication in the case when the components of ω are rational since then I can choose n in (6.17) sufficiently large such that the components of $n\omega$ are integers. Then (6.17) becomes, due to the 2π-periodicity of $\Psi_{\omega,A}(n;\cdot)$ and G, an eigenvalue value problem for $G(\phi)$:

$$G(\phi) = \Psi_{\omega,A}(n;\phi)G(\phi) . \tag{6.18}$$

It also follows that if the components of ω are not rational then, by rational approximation of ω, one obtains an approximation of an invariant polarization field by solutions of eigenvalue problems. □

By (6.5),(6.16) I get the following equation of motion for a polarization field $\mathcal{S}_G$

$$\mathcal{S}_G(n+1,\phi) = A(\phi - 2\pi\omega)\mathcal{S}_G(n,\phi - 2\pi\omega) . \tag{6.19}$$

If $\mathcal{S}_G$ is a polarization field then $\mathcal{S}_G(0,\cdot) = G(\cdot) \in \mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3)$ and $\mathcal{S}_G(n,\cdot) \in \mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3)$. Clearly, the equation of motion (6.19) for $\mathcal{S}_G$ is linear and autonomous. Defining the function $L^{(PF)}_{\omega,A} : \mathbb{Z} \times \mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3) \to \mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3)$ by

$$L^{(PF)}_{\omega,A}(n;G) := \mathcal{S}_G(n,\cdot) = \Psi_{\omega,A}(n;\cdot - 2\pi n\omega)G(\cdot - 2\pi n\omega) , \tag{6.20}$$

it follows easily from (6.6),(6.16) that $L^{(PF)}_{\omega,A}$ is a $\mathbb{Z}$-action on $\mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3)$, i.e., that $(\mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3), L^{(PF)}_{\omega,A})$ is a $\mathbb{Z}$-space. Thus by (6.16)

$$\mathcal{S}_G(n,\cdot) = L^{(PF)}_{\omega,A}(n-m;\mathcal{S}_G(m,\cdot)) . \tag{6.21}$$

Loosely speaking, $L^{(PF)}_{\omega,A}$ is the transport map associated with (6.19). Clearly, every $G \in \mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3)$ gives a unique polarization field $\mathcal{S}_G$ for a given spin-orbit torus. In particular, each d-dimensional spin-orbit torus has as many polarization fields as the set $\mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$ has elements. We see that the role which the $\mathbb{Z}$-action $L^{(PF)}_{\omega,A}$ plays for polarization fields, is analogous to the role which the $\mathbb{Z}$-action $L_{\omega,A}$ plays for spin-orbit trajectories. Note also that G is a fixpoint of $L^{(PF)}_{\omega,A}$ iff the polarization field $\mathcal{S}_G$ is invariant. Since $L^{(PF)}_{\omega,A}$ is a group action of the group $\mathbb{Z}$ one easily concludes:

Proposition 6.3 *Let (ω, A) be a spin-orbit torus. A polarization field $\mathcal{S}_G$ of (ω, A) is invariant iff*

$$L^{(PF)}_{\omega,A}(1; G) = G \; . \tag{6.22}$$

In other words, $\mathcal{S}_G$ is invariant, iff for all ϕ,

$$G(\phi) = A(\phi - 2\pi\omega)G(\phi - 2\pi\omega) \; . \tag{6.23}$$

□

Proof of Theorem 9.5: See Section F.30. □

Note that (6.23) will be interpreted by Theorem 9.5b as a symmetry property of (ω, A) along the lines of reduction theory.

A polarization field $\mathcal{S}_G$ is a spin field iff $|G(\phi)| = 1$ for all ϕ. Defining the 2-sphere $\mathbb{S}^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$ and equipping it with the relative topology from $\mathbb{R}^3$ we see that the set $\mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$ of 2π–periodic and continuous functions from $\mathbb{R}^d$ into $\mathbb{S}^2$ is equal to the set of 2π–periodic, normalized (w.r.t. the Euclidean norm), and continuous functions from $\mathbb{R}^d$ into $\mathbb{R}^3$. Thus for every spin field $\mathcal{S}_G$ we have $\mathcal{S}_G(n, \cdot) \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$. Clearly each ISF is a polarization field.

Due to Definition 6.2, every polarization field $\mathcal{S}_G$ fulfills three different conditions: the 'dynamical' condition (6.16), the 'kinematical' condition that G is 2π–periodic, and the 'regularity' condition that G is continuous. In contrast to the dynamical and kinematical conditions, the regularity condition is a matter of choice. The regularity of G can basically vary between the extremes 'no regularity condition' and 'G being real analytic'. In this work I choose G to be continuous since the spin-orbit tori are built on continuity, i.e., the $\Psi_{\omega,A}(n;\cdot)$ are continuous functions.

Since the equation of motion (6.19) for $\mathcal{S}_G$ is linear, $L^{(PF)}_{\omega,A}(n;\cdot)$ is a homomorphism of the additive group $\mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3)$, i.e., for $n\in\mathbb{Z}, G, G'\in\mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3)$,

$$L^{(PF)}_{\omega,A}(n;G+G') = L^{(PF)}_{\omega,A}(n;G) + L^{(PF)}_{\omega,A}(n;G')\ . \tag{6.24}$$

Eq. (6.24) allows, by the technique of twisted cocycles [HK1, HK2, Zi1], to define cohomology groups for any spin-orbit torus, which give further insight into $\mathcal{SOT}(d,\omega)$ in general and into the ISF conjecture in particular [He]. However this is beyond the scope of the present work.

6.4 Homotopy Theory relevant for spin-orbit tori

Throughout this work I will see some impact of Homotopy Theory on the theory of spin-orbit tori and in this section I introduce some basic features (the details are worked out in Appendix C).

Let X be a path-connected topological space. In the context of spin-orbit tori, one is especially interested in $X = SO(3)$ and $X = \mathbb{S}^2$ (recall that spin transfer matrices are $SO(3)$-valued functions and that spin fields are $\mathbb{S}^2$-valued functions). The use of Homotopy Theory for $\mathcal{C}_{per}(\mathbb{R}^d,X)$ is twofold. Firstly, I use it by applying the Homotopy Lifting Theorem (see Lemma C.6 in Section C.1) which in turn is used in many of those proofs of this work which involve the sets $\mathcal{C}_{per}(\mathbb{R}^d,X)$. Secondly, Homotopy Theory gives us the useful equivalence relation $\simeq^{2\pi}_X$ on $\mathcal{C}_{per}(\mathbb{R}^d,X)$, as follows. To explain this equivalence relation I first note, by Proposition C.4, that any two functions in $\mathcal{C}_{per}(\mathbb{R}^d,X)$ are homotopic w.r.t. X, i.e., $[\mathbb{R}^d,X]$ is a singleton.

In other words, the equivalence relation $\simeq_X$ on $\mathcal{C}_{per}(\mathbb{R}^d, X)$ is of little use. However, since the functions in $\mathcal{C}_{per}(\mathbb{R}^d, X)$ are 2π-periodic, one can associate, as detailed in Section C.3, every function $g \in \mathcal{C}_{per}(\mathbb{R}^d, X)$ with a function $G := FAC_d(g; X) \in \mathcal{C}(\mathbb{T}^d, X)$ which is uniquely determined by g via the relation $G \circ p_{4,d} = g$. Thus I call two functions $g_0, g_1 \in \mathcal{C}_{per}(\mathbb{R}^d, X)$ '2π-homotopic w.r.t. X', written $g_0 \simeq_X^{2\pi} g_1$, if $FAC_d(g_0; X), FAC_d(g_1; X)$ are homotopic w.r.t. X, i.e., if $FAC_d(g_0; X) \simeq_X FAC_d(g_1; X)$. Clearly $\simeq_X^{2\pi}$ is an equivalence relation on $\mathcal{C}_{per}(\mathbb{R}^d, X)$ and I denote the set of equivalence classes by $[\mathbb{R}^d, X]_{2\pi}$. Obviously the function which maps the $\simeq_X$-equivalence class of a $F \in \mathcal{C}(\mathbb{T}^d, X)$ to the $\simeq_X^{2\pi}$-equivalence class of $F \circ p_{4,d}$, is a bijection from $[\mathbb{T}^d, X]$ onto $[\mathbb{R}^d, X]_{2\pi}$. Thus every statement about $[\mathbb{R}^d, X]_{2\pi}$ corresponds to a statement about $[\mathbb{T}^d, X]$.

The point to be made here is that for the topological spaces X of interest, in general two functions in $\mathcal{C}(\mathbb{T}^d, X)$ are not homotopic w.r.t. X whence, in general, two functions in $\mathcal{C}_{per}(\mathbb{R}^d, X)$ are not 2π-homotopic w.r.t. X, i.e., $[\mathbb{R}^d, X]_{2\pi}$ is not a singleton. In particular we will see below that, for no positive integer d, is $[\mathbb{R}^d, SO(3)]_{2\pi}$ a singleton and that, by Proposition C.18c and Theorem C.24, $[\mathbb{R}^d, \mathbb{S}^2]_{2\pi}$ is not a singleton for any $d \geq 2$. The meaning of this is, loosely speaking, that, among the functions in $\mathcal{C}_{per}(\mathbb{R}^d, X)$, the ones which are especially simple are the g which are '2π-nullhomotopic w.r.t. X', i.e., for which $FAC_d(g; X)$ is nullhomotopic w.r.t. X (the latter condition means that $FAC_d(g; X)$ is homotopic w.r.t. X to a constant function). Note that, by Proposition C.18c, all 2π-nullhomotopic functions in $\mathcal{C}_{per}(\mathbb{R}^d, X)$ are 2π-homotopic w.r.t. X, i.e., belong to the same element of $[\mathbb{R}^d, X]_{2\pi}$. Thus if $[\mathbb{R}^d, X]_{2\pi}$ is not a singleton then $\mathcal{C}_{per}(\mathbb{R}^d, X)$ contains functions which are not 2π-nullhomotopic w.r.t. X. As we will see in this work, the fact that $[\mathbb{R}^d, SO(3)]_{2\pi}$ and, for $d \geq 2$, $[\mathbb{R}^d, \mathbb{S}^2]_{2\pi}$ are not singletons, contributes to the structural richness of the sets $\mathcal{SOT}(d, \omega)$. Note that, in the context of polarized beams in storage rings, the case $d = 3$ is the most important one whereas the cases $d = 1, 2$ come next in terms of importance.

I wrap up this brief section by mentioning several important facts and concepts valid for the case $X = SO(3)$ and it first of all has to be pointed out that in my study of $SO(3)$-valued functions in Appendix C the 'quaternion formalism' is employed which consists in representing $SO(3)$-valued functions by $\mathbb{S}^3$-valued functions. For every positive integer d there is a function $Ind_{3,d} : \mathcal{C}_{per}(\mathbb{R}^d, SO(3)) \to \{1, -1\}^d$, defined by Definition C.14 and called the '$SO(3)$-index', which, due to Proposition C.18e, has the property that, if $g_0, g_1 \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ and $g_0 \simeq^{2\pi}_{SO(3)} g_1$, then $Ind_{3,d}(g_0) = Ind_{3,d}(g_1)$. Since, by Theorem C.15a, the function $Ind_{3,d}$ is onto $\{1, -1\}^d$ one observes that $[\mathbb{R}^d, SO(3)]_{2\pi}$ is not a singleton. Moreover, for $d = 1, 2$, the function $Ind_{3,d}$ completely determines $[\mathbb{R}^d, SO(3)]_{2\pi}$ since, by Theorem C.22c, we have, for $g_0, g_1 \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$, that $g_0 \simeq^{2\pi}_{SO(3)} g_1$ iff $Ind_{3,d}(g_0) = Ind_{3,d}(g_1)$. For the most important case, $d = 3$, the structure of $[\mathbb{R}^d, SO(3)]_{2\pi}$ is even richer. In fact, Definition C.21 gives a function $DEG : \mathcal{C}_{per}(\mathbb{R}^3, SO(3)) \to \mathbb{Z}$, which is onto $\mathbb{Z}$ and, due to Theorem C.22f, has the property that, for $g_0, g_1 \in \mathcal{C}_{per}(\mathbb{R}^3, SO(3))$, we have $g_0 \simeq^{2\pi}_{SO(3)} g_1$ iff $DEG(g_0) = DEG(g_1)$ and $Ind_{3,3}(g_0) = Ind_{3,3}(g_1)$. Thus, for $d = 3$, $[\mathbb{R}^d, SO(3)]_{2\pi}$ has infinitely many elements. One also concludes that, for $d = 1, 2, 3$, the $SO(3)$-index and the function DEG are sufficient to determine the equivalence class of every $g \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ w.r.t. the equivalence relation $\simeq^{2\pi}_{SO(3)}$ whence to determine the equivalence class of every $F \in \mathcal{C}(\mathbb{T}^d, SO(3))$ w.r.t. the equivalence relation $\simeq_{SO(3)}$.

Before I state the following proposition, note that I consider $\{1, -1\}$ as a multiplicative group with identity 1 and $\{1, -1\}^d$ as the d-fold direct product of the group $\{1, -1\}$. The following proposition is the most basic result of how Homotopy Theory impacts spin-orbit tori via the $SO(3)$-index.

Proposition 6.4 *If $(\omega, A) \in \mathcal{SOT}(d, \omega)$ then, for an arbitrary integer n, we have*

$$Ind_{3,d}(\Psi_{\omega,A}(n; \cdot)) = (Ind_{3,d}(A))^n \,, \tag{6.25}$$

where $(Ind_{3,d}(A))^n$ *denotes the* n*-th power of* $Ind_{3,d}(A)$ *w.r.t. the group multiplication in* $\{1, -1\}^d$.

Proof of Proposition 6.4: See Section F.1. $\square$

Chapter 7

Transforming spin-orbit tori

In this chapter I study the right group action $R_{d,\omega}$ on $\mathcal{SOT}(d,\omega)$ for the group $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ and the associated equivalence relation $\sim_{d,\omega}$ by which two spin-orbit tori $\mathcal{SOT}(d,\omega)$ are equivalent iff they belong to the same $R_{d,\omega}$-orbit. The right group action $R_{d,\omega}$ is an outgrowth of the observation (see Section 7.1) that spin-orbit tori can be transformed into each other in a natural way. In fact in each $\mathcal{SOT}(d,\omega)$ one has a large family of pairs of spin-orbit tori whose topological $\mathbb{Z}$-spaces $(\mathbb{R}^{d+3}, L_{\omega,A})$ are conjugate by conjugating homeomorphisms which form a family L_T labelled by the $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. In particular I obtain in Section 7.1 a transformation law for spin-orbit tori and polarization fields.

The right group action $R_{d,\omega}$ allows to define the spin tune (spin tune of first kind) in an elegant way. We will see that two spin-orbit tori which belong to the same $R_{d,\omega}$-orbit, share important properties, e.g., they have the same spin tunes of first kind (see Proposition 7.12) and either both of them have an ISF or both of them have no ISF (see Theorem 7.3e). In other words, spin-orbit tori, whose topological $\mathbb{Z}$-spaces $(\mathbb{R}^{d+3}, L_{\omega,A})$ are conjugate by a homeomorphism L_T, resemble each other. Thus to a large extent the study of $\mathcal{SOT}(d,\omega)$ reduces to the study of just one spin-orbit torus per $R_{d,\omega}$-orbit.

This, of course, raises the question, of whether an $R_{d,\omega}$-orbit contains spin-orbit tori which are more ‘simply structured’ than others. Indeed (see also Section 7.6) it is widely believed and based on numerical evidence that, generically, the spin-orbit tori of practical relevance are ‘weak coboundaries’ (see Definition 7.6) which means that each of them lies on the same $R_{d,\omega}$-orbit as a ‘weakly trivial’ spin-orbit torus (see Definition 7.4). Thus, generically, many features of spin-orbit tori can be studied on weakly trivial spin-orbit tori, which indeed are simply structured. Note also that the $SO(3)$-indices and the $SO_3(2)$-indices associated with a weakly trivial spin-orbit torus carry important topological information (see Proposition 7.5). There is strong evidence that, generically, the spin-orbit tori of practical relevance are not only weak coboundaries but also ‘almost coboundaries’ (see Definition 7.6). As their name suggests, almost coboundaries lie on the same $R_{d,\omega}$-orbit as ‘almost trivial’ spin-orbit tori. Most importantly, almost coboundaries are those spin-orbit tori which carry spin tunes (in fact, spin tunes of first kind - see Definition 7.11). ‘Coboundaries’ (see Definition 7.6) are those almost coboundaries which are on spin-orbit resonance of first kind. Coboundaries, by definition, lie on the same $R_{d,\omega}$-orbit as ‘trivial’ spin-orbit tori, which indeed are the simplest spin-orbit tori of all (see Definition 7.4).

7.1 Introducing the transformations of spin-orbit tori and the right group action $R_{d,\omega}$ on $\mathcal{SOT}(d,\omega)$

In this section I introduce the right group action $R_{d,\omega}$ and the associated equivalence relation $\sim_{d,\omega}$.

The motivation for $R_{d,\omega}$ comes from the practical need to transform spin trajectories in order to simplify (analytical and numerical) computations. The archetypical way to transform a spin trajectory goes, in the context of spin-orbit tori, as follows. Let a spin-orbit torus (ω, A) be given with a spin trajectory $S(\cdot)$ over some ϕ_0. Then a function $t : \mathbb{Z} \to SO(3)$ transforms $S(\cdot)$ into the function $S' : \mathbb{Z} \to \mathbb{R}^3$ via $S'(n) := t^T(n)S(n)$ (using t^T instead of t is just a convention). Of course, since $S(\cdot)$ satisfies the equation of motion (6.8), one observes that $S'(\cdot)$ satisfies the equation of motion

$$S'(n+1) = t^T(n+1)A(\phi_0 + 2\pi n\omega)t(n)S'(n) \ , \tag{7.1}$$

where $n \in \mathbb{Z}$. Clearly $S'(\cdot)$ has many features of a spin motion, e.g., $|S'(n)| = |S(n)|$ is independent of n and $S'(n)$ is uniquely determined by $S'(0)$ and n. Perhaps surprisingly however, in general $S'(\cdot)$ is not a spin trajectory of any spin-orbit torus! This follows from the fact that $A(\phi_0 + 2\pi n\omega)$ is an ω-quasiperiodic function of n whereas $t^T(n+1)A(\phi_0 + 2\pi n\omega)t(n)$ in general is not a quasiperiodic function of n at all, since t may not be quasiperiodic. Thus in general there is no spin-orbit torus (ω', A') with the spin trajectory $S'(\cdot)$, i.e., which satisfies $t^T(n+1)A(\phi_0+2\pi n\omega)t(n) = A'(\phi_0 + 2\pi n\omega')$ since $A'(\phi_0 + 2\pi n\omega')$ is a ω'-quasiperiodic function of n while, in general, $t^T(n+1)A(\phi_0 + 2\pi n\omega)t(n)$ is not a quasiperiodic function of n. Note that quasiperiodic functions play a major role in Chapter 8 and are defined in Section D.1.

Part d) of the following proposition now comes as a relief.

Proposition 7.1 *a) Let $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. Then the function $L_T : \mathbb{R}^{d+3} \to \mathbb{R}^{d+3}$, defined by*

$$L_T(\phi, S) := (\phi, T^T(\phi)S) \ , \tag{7.2}$$

is a homeomorphism onto $\mathbb{R}^{d+3}$ and its inverse L_T^{-1} is defined by $L_T^{-1}(\phi, S) := (\phi, T(\phi)S)$, i.e., $L_T^{-1} = L_{T^T}$.

b) Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$ *and* $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$*. Then, for* $n \in \mathbb{Z}, \phi \in \mathbb{R}^d, S \in \mathbb{R}^3$,

$$\Big(L_T \circ L_{\omega,A}(n;\cdot) \circ L_T^{-1}\Big) \begin{pmatrix} \phi \\ S \end{pmatrix} = \begin{pmatrix} \phi + 2\pi n\omega \\ T^T(\phi + 2\pi n\omega)\Psi_{\omega,A}(n;\phi)T(\phi)S \end{pmatrix} . \quad (7.3)$$

Moreover $(\omega, A') \in \mathcal{SOT}(d,\omega)$ *where*

$$A'(\phi) := T^T(\phi + 2\pi\omega)A(\phi)T(\phi) . \quad (7.4)$$

Also

$$\Psi_{\omega,A'}(n;\phi) = T^T(\phi + 2\pi n\omega)\Psi_{\omega,A}(n;\phi)T(\phi) . \quad (7.5)$$

Furthermore, L_T *is a continuous* $\mathbb{Z}$*-map from the topological* $\mathbb{Z}$*-space* $(\mathbb{R}^{d+3}, L_{\omega,A})$ *to the topological* $\mathbb{Z}$*-space* $(\mathbb{R}^{d+3}, L_{\omega,A'})$*, i.e., for* $n \in \mathbb{Z}$,

$$L_{\omega,A'}(n;\cdot) = L_T \circ L_{\omega,A}(n;\cdot) \circ L_T^{-1} . \quad (7.6)$$

Thus the topological $\mathbb{Z}$*-spaces* $(\mathbb{R}^{d+3}, L_{\omega,A})$ *and* $(\mathbb{R}^{d+3}, L_{\omega,A'})$ *are conjugate.*

c) (Transformation rule of spin-orbit trajectories) Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$ *and* $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$*. If* $\begin{pmatrix} \phi(\cdot) \\ S(\cdot) \end{pmatrix}$ *is a spin-orbit trajectory of the spin-orbit torus* (ω, A)*, then* $\begin{pmatrix} \phi(\cdot) \\ S'(\cdot) \end{pmatrix}$ *is a spin-orbit trajectory of the spin-orbit torus* (ω, A') *where* A' *is given by (7.4) and where, for* $n \in \mathbb{Z}$,

$$\begin{pmatrix} \phi(n) \\ S'(n) \end{pmatrix} := L_T(\phi(n), S(n)) = \begin{pmatrix} \phi(n) \\ T^T(\phi(n))S(n) \end{pmatrix} . \quad (7.7)$$

d) (Transformation rule of spin trajectories) Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$ *and* $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$*. Let also* $\phi_0 \in \mathbb{R}^d$ *and let* $t : \mathbb{Z} \to SO(3)$ *be defined by* $t(n) := T(\phi_0 + 2\pi n\omega)$*. If* $S(\cdot)$ *is a spin trajectory, over* ϕ_0*, of the spin-orbit torus* (ω, A) *then* $S'(\cdot)$*, defined by* $S'(n) := t^T(n)S(n)$*, is a spin trajectory, over* ϕ_0*, of the spin-orbit torus* (ω, A') *where* A' *is given by (7.4).*

Proof of Proposition 7.1: See Section F.2. □

With Proposition 7.1b we see, for every $\omega \in \mathbb{R}^d$, that every $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ associates any $(\omega, A) \in \mathcal{SOT}(d, \omega)$ with onother $(\omega, A') \in \mathcal{SOT}(d, \omega)$. This I cast into the following definition:

Definition 7.2 *Let $\omega \in \mathbb{R}^d$. I define the function $R_{d,\omega} : \mathcal{C}_{per}(\mathbb{R}^d, SO(3)) \times \mathcal{SOT}(d, \omega) \to \mathcal{SOT}(d, \omega)$ by $R_{d,\omega}(T; \omega, A) := (\omega, A')$ where $(\omega, A) \in \mathcal{SOT}(d, \omega), T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$, and where $A' \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ is given by (7.4). If $R_{d,\omega}(T; \omega, A) = (\omega, A')$ then I call T a 'transfer field from (ω, A) to (ω, A')'.* □

The following theorem states the basic properties of $R_{d,\omega}$.

Theorem 7.3 *a) Let $(\omega, A), (\omega, A') \in \mathcal{SOT}(d, \omega)$ and $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ such that $R_{d,\omega}(T; \omega, A) = (\omega, A')$. Then (7.5) holds for all $n \in \mathbb{Z}, \phi \in \mathbb{R}^d$.*

b) Let $\omega \in \mathbb{R}^d$. Then, for $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$, $(\omega, A) \in \mathcal{SOT}(d, \omega)$,

$$L_{R_{d,\omega}(T;\omega,A)} = L_T \circ L_{\omega,A}(n; \cdot) \circ L_T^{-1} \ . \tag{7.8}$$

Furthermore $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ is a group under pointwise multiplication of $SO(3)$-valued functions and $R_{d,\omega}$ is a right $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$-action on $\mathcal{SOT}(d, \omega)$.

c) (Transformation rule of spin-orbit trajectories) Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$ and $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. If $\begin{pmatrix} \phi(\cdot) \\ S(\cdot) \end{pmatrix}$ is a spin-orbit trajectory of the spin-orbit torus (ω, A), then $\begin{pmatrix} \phi(\cdot) \\ S'(\cdot) \end{pmatrix}$, defined by (7.7), is a spin-orbit trajectory of the spin-orbit torus $R_{d,\omega}(T; \omega, A)$.

d) (Transformation rule of polarization fields) Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$ and $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. Let also $\mathcal{S}_G$ be a polarization field of the spin-orbit torus (ω, A).

Then $\mathcal{S}'$*, defined by*

$$\mathcal{S}'(n,\phi) := T^T(\phi)\mathcal{S}_G(n,\phi) \ , \tag{7.9}$$

is a polarization field of the spin-orbit torus $R_{d,\omega}(T;\omega,A)$ *and the generator of* $\mathcal{S}'$ *is* $T^T G$*. Thus for every* $n\in\mathbb{Z}, G\in\mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3)$

$$L^{(PF)}_{\omega,A'}(n;G) = T^T L^{(PF)}_{\omega,A}(n;TG) \ . \tag{7.10}$$

If the polarization field $\mathcal{S}_G$ *is invariant, then so is* $\mathcal{S}'$*. If the polarization field* $\mathcal{S}_G$ *is a spin field, then so is* $\mathcal{S}'$*.*

e) Let $(\omega,A),(\omega,A')\in\mathcal{SOT}(d,\omega)$ *belong to the same* $R_{d,\omega}$*-orbit. Then either both spin-orbit tori have an ISF or neither of them.*

f) Let $(\omega,A),(\omega,A')\in\mathcal{SOT}(d,\omega)$ *belong to the same* $R_{d,\omega}$*-orbit. Then, for every integer* n*,* $\Psi_{\omega,A}(n;\cdot),\Psi_{\omega,A'}(n;\cdot)$ *have the same* $SO(3)$*-index, i.e.,* $Ind_{3,d}(\Psi_{\omega,A}(n;\cdot)) = Ind_{3,d}(\Psi_{\omega,A'}(n;\cdot))$*. If* $d=1,2$ *then, for every integer* n*,* $\Psi_{\omega,A}(n;\cdot)\simeq^{2\pi}_{SO(3)}\Psi_{\omega,A'}(n;\cdot)$*.*

Proof of Theorem 7.3: See Section F.3. □

If $(\omega,A),(\omega,A')\in\mathcal{SOT}(d,\omega)$ lie on the same $R_{d,\omega}$-orbit then I write $(\omega,A)\sim_{d,\omega}(\omega,A')$. It follows from Theorem 7.3b that $\sim_{d,\omega}$ is an equivalence relation on $\mathcal{SOT}(d,\omega)$. It also follows from Theorem 7.3b that, for each $T\in\mathcal{C}_{per}(\mathbb{R}^d,SO(3))$, the function $R_{d,\omega}(T;\cdot)$ is a bijection from $\mathcal{SOT}(d,\omega)$ onto $\mathcal{SOT}(d,\omega)$. Clearly each $R_{d,\omega}(T;\cdot)$ transforms spin-orbit tori into spin-orbit tori and the associated transformation of spin-orbit trajectories and polarization fields is given by parts c),d) of Theorem 7.3 respectively.

Since $\mathcal{C}_{per}(\mathbb{R}^d,SO(3))$ is a group under pointwise multiplication of $SO(3)$-valued functions, the constant function in $\mathcal{C}_{per}(\mathbb{R}^d,SO(3))$ whose constant value is $I_{3\times3}$, is the unit element of the group. If there is no danger of confusion, I denote the unit element by $I_{3\times3}$. Furthermore the inverse of $f\in\mathcal{C}_{per}(\mathbb{R}^d,SO(3))$ is the transpose f^T

since $(f^T f)(\phi) = f^T(\phi)f(\phi) = I_{3\times 3}$. Since the group $SO(3)$ is not Abelian, so is the group $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$.

As announced at the beginning of Chapter 7, spin-orbit tori on the same $R_{d,\omega}$-orbit share some important properties and with parts e,f of Theorem 7.3 we have got a first glimpse on that and more in that vein will follow. This raises the following issue. While, by Proposition 7.1b, spin-orbit tori on the same $R_{d,\omega}$-orbit have conjugate topological $\mathbb{Z}$-spaces $(\mathbb{R}^{d+3}, L_{\omega,A})$ this does not exclude more general conjugacy relations in $\mathcal{SOT}(d,\omega)$. Although I here cannot pursue more general conjugacy relations, it is in fact conceivable that there are pairs of spin-orbit tori in $\mathcal{SOT}(d,\omega)$ whose topological $\mathbb{Z}$-spaces $(\mathbb{R}^{d+3}, L_{\omega,A})$ are conjugate but which do not lie on the same $R_{d,\omega}$-orbit. Nevertheless it is questionable whether those pairs of spin-orbit tori would share properties like the one in Theorem 7.3e.

Since the group $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ is not Abelian, it is easy to see that $R_{d,\omega}$ is *not* a left $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$-action on $\mathcal{SOT}(d,\omega)$. However, as every right action has its 'dual' left action, I could use the left $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$-action $L_{d,\omega}$ on $\mathcal{SOT}(d,\omega)$ defined by $L_{d,\omega}(T;\omega,A) := R_{d,\omega}(T^T;\omega,A)$ and the subsequent theory would be just 'dual' to the theory based on $R_{d,\omega}$. Nevertheless I stick, for convenience, with $R_{d,\omega}$.

Remark:

(1) That $R_{d,\omega}$ is so useful in this work is due to the fact that the equations of motion (6.1), (6.2) are autonomous. In a more general situation where the ring is not a storage ring but where the beam is accelerated, (6.1), (6.2) maybe generalized to a non-autonomous system of the form

$$\phi(n+1) = \phi(n) + 2\pi\omega\ , \qquad S(n+1) = A(n;\phi(n))S(n)\ . \quad (7.11)$$

Accordingly the definition of $\mathcal{SOT}(d,\omega)$ would be modified and the right group action $R_{d,\omega}$ would be modified to a right G-action where G consists of functions $T : \mathbb{Z}\times\mathbb{R}^d \to SO(3)$ where $T(n,\cdot) \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. □

7.2 Introducing weakly trivial spin-orbit tori

As mentioned at the beginning of Chapter 7, simply structured spin-orbit tori will play an important role in this work and the following definition specifies what a 'simply structured' spin-orbit torus is.

Definition 7.4 *(Trivial, almost trivial, weakly trivial spin-orbit tori) A spin-orbit torus (ω, A) is called 'trivial' if $\Psi_{\omega,A}(n;\phi) = I_{3\times3}$. The set of trivial spin-orbit tori in $\mathcal{SOT}(d,\omega)$ is denoted by $\mathcal{T}(d,\omega)$. A spin-orbit torus (ω, A) is called 'almost trivial' if $\Psi_{\omega,A}$ is $SO_3(2)$-valued and if, for every integer n, $\Psi_{\omega,A}(n;\phi)$ is independent of ϕ where $SO_3(2) \subset SO(3)$ is defined by Definition C.2. I denote the set of almost trivial spin-orbit tori in $\mathcal{SOT}(d,\omega)$ by $\mathcal{AT}(d,\omega)$. A spin-orbit torus (ω, A) is called 'weakly trivial' if $\Psi_{\omega,A}$ is $SO_3(2)$-valued and the set of weakly trivial spin-orbit tori in $\mathcal{SOT}(d,\omega)$ is denoted by $\mathcal{WT}(d,\omega)$.* □

It is clear by (6.4) that a spin-orbit torus (ω, A) is trivial iff $A = I_{3\times3}$.

I now draw some simple consequences from Definition 7.4. Firstly, for each $\omega \in \mathbb{R}^d$, there exists exactly one trivial spin-orbit torus (ω, A), i.e., $\mathcal{T}(d,\omega) = \{(\omega, I_{3\times3})\}$. Secondly

$$\mathcal{T}(d,\omega) \subset \mathcal{AT}(d,\omega) \subset \mathcal{WT}(d,\omega) \subset \mathcal{SOT}(d,\omega)\ . \tag{7.12}$$

Thirdly it is clear by Definition 6.2 that every weakly trivial spin-orbit torus has the constant ISF's $\mathcal{S}_G = e^3$ and $\mathcal{S}_G = -e^3$ where e^i denotes the i-th unit vector (see Definition C.2).

For the following proposition, I note that the topology of $SO_3(2)$ is defined as the relative topology from $\mathbb{R}^{3\times3}$ (see also Definition C.2). Thus if $(\omega, A) \in \mathcal{WT}(d,\omega)$ then, for every $n \in \mathbb{Z}$, the function $\Psi_{\omega,A}(n;\cdot)$ belongs to $\mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$ whence has a unique phase function (which is an element of $\mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$) and has a unique

$SO_3(2)$-index (which is an element of $\mathbb{Z}^d$). Note that the $SO_3(2)$-index is defined by Definition C.12. Note also that, for $\mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$, each of the d components of $Ind_{2,d}(g)$ can be interpreted, in an obvious way, as a winding number in the plane $\mathbb{R}^2$. However this aspect of the $SO_3(2)$-index plays no role in this work. Denoting the fractional part of a real number x by $\lfloor x \rfloor$, I obtain

Proposition 7.5 *a) (Structure of weakly trivial spin-orbit tori) Let $(\omega, A) \in \mathcal{WT}(d, \omega)$. Then, for every positive integer n,*

$$\Psi_{\omega,A}(n;\phi) = \exp\left(\mathcal{J}[nN^T\phi + \pi n(n-1)N^T\omega + 2\pi\sum_{j=0}^{n-1} g(\phi + 2\pi j\omega)]\right) , \quad (7.13)$$

where $N := Ind_{2,d}(A), g := PHF(A)$ and $\mathcal{J}$ is defined by (C.1). Also, for every $n \in \mathbb{Z}$,

$$Ind_{2,d}(\Psi_{\omega,A}(n;\cdot)) = nInd_{2,d}(A) . \quad (7.14)$$

Thus defining $f : \mathbb{Z} \times \mathbb{R}^d \to \mathbb{R}$ by $f(n,\cdot) := PHF(\Psi_{\omega,A}(n;\cdot))$, I have $f(1,\cdot) = g(\cdot)$ and, for every $n \in \mathbb{Z}$,

$$\Psi_{\omega,A}(n;\phi) = \exp(\mathcal{J}[nN^T\phi + 2\pi f(n,\phi)]) . \quad (7.15)$$

Moreover $\Psi_{\omega,A}(n;\cdot)$ is 2π-nullhomotopic w.r.t. $SO(3)$ iff $Ind_{3,d}(\Psi_{\omega,A}(n;\cdot)) = (1, ..., 1)^T$. Furthermore the $SO(3)$-index of $\Psi_{\omega,A}(n;\cdot)$ reads as $Ind_{3,d}(\Psi_{\omega,A}(n;\cdot))$ $= ((-1)^{nN_1}, ..., (-1)^{nN_d})^T$.

b) (Structure of almost trivial spin-orbit tori) If $(\omega, A) \in \mathcal{AT}(d, \omega)$, then, for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d$,

$$\Psi_{\omega,A}(n;\phi) = \Psi_{\omega,A}(n;0) = \exp(\mathcal{J}2\pi n\nu) , \quad (7.16)$$

where $\nu := PH(A)$ (recall Definition C.2). Moreover if $(\omega, A) \in \mathcal{AT}(d, \omega)$ then, for every $n \in \mathbb{Z}$, $Ind_2(\Psi_{\omega,A}(n;\cdot)) = 0$ and $PHF(\Psi_{\omega,A}(n;\cdot))$ is the constant function *in* $\mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$ *whose value is* $\lfloor n\nu \rfloor$ *where* $\nu := PH(A)$. *Furthermore, a* $(\omega, A) \in \mathcal{AT}(d, \omega)$ *is trivial iff* $PH(A) = 0$.

c) (The one-turn criterion) Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$. *Then* $(\omega, A) \in \mathcal{WT}(d, \omega)$ *iff* A *is* $SO_3(2)$*-valued. Moreover* $(\omega, A) \in \mathcal{AT}(d, \omega)$ *iff* A *is* $SO_3(2)$*-valued and constant.*

d) Let $(\omega, A), (\omega, A') \in \mathcal{WT}(d, \omega)$. *If* n *is an even integer then* $\Psi_{\omega,A}(n;\cdot) \simeq^{2\pi}_{SO(3)} \Psi_{\omega,A'}(n;\cdot)$. *If* n *is an odd integer then* $\Psi_{\omega,A}(n;\cdot) \simeq^{2\pi}_{SO(3)} \Psi_{\omega,A'}(n;\cdot)$ *iff* $Ind_{3,d}(A) = Ind_{3,d}(A')$. *For every integer* n, $(\omega, A) \sim_{d,\omega} (\omega, A')$ *implies* $\Psi_{\omega,A}(n;\cdot) \simeq^{2\pi}_{SO(3)} \Psi_{\omega,A'}(n;\cdot)$.

Proof of Proposition 7.5: See Section F.4. □

Note that the last claim in Proposition 7.5a confirms Proposition 6.4. Note also that, by Proposition 7.5c and (6.4), there are as many weakly trivial spin-orbit tori in every $\mathcal{SOT}(d, \omega)$ as there are elements in $\mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$ and that there are as many almost trivial spin-orbit tori in every $\mathcal{SOT}(d, \omega)$ as there are elements in $[0, 1)$.

7.3 Introducing weak coboundaries

Recalling Section 6.2, given a spin-orbit torus (ω, A) in $\mathcal{SOT}(d, \omega)$, the function $\Psi_{\omega,A}$ is a $SO(3)$-cocycle over the topological $\mathbb{Z}$-space $(\mathbb{R}^d, L_\omega)$. This terminology comes from Dynamical Systems Theory and, in fact, from this terminology I also borrow the terms 'coboundary' and 'almost coboundary' which will be introduced now (the weaker notion 'weak coboundary' is my terminology). Note also that, in this terminology, if $(\omega, A), (\omega, A')$ lie on the same $R_{d,\omega}$-orbit then the $SO(3)$-cocycles $\Psi_{\omega,A}, \Psi_{\omega,A'}$ are called 'cohomologous'.

Definition 7.6 *(Coboundary, almost coboundary, weak coboundary)*

A spin-orbit torus $(\omega, A) \in \mathcal{SOT}(d, \omega)$ *is called a 'coboundary' if it belongs to the* $R_{d,\omega}$*-orbit of the trivial spin-orbit torus* $(\omega, I_{3\times 3})$. *I denote the set of coboundaries* *in $\mathcal{SOT}(d,\omega)$ by $\mathcal{CB}(d,\omega)$. A spin-orbit torus $(\omega, A) \in \mathcal{SOT}(d,\omega)$ is called an 'almost coboundary' if it belongs to the $R_{d,\omega}$-orbit of a spin-orbit torus in $\mathcal{AT}(d,\omega)$. I denote the set of almost coboundaries in $\mathcal{SOT}(d,\omega)$ by $\mathcal{ACB}(d,\omega)$. A spin-orbit torus $(\omega, A) \in \mathcal{SOT}(d,\omega)$ is called a 'weak coboundary' if it belongs to the $R_{d,\omega}$-orbit of a spin-orbit torus in $\mathcal{WT}(d,\omega)$. I denote the set of weak coboundaries in $\mathcal{SOT}(d,\omega)$ by $\mathcal{WCB}(d,\omega)$.* □

Thus a spin-orbit torus (ω, A) is called a 'coboundary' iff $\Psi_{\omega,A}$ is a coboundary in the terminology of Dynamical Systems Theory and is called an 'almost coboundary' iff $\Psi_{\omega,A}$ is an almost coboundary in the terminology of Dynamical Systems Theory. Note also that the terminology coboundary is also borrowed from Nonabelian Group Cohomology.

Recalling Section 7.1, $\sim_{d,\omega}$ is an equivalence relation on $\mathcal{SOT}(d,\omega)$ whence, by Definitions 7.4,7.6,

$$\mathcal{T}(d,\omega) \subset \mathcal{CB}(d,\omega) \ , \quad \mathcal{AT}(d,\omega) \subset \mathcal{ACB}(d,\omega) \ , \quad \mathcal{WT}(d,\omega) \subset \mathcal{WCB}(d,\omega) \ , \tag{7.17}$$

$$\mathcal{CB}(d,\omega) \subset \mathcal{ACB}(d,\omega) \subset \mathcal{WCB}(d,\omega) \subset \mathcal{SOT}(d,\omega) \ . \tag{7.18}$$

For the relevance of coboundaries, almost coboundaries, and weak coboundaries, see Section 7.6.

Proposition 7.7 *a) Let $(\omega, A) \in \mathcal{WCB}(d,\omega)$ and $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ with $(\omega, A') := R_{d,\omega}(T;\omega, A) \in \mathcal{WT}(d,\omega)$. If $N := Ind_{2,d}(A')$ then $Ind_{3,d}(\Psi_{\omega,A}(n;\cdot)) = ((-1)^{nN_1}, ..., (-1)^{nN_d})^T$ for arbitrary integer n.*

b) Let $(\omega, A) \in \mathcal{ACB}(d,\omega)$. Then, for every $n \in \mathbb{Z}$, $\Psi_{\omega,A}(n;\cdot)$ is 2π-nullhomotopic w.r.t. $SO(3)$ and $Ind_{3,d}(\Psi_{\omega,A}(n;\cdot)) = (1, ..., 1)^T$.

Proof of Proposition 7.7: See Section F.5. □

Lemma 7.8 *a) Let R be in $SO(3)$ and $Re^3 = e^3$. Then $R \in SO_3(2)$.*

b) A spin-orbit torus (ω, A) is weakly trivial iff $A(\phi)e^3 = e^3$.

Proof of Lemma 7.8: See Section F.6. □

The following theorem expresses the most important property of weak coboundaries.

Theorem 7.9 *Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$. Then, for every $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$, we have $R_{d,\omega}(T; \omega, A) \in \mathcal{WT}(d, \omega)$ iff the third column, Te^3, of T is the generator of an ISF of (ω, A). Moreover $(\omega, A) \in \mathcal{WCB}(d, \omega)$ iff there exists a $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ such that Te^3 is the generator of an ISF of (ω, A).*

Proof of Theorem 7.9: See Section F.7. □

Theorem 7.9 shows that the existence of an ISF is a necessary condition for a spin-orbit torus to be a weak coboundary. However Theorem 7.10, below, shows that this is not a sufficient condition.

As we just learned from Theorem 7.9, every weak coboundary has an ISF. I now address the converse question: is a spin-orbit torus a weak coboundary, if it has an ISF? A partial answer is given by the following theorem which uses some concepts introduced in Section 6.4 and which are borrowed from Homotopy Theory.

Theorem 7.10 *Let $G \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$ and let $(\omega, A) \in \mathcal{SOT}(d, \omega)$ such that G is the generator of an ISF $\mathcal{S}_G$ of (ω, A). Then the following hold.*

a) If G is 2π-nullhomotopic w.r.t. $\mathbb{S}^2$ then $(\omega, A) \in \mathcal{WCB}(d, \omega)$ and a $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ exists such that $R_{d,\omega}(T; \omega, A) \in \mathcal{WT}(d, \omega)$ and $G = Te^3$.

b) If $d = 1$ then $(\omega, A) \in \mathcal{WCB}(1, \omega)$ and a $T \in \mathcal{C}_{per}(\mathbb{R}, SO(3))$ exists such that $R_{1,\omega}(T; \omega, A) \in \mathcal{WT}(1, \omega)$ and $G = Te^3$.

c) If $d = 2$ then a $T \in \mathcal{C}_{per}(\mathbb{R}^2, SO(3))$ exists such that $R_{2,\omega}(T;\omega, A) \in \mathcal{WT}(2,\omega)$ and $G = Te^3$ iff G is 2π-nullhomotopic w.r.t. $\mathbb{S}^2$.

Proof of Theorem 7.10: See Section F.8. □

Let $G \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$ and let $(\omega, A) \in \mathcal{SOT}(d,\omega)$ such that G is the generator of an ISF of (ω, A). It is clear by Theorem 7.10a that if (ω, A) is not a weak coboundary, then G is not 2π-nullhomotopic w.r.t. $\mathbb{S}^2$. That this situation does occur, is the content of Theorem 8.17 (of course, due to Theorem 7.10b, this situation only occurs if $d \geq 2$).

Let $G \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$ and let $(\omega, A) \in \mathcal{SOT}(d,\omega)$ such that G is the generator of an ISF of (ω, A). If $S_0 \in \mathbb{S}^2$ exists such that neither S_0 nor $-S_0$ belong to the image G then it follows easily from Theorem 7.9 that $(\omega, A) \in \mathcal{WCB}(d,\omega)$ (and thus, by Theorem 7.10c, that, for $d = 2$, G is 2π-nullhomotopic w.r.t. $\mathbb{S}^2$). This also implies that if $(\omega, A) \in \mathcal{SOT}(d,\omega)$ has an ISF $\mathcal{S}_G$ then the question, whether $(\omega, A) \in \mathcal{WCB}(d,\omega)$, is connected with the issue of 'how complete' the image of G covers the sphere $\mathbb{S}^2$.

7.4 Introducing spin tune and spin-orbit resonance of first kind

Definition 7.11 *(Spin tune of first kind, spin-orbit resonance of first kind) Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$. Then the subset $\Xi_1(\omega, A)$ of $[0,1)$ is defined by*

$$\Xi_1(\omega, A) := \{PH(A') : (\omega, A') \in \mathcal{AT}(d,\omega) \;\&\; (\omega, A') \sim_{d,\omega} (\omega, A)\} \,. \tag{7.19}$$

I call ν a 'spin tune of first kind of (ω, A)' if $\nu \in \Xi_1(\omega, A)$.

I say that (ω, A) is 'on spin-orbit resonance of first kind' iff $0 \in \Xi_1(\omega, A)$. I say *that* (ω, A) *is 'off spin-orbit resonance of first kind' iff* $\Xi_1(\omega, A)$ *is nonempty and* $0 \notin \Xi_1(\omega, A)$. □

Definition 7.11 will be discussed, in the Physics context, in Section 7.6.

It is clear that if $(\omega, A) \in \mathcal{AT}(d, \omega)$ then, since $(\omega, A) \sim_{d,\omega} (\omega, A)$, $PH(A) \in \Xi_1(\omega, A)$. Of course, $\Xi_1(\omega, A)$ is nonempty iff (ω, A) is an almost coboundary. Thus (ω, A) has no spin tune of first kind iff (ω, A) is not an almost coboundary.

By Proposition 7.5 it is clear that there is a vast supply of spin-orbit tori which have spin tunes of first kind. On the other hand in Section 7.7 I will find a vast supply of spin-orbit tori which have no spin tune of first kind (see Remark 1 in Section 8.5).

In Section 8.4 (see Proposition 8.9a) we will observe that the sets $\Xi_1(\omega, A)$ have a simple structure and (see Proposition 8.10c) I will show that the definition of the spin-orbit resonance of first kind is equivalent to the familiar condition (8.15). These results, as several others, go beyond Chapter 7 since they rely on the machinery of quasiperiodic functions worked out in Chapter 8.

Proposition 7.12 *a) Let* $(\omega, A), (\omega, A') \in \mathcal{SOT}(d, \omega)$*. If* $(\omega, A) \sim_{d,\omega} (\omega, A')$*, then* $\Xi_1(\omega, A) = \Xi_1(\omega, A')$*. If* $(\omega, A) \in \mathcal{ACB}(d, \omega)$ *then* $(\omega, A) \sim_{d,\omega} (\omega, A')$ *iff* $\Xi_1(\omega, A) = \Xi_1(\omega, A')$*.*

b) $(\omega, A) \in \mathcal{SOT}(d, \omega)$ *is on spin-orbit resonance of first kind iff* $(\omega, A) \in \mathcal{CB}(d, \omega)$*. Moreover,* $(\omega, A) \in \mathcal{SOT}(d, \omega)$ *is off spin-orbit resonance of first kind iff* $(\omega, A) \in (\mathcal{ACB}(d, \omega) \setminus \mathcal{CB}(d, \omega))$*.*

c) Let $(\omega, A), (\omega, A') \in \mathcal{SOT}(d, \omega)$ *with* $(\omega, A) \sim_{d,\omega} (\omega, A')$*. Then either both spin-orbit tori are coboundaries or neither of them, and either both are almost coboundaries or neither of them, and either both are weak coboundaries or neither of them.*

d) Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$*. Then* $(\omega, A) \in \mathcal{ACB}(d, \omega)$ *iff there exists a* $(\omega, A') \in$

$\mathcal{SOT}(d,\omega)$ such that $\Psi_{\omega,A}(n;\phi)$ is independent of ϕ and $(\omega,A) \sim_{d,\omega} (\omega,A')$.

Proof of Proposition 7.12: See Section F.9. □

Propositions 7.12a, 7.12c give again properties shared by spin-orbit tori which belong to the same $R_{d,\omega}$-orbit.

It follows from Proposition 7.12d that (ω, A) is an almost coboundary iff $\Psi_{\omega,A}$ is an almost coboundary in the terminology of [KR].

Concerning Proposition 7.12d, I also note that, by (6.4), $\Psi_{\omega,A}(n;\phi)$ is independent of ϕ for all integers n iff $A(\phi)$ is independent of ϕ. Moreover it is easy to see that if $A(\phi)$ is independent of ϕ, then the function $\Psi_{\omega,A}(n)$ of n is a group homomorphism from the additive group $\mathbb{Z}$ into the multiplicative group $SO(3)$, i.e., $\Psi_{\omega,A}(n+m) = \Psi_{\omega,A}(n)\Psi_{\omega,A}(m)$. In particular this is the case for almost trivial (ω, A).

7.5 Yokoya's uniqueness theorem

If a spin-orbit torus has an ISF $\mathcal{S}_G$ then also $-\mathcal{S}_G$ is an ISF. Thus for spin-orbit tori which have an ISF, the question arises of whether they have more than two ISF's. The following celebrated theorem gives a partial answer (its importance is pointed out in Section 7.6).

Theorem 7.13 *(Yokoya's uniqueness theorem) Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$ and let $(1,\omega)$ be nonresonant. Let (ω, A) have an ISF $\mathcal{S}_G$ and an ISF which is different from $\mathcal{S}_G$ and $-\mathcal{S}_G$. Then (ω, A) is on spin-orbit resonance of first kind.*

Proof of Theorem 7.13: See Section F.10. □

7.6 Putting weak coboundaries into perspective

I now can begin to put things into perspective. On the basis of numerical and experimental evidence from storage rings, it is widely believed that the practically relevant spin-orbit tori are almost coboundaries (whence weak coboundaries) which is a strong motivation for many of the concepts introduced in Chapter 7. Part of the numerical evidence comes from the code SPRINT which, among other things, contains a numerical procedure which transforms a given almost coboundary into a weakly trivial spin-orbit torus and then transforms this weakly trivial spin-orbit torus into an almost trivial spin-orbit torus which then yields a spin tune of first kind (for more details on this code, see Section 8.5).

Nevertheless one knows of counterexamples, since one has discovered [BV], by numerical means, spin-orbit tori on orbital resonance which do not have an ISF, i.e., which, by Theorem 7.9, are not weak coboundaries (and these results were subsequently confirmed by analytical means). However, I am not aware of a spin-orbit torus off orbital resonance which does not have an ISF. It is therefore useful here to state the following conjecture, which I call the 'ISF-conjecture': 'If a spin-orbit torus (ω, A) is off orbital resonance, then it has an ISF'. While, at least to my knowledge, the ISF-conjecture is unsettled, it is definitely true that spin-orbit tori exist off orbital resonance, which are not weak coboundaries (see Theorem 8.17).

Spin tunes of first kind are important tools in the simulation and analysis of polarized beams in storage rings since spin-orbit resonances of first kind impose serious limitations on the polarization in a storage ring. On the other hand, by Theorem 7.13, one sees that, off orbital resonance and off spin-orbit resonance of first kind, the invariant spin field is unique up to a sign, i.e., only two ISF's exist in that situation. Thus in this case one can expect that the invariant spin field is an important characteristic of (ω, A) and so it perhaps comes as no surprise that, off orbital resonance and off spin-orbit resonance of first kind, the invariant spin field allows to compute the maximal possible polarization in a storage ring [BEH04, Hof, Vo]. This makes the invariant spin field an important tool in the statistical treatment of spin-orbit motion.

This is the right place to make also some remarks on the relation of the concept of spin tune of first kind with other works. Let $(\omega, A) \in \mathcal{WCB}(d, \omega)$ and $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. Then, in the context of the flow formalism, T is called, in the terminology of [BEH04], an 'invariant frame field' of (ω, A) if $R_{d,\omega}(T; \omega, A) \in \mathcal{WT}(d, \omega)$ and T is called a 'uniform invariant frame field' of (ω, A) if $R_{d,\omega}(T; \omega, A) \in \mathcal{AT}(d, \omega)$. The point to be made here is that in Yokoya's fundamental paper [Yo1], uniform invariant frame fields are used (in the context of the flow formalism) to define spin tunes so that indeed spin tunes of first kind are reincarnations of Yokoya's spin tunes. In contrast, the spin tunes, defined for the flow formalism in [BEH04] and their counterparts in the map formalism (introduced in Section 8.4 of the present work), are the spin tunes of second kind which are based on the tool of quasiperiodic functions and are nonetheless essentially equal to the spin tunes of first kind. In fact, by Proposition 8.9a, the spin tunes of first and second kind are identical for almost coboundaries. In this work the main purpose of the spin tunes of second kind is to enhance the knowledge of the spin tunes of first kind. Note also that [Yo1] builds on earlier work by Derbenev and Kondratenko [DK72, DK73] and that [BEH04] can be roughly characterized as refining [Yo1] by employing quasiperiodic functions. In turn, the present work refines [BEH04] by employing right and left group actions allowing thus to systematically build up a transformation theory of spin-orbit tori.

7.7 Transformations between weakly trivial spin-orbit tori

Clearly each $\mathcal{SOT}(d,\omega)$ is the disjoint union of the $R_{d,\omega}$-orbits. Thus of obvious interest is the issue of how this foliation looks, e.g., how it depends on d and ω. Since (recall Section 7.6) I am mainly interested in almost coboundaries (or, slightly more generally, weak coboundaries), I will only study the subset of $\mathcal{SOT}(d,\omega)$ which consists of the $R_{d,\omega}$-orbits of weak coboundaries. Thus I have to deal with the following question: when do two weakly trivial spin-orbit tori in $\mathcal{SOT}(d,\omega)$ belong to the same $R_{d,\omega}$-orbit? Perhaps surprisingly, this question can be pursued rather easily. As a matter of fact I only treat the generic case where spin-orbit tori are off orbital resonance (the case on orbital resonance can be tackled by the same techniques). Therefore in this section I state and prove Theorem 7.14 which gives sufficient and necessary conditions for two weakly trivial spin-orbit tori to be on the same $R_{d,\omega}$-orbit. I also point out (see Remark 1 of this section) how these conditions are related to small-divisor problems and Diophantine sets of orbital tunes. Corollary 7.15 then shows how things further simplify if one of the spin-orbit tori is almost trivial. In Sections 8.4, 8.5 I will, by using the machinery of quasiperiodic functions, obtain results related with, and going beyond, Theorem 7.14 and Corollary 7.15. In particular in Section 8.5 I will see the practical importance of the material from the present section.

Defining

$$\mathcal{J}' := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \tag{7.20}$$

and using

$$\mathcal{J}'\mathcal{J}\mathcal{J}' = -\mathcal{J} , \tag{7.21}$$

I obtain:

Theorem 7.14 *Let $(1,\omega)$ be nonresonant and $(\omega, A_i) \in \mathcal{WT}(d,\omega)$ where $i = 1,2$. Thus, by Proposition 7.5a, I have, for $\phi \in \mathbb{R}^d$, $i = 1,2$,*

$$A_i(\phi) = \exp(\mathcal{J}[M_i^T\phi + 2\pi f_i(\phi)]) \; , \tag{7.22}$$

where $M_i := Ind_2(A_i)$, $f_i := PHF(A_i)$. Then, abbreviating the zeroth Fourier coefficient by $f_{i,0} := (1/2\pi)^d \int_0^{2\pi}\cdots\int_0^{2\pi} f_i(\phi)d\phi_1\cdots d\phi_d$ and defining $\tilde{f}_i := f_i - f_{i,0} \in \mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R})$, the following hold:

a) If $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$ such that $R_{d,\omega}(T;\omega,A_1) = (\omega,A_2)$ then, after abbreviating $N := Ind_2(T)$, $g := PHF(T)$, I get

$$M_1 = M_2 \; , \tag{7.23}$$

$$(f_{1,0} - f_{2,0} - N^T\omega) \in \mathbb{Z} \; , \tag{7.24}$$

and, for all $\phi \in \mathbb{R}^d$,

$$g(\phi + 2\pi\omega) - g(\phi) = \tilde{f}_1(\phi) - \tilde{f}_2(\phi) \; . \tag{7.25}$$

If $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$ such that $R_{d,\omega}(T\mathcal{J}';\omega,A_1) = (\omega,A_2)$ then, after abbreviating $N := Ind_2(T)$, $g := PHF(T)$, I get

$$M_1 = -M_2 \; , \tag{7.26}$$

$$(f_{1,0} + f_{2,0} - N^T\omega) \in \mathbb{Z} \; , \tag{7.27}$$

and, for all $\phi \in \mathbb{R}^d$,

$$g(\phi + 2\pi\omega) - g(\phi) = \tilde{f}_1(\phi) + \tilde{f}_2(\phi) \; . \tag{7.28}$$

b) If $(\omega,A_1) \sim_{d,\omega} (\omega,A_2)$ then a $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$ exists such that either $R_{d,\omega}(T;\omega,A_1) = (\omega,A_2)$ or $R_{d,\omega}(T\mathcal{J}';\omega,A_1) = (\omega,A_2)$.

c) $(\omega, A_1) \sim_{d,\omega} (\omega, A_2)$ *iff the following criterion holds:*

Either

$M_1 = M_2$ *and* $g \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$, $N \in \mathbb{Z}^d$ *exist such that (7.24), (7.25) hold,*

or

$M_1 = -M_2$ *and* $g \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$, $N \in \mathbb{Z}^d$ *exist such that (7.27), (7.28) hold.*

In the former case $R_{d,\omega}(T; \omega, A_1) = (\omega, A_2)$ *where*

$$T(\phi) := \exp(\mathcal{J}[N^T\phi + 2\pi g(\phi)]) \ , \tag{7.29}$$

and in the latter case $R_{d,\omega}(T\mathcal{J}'; \omega, A_1) = (\omega, A_2)$ *where* T *is given by eq. (7.29).*

Proof of Theorem 7.14: See Section F.11. □

Note that the nontrivial part of the proof of Theorem 7.14 is part b).

Remarks:

(1) Perhaps the most important conclusion from Theorem 7.14 is that the spin-orbit tori $(\omega, A_1), (\omega, A_2)$ need not belong to the same $R_{d,\omega}$-orbit. To make this point clear, let $(1, \omega)$ be nonresonant and let me adopt the notation of Theorem 7.14.

If $M_1^2 - M_2^2 \neq 0$, $f_{1,0} - f_{2,0} \notin Y_\omega$, and $f_{1,0} + f_{2,0} \notin Y_\omega$ then, by Theorem 7.14c, one has $(\omega, A_1) \not\sim_{d,\omega} (\omega, A_2)$ (recall the definition (D.1) of Y_ω). In addition, a small divisor problem enhances this effect as follows. Even if $M_1 - M_2 = 0$ and $f_{1,0} - f_{2,0} \in Y_\omega$, in general one cannot solve eq. (7.25) for g since the Fourier coefficients of a provisional g are in general hampered by a small divisor problem preventing them to decay sufficiently fast to make g an element of $\mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$. Note also that these Fourier coefficients are, except for the zeroth Fourier coefficient, uniquely determined by $\tilde{f}_1, \tilde{f}_2$. Analogously, even if

$M_1 + M_2 = 0$ and $f_{1,0} + f_{2,0} \in Y_\omega$, in general one cannot solve eq. (7.28) for g due to an analogous small divisor problem. Note however that if one restricts ω to some appropriate Diophantine sets, then one can solve eq. (7.25),(7.28) (whence, in that case, $(\omega, A_1) \sim_{d,\omega} (\omega, A_2)$). For further details on Diophantine sets and related references, see [DEV].

We conclude, for nonresonant $(1, \omega)$, that the right group action $R_{d,\omega}$ is not transitive (recall the definition of 'transitive' in Appendix B). This comes as a relief since $\sim_{d,\omega}$ would be rather useless if all spin-orbit tori in $\mathcal{SOT}(d, \omega)$ would lie on the same $R_{d,\omega}$-orbit. Note also that, even without Theorem 7.14, it is obvious that the $R_{d,\omega}$-orbits of (ω, A_1) and (ω, A_2) contain many spin-orbit tori.

Of course, by the definition of weak coboundaries, I also conclude for nonresonant $(1, \omega)$ that weak coboundaries in $\mathcal{SOT}(d, \omega)$ need not belong to the same $R_{d,\omega}$-orbit.

(2) Let me again adopt the notation of Theorem 7.14 and let $(1, \omega)$ be nonresonant and $(\omega, A_1) \sim_{d,\omega} (\omega, A_2)$. Theorem 7.14b does *not* claim that *every* $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ with $R_{d,\omega}(T; \omega, A_1) = (\omega, A_2)$ is either in $\mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$ or of the form $T = T'\mathcal{J}'$ with $T' \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$. However the proof of Theorem 7.14b implies that, if $(\omega, A_1), (\omega, A_2)$ are not coboundaries, then *every* $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ with $R_{d,\omega}(T; \omega, A_1) = (\omega, A_2)$ is of this simple form, i.e., either $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$ or $T = T'\mathcal{J}'$ with $T' \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$. □

Note also that Theorem 7.14c confirms Proposition 7.5d.

The following corollary reconsiders the situation of Theorem 7.14 in the special case when the spin-orbit torus (ω, A_2) is almost trivial.

Corollary 7.15 *Let $(1, \omega)$ be nonresonant and $(\omega, A_1) \in \mathcal{WT}(d, \omega)$, $(\omega, A_2) \in \mathcal{AT}(d, \omega)$.*

Thus, by Proposition 7.5, we have, for $\phi \in \mathbb{R}^d$,

$$A_1(\phi) = \exp(\mathcal{J}[M_1^T\phi + 2\pi f_1(\phi)]) \ , \tag{7.30}$$

$$A_2(\phi) = \exp(\mathcal{J}2\pi\nu) \ , \tag{7.31}$$

where $M_1 := Ind_2(A_1)$, $f_1 := PHF(A_1)$, $\nu := PH(A_2) \in [0,1)$. *Then, abbreviating the zeroth Fourier coefficient of* f_1 *by* $f_{1,0} := (1/2\pi)^d \int_0^{2\pi} \cdots \int_0^{2\pi} f_1(\phi)d\phi_1 \cdots d\phi_d$ *and defining* $\tilde{f}_1 := f_1 - f_{1,0} \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$, *the following hold:*

a) If $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$ *such that* $R_{d,\omega}(T;\omega, A_1) = (\omega, A_2)$ *then, after abbreviating* $N := Ind_2(T)$, $g := PHF(T)$, *I get*

$$M_1 = 0 \ , \tag{7.32}$$

$$(f_{1,0} - \nu - N^T\omega) \in \mathbb{Z} \ , \tag{7.33}$$

and, for all $\phi \in \mathbb{R}^d$,

$$g(\phi + 2\pi\omega) - g(\phi) = \tilde{f}_1(\phi) \ . \tag{7.34}$$

If $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$ *such that* $R_{d,\omega}(T\mathcal{J}';\omega, A_1) = (\omega, A_2)$ *then we have eq. (7.32) and, after abbreviating* $N := Ind_2(T)$, $g := PHF(T)$, *I get*

$$(f_{1,0} + \nu - N^T\omega) \in \mathbb{Z} \ , \tag{7.35}$$

and, for all $\phi \in \mathbb{R}^d$, *I get eq. (7.34).*

b) $(\omega, A_1) \sim_{d,\omega} (\omega, A_2)$ *iff the following criterion holds:*

Either

$M_1 = 0$ *and* $g \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$, $N \in \mathbb{Z}^d$ *exist such that (7.33), (7.34) hold,*

or

$M_1 = 0$ *and* $g \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$, $N \in \mathbb{Z}^d$ *exist such that (7.34), (7.35) hold.*

In the former case $R_{d,\omega}(T;\omega,A_1)=(\omega,A_2)$ *where* T *is given by eq. (7.29) and in the latter case* $R_{d,\omega}(T\mathcal{J}';\omega,A_1)=(\omega,A_2)$ *where* T *is given by eq. (7.29).*

Proof of Corollary 7.15: See Section F.12. □

Chapter 8

Quasiperiodic functions as tools for studying spin-orbit tori

Quasiperiodic functions on $\mathbb{Z}$ come up naturally for spin-orbit tori since, as already pointed out at the beginning of Section 7.1, the expression $A(\phi_0 + 2\pi n\omega)$, occurring in the equation of spin motion (6.8), is an ω-quasiperiodic function of n. Note that quasiperiodic functions are defined in Section D.1. In Sections 8.1-8.4 I develop the basic machinery of quasiperiodic functions needed for spin-orbit tori. While some of the results of Sections 8.1-8.4 are interesting per se (notably Theorems 8.1,8.3,8.5), their main purpose is to improve, in Sections 8.5-8.7, on the themes which I started in Chapter 7. Thus the tranformation theory of spin-orbit tori, developed in Chapter 7, stays in the foreground also in the present section. In particular I stick with my credo mentioned in Section 7.6, that the emphasis is on weak coboundaries.

8.1 Relations between polarization fields and spin trajectories

The following theorem is about the characteristic curves of polarization fields.

Theorem 8.1 *a) Let* $(\omega, A) \in \mathcal{SOT}(d, \omega)$*. Let* $\mathcal{S}_G$ *be a polarization field for this spin-orbit torus and let* $\phi_0 \in \mathbb{R}^d$*. Then the 'characteristic curve'* $S : \mathbb{Z} \to \mathbb{R}^3$*, defined by* $S(n) := \mathcal{S}_G(n, \phi_0 + 2\pi n\omega)$*, is a spin trajectory over* ϕ_0 *for* (ω, A)*. If the polarization field* $\mathcal{S}_G$ *is invariant, then* $S(n) = G(\phi_0 + 2\pi n\omega)$ *and the spin trajectory* S *is* ω*-quasiperiodic.*

b) Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$ *and let* $(1, \omega)$ *be nonresonant (for the definition of 'nonresonant', see Section D.1). Let* (ω, A) *have, for some* $\phi_0 \in \mathbb{R}^d$*, an* ω*–quasiperiodic spin trajectory* S *over* ϕ_0*. Then* (ω, A) *has a unique invariant polarization field* $\mathcal{S}_G$ *such that, for all integers* n*,*

$$S(n) = G(\phi_0 + 2\pi n\omega) \ . \tag{8.1}$$

If in addition S *is normalized to* 1*, i.e.,* $|S(n)| = 1$ *then* $\mathcal{S}_G$ *is an ISF of* (ω, A)*.*

Proof of Theorem 8.1: See Section F.13. □

Note that by Theorem 8.1, and off orbital resonance, a nonzero ω–quasiperiodic spin trajectory over ϕ_0 exists for *every* ϕ_0, if a nonzero ω–quasiperiodic spin trajectory exists over *some* ϕ_0.

Since for every spin trajectory S the function $|S|$ is constant, it follows from Theorem 8.1b that if, off orbital resonance, at least one nonzero ω–quasiperiodic spin trajectory exists, then (ω, A) has an ISF.

In spite of Theorem 8.1b, every spin trajectory S over a ϕ_0 is the characteristic curve of infinitely many polarization fields. In fact, every polarization field $\mathcal{S}_G$ for which $G(\phi_0) = S(0)$ also satisfies, for every integer n, $S(n) = \mathcal{S}_G(n, \phi_0 + 2\pi n\omega) = \Psi_{\omega,A}(n; \phi_0)G(\phi_0)$. However it follows from Theorem 8.1b that, in the special case when $(1, \omega)$ is nonresonant and S is ω–quasiperiodic, there is among those infinitely many polarization fields $\mathcal{S}_G$, which satisfy $S(n) = \mathcal{S}_G(n, \phi_0 + 2\pi n\omega)$, only one that is invariant.

Recalling Section 7.6, I do not try to solve the ISF-conjecture. Thus by Theorem 8.1b I leave open the question of whether nonzero ω–quasiperiodic spin trajectories exist off orbital resonance.

However, as mentioned in Section 7.6, relevant spin-orbit tori are almost coboundaries whence, by Theorem 7.9 and Theorem 8.1a, they have nonzero ω–quasiperiodic spin trajectories.

Since, for every spin trajectory S, $|S|$ is constant, it follows from Theorem 8.1b that if, off orbital resonance, at least one nonzero ω–quasiperiodic spin trajectory exists, then (ω, A) has an ISF.

Moreover, it follows from the proof of Theorem 8.1b that the invariant polarization field $\mathcal{S}_G$ is uniquely determined by $S(0)$. One takes advantage of this fact if one computes the ISF by the technique of stroboscopic averaging (for remarks on stroboscopic averaging, see Section 8.5).

8.2 Simple precession frames

With the right group action $R_{d,\omega}$ introduced in Chapter 7, we arrive, in the present section, at the concept of the simple precession frame. We recall from Definition 7.2 that if $(\omega, A), (\omega, A') \in \mathcal{SOT}(d, \omega)$ and $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ such that $R_{d,\omega}(T; \omega, A) = (\omega, A')$ then (7.4) holds. Thus picking a $\phi_0 \in \mathbb{R}^d$, the function

$t : \mathbb{Z} \to SO(3)$, defined by $t(n) := T(\phi_0 + 2\pi n\omega)$, satisfies

$$t^T(n+1)A(\phi_0 + 2\pi n\omega)t(n) = A'(\phi_0 + 2\pi n\omega) \; . \tag{8.2}$$

Let in addition $(\omega, A') \in \mathcal{WT}(d, \omega)$. Then by Lemma 7.8b, eq. (8.2) implies

$$t(n+1)e^3 = A(\phi_0 + 2\pi n\omega)t(n)e^3 \; . \tag{8.3}$$

Comparing (6.8),(8.3), one finds that the third column of t is a spin trajectory of (ω, A) over ϕ_0. This leads to the following definition.

Definition 8.2 *(Simple precession frame)*

Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$ and let $\phi_0 \in \mathbb{R}^d$. A function $t : \mathbb{Z} \to SO(3)$ is called a 'simple precession frame (SPF) of (ω, A) over ϕ_0' if its third column is a spin trajectory over ϕ_0, i.e., if (8.3) holds for all integers n. □

By the remarks before Definition 8.2 it is clear that if $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ and $R_{d,\omega}(T; \omega, A) \in \mathcal{WT}(d, \omega)$ then $T(\phi_0 + 2\pi n\omega)$, as a function of n, is an SPF over ϕ_0. Thus the 'characteristic curves' of T are SPF's (for more details on this, see Theorem 8.3 below).

If t is an SPF over ϕ_0 then, by (8.3), $e^3 = t^T(n+1)A(\phi_0 + 2\pi n\omega)t(n)e^3$. Hence, by Lemma 7.8a, a function $\lambda : \mathbb{Z} \to [0, 1)$ exists such that for all n

$$t^T(n+1)A(\phi_0 + 2\pi n\omega)t(n) = \exp(2\pi\lambda(n)\mathcal{J}) \; . \tag{8.4}$$

Clearly λ is unique. I call λ the 'differential phase function' of t. We see that t 'transforms' $A(\phi_0 + 2\pi n\omega)$ via (8.4) into the matrix $\exp(2\pi\lambda(n)\mathcal{J})$ which has a simple block diagonal form and this is the origin of the term 'simple'. Defining the function $\mu : \mathbb{Z} \to [0, 1)$ by

$$\mu(n) := \begin{cases} 0 & \text{if } n = 0 \\ \lfloor \lambda(0) + ... + \lambda(n-1) \rfloor & \text{if } n > 0 \\ \lfloor -\lambda(-1) - ... - \lambda(n) \rfloor & \text{if } n < 0 \end{cases} \; , \tag{8.5}$$

I obtain, by (6.4),(8.4), that, for $n \in \mathbb{Z}$,

$$\Psi_{\omega,A}(n;\phi_0) = t(n)\exp(2\pi\mu(n)\mathcal{J})t^T(0)\ . \tag{8.6}$$

Note that μ is uniquely determined by $\Psi_{\omega,A}, \phi_0, t$ via (8.6) and satisfies $\lfloor \mu(n+1) - \mu(n) \rfloor = \lambda(n)$ so that I call μ the 'integral phase function' of t. Clearly a function $t : \mathbb{Z} \to SO(3)$ is an SPF over ϕ_0 iff a function $\mu : \mathbb{Z} \to \mathbb{R}$ exists such that (8.6) holds for all integers n.

Remarks:

(1) Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$ and let $\phi_0 \in \mathbb{R}^d$. If f is an arbitrary function $f : \mathbb{Z} \to \mathbb{R}$ and if R is a constant $SO(3)$–matrix then, by using (8.4) and the remarks on (6.5), the function t, defined by $t(n) := \Psi_{\omega,A}(n;\phi_0)R\exp(-\mathcal{J}2\pi f(n))$, is an SPF over ϕ_0 with the differential phase function $\lambda(n) = \lfloor f(n+1) - f(n) \rfloor$. One sees by this construction that, for every ϕ_0, a large abundance of SPF's, over ϕ_0, exists.

(2) I here discuss a sometimes useful property of SPF's. Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$ and let t be an SPF of (ω, A) over some ϕ_0 with differential phase function λ. Let j be an integer and let the function $t' : \mathbb{Z} \to SO(3)$ be defined by $t'(n) := t(n+j)$. It follows from (8.3) that for all integers n

$$t'(n+1)e^3 = t(n+1+j)e^3 = A(\phi_0 + 2\pi(n+j)\omega)t(n+j)e^3$$
$$= A(\phi_0 + 2\pi(n+j)\omega)t'(n)e^3\ ,$$

whence, by (6.8), the third column of t' is a spin trajectory over $\phi_0 + 2\pi j\omega$. Thus t' is an SPF over $\phi_0 + 2\pi j\omega$. I also obtain from (8.4) that for all n $t'^T(n+1)A(\phi_0+2\pi(n+j)\omega)t'(n) = t^T(n+1+j)A(\phi_0+2\pi(n+j)\omega)t(n+j) = \exp(2\pi\lambda(n+j)\mathcal{J})$. Hence the differential phase function λ' of t' is given by $\lambda'(n) := \lambda(n+j)$. If t is ω–quasiperiodic and $\tilde{t}$ is an ω-generator of t then $\tilde{t}(\cdot + 2\pi j\omega)$ is an ω-generator of t' whence t' is ω–quasiperiodic. □

Since an ω–quasiperiodic SPF t is $SO(3)$-valued, it follows from Definition D.1 that t has an ω-generator $\tilde{t}$ which is $\mathbb{R}^{3\times 3}$-valued, albeit in general not $SO(3)$-valued. Nevertheless, the situation simplifies when $(1,\omega)$ is nonresonant, as Part b) of the following theorem shows.

Theorem 8.3 *a) Let $(\omega, A) \in \mathcal{WCB}(d,\omega)$ and $(\omega, A') := R_{d,\omega}(T;\omega, A) \in \mathcal{WT}(d,\omega)$ with $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. Then, for an arbitrary $\phi_0 \in \mathbb{R}^d$ the function $t : \mathbb{Z} \to SO(3)$, defined by $t(n) := T(\phi_0 + 2\pi n\omega)$, is an ω–quasiperiodic SPF of (ω, A) over ϕ_0. Furthermore the differential phase function λ of t satisfies, for $n \in \mathbb{Z}$,*

$$\begin{aligned}\lambda(n) &= \lfloor \frac{N_1^T\phi_0}{2\pi} + N_n^T\omega + f(1, \phi_0 + 2\pi n\omega)\rfloor \\ &= \lfloor \frac{N_1^T\phi_0}{2\pi} + nN_1^T\omega + f(1, \phi_0 + 2\pi n\omega)\rfloor \,, \end{aligned} \tag{8.7}$$

and the integral phase function μ of t satisfies, for $n \in \mathbb{Z}$,

$$\mu(n) = \lfloor \frac{N_n^T\phi_0}{2\pi} + f(n,\phi_0)\rfloor = \lfloor \frac{nN_1^T\phi_0}{2\pi} + f(n,\phi_0)\rfloor \,, \tag{8.8}$$

where $N_n := Ind_2(\Psi_{\omega,A'}(n;\cdot)), f(n,\cdot) := PHF(\Psi_{\omega,A'}(n;\cdot))$.

b) Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$ and let $(1,\omega)$ be nonresonant. Let also (ω, A) have an ω–quasiperiodic SPF t over some ϕ_0. Then a unique $T \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^{3\times 3})$ exists such that, for all integers n, $t(n) = T(\phi_0 + 2\pi n\omega)$. Moreover $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. Furthermore, $(\omega, A) \in \mathcal{WCB}(d,\omega)$ and $R_{d,\omega}(T;\omega, A) \in \mathcal{WT}(d,\omega)$.

Proof of Theorem 8.3: See Section F.14. □

As mentioned in Section 7.6, relevant spin-orbit tori are weak coboundaries whence, by Theorem 8.3a, they have ω–quasiperiodic SPF's. However as Theorem 8.17 shows there are spin-orbit tori off orbital resonance which are not weak coboundaries whence, by Theorem 8.3b, they have no ω–quasiperiodic SPF.

8.3 Uniform precession frames

In this section I introduce 'uniform precession frames' which are special SPF's. As one shall see in the next section, uniform precession frames lead to the definition of the 'spin tune of second kind'.

Definition 8.4 *(Uniform precession frame)*

Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$ and let $\phi_0 \in \mathbb{R}^d$. Let also t be a simple precession frame of (ω, A) over ϕ_0 and let its differential phase function be denoted by λ. Then t is called a 'uniform precession frame (UPF) over ϕ_0' if $\lambda(n)$ is independent of n. The constant value, say ν, of λ is then called the 'uniform precession rate (UPR) of t'. Thus by (8.4)

$$t^T(n+1)A(\phi_0 + 2\pi n\omega)t(n) = \exp(2\pi\nu\mathcal{J}) , \tag{8.9}$$

and, by (8.5), the integral phase function μ of t reads as $\mu(n) = \lfloor n\nu \rfloor$ and whence by (8.6)

$$\Psi_{\omega,A}(n;\phi_0) = t(n)\exp(\mathcal{J}2\pi n\nu)t^T(0) . \tag{8.10}$$

I denote by $\Xi_2(\omega, A, \phi_0)$ the set of those UPR's which correspond to an ω–quasiperiodic UPF over ϕ_0 and I define $\Xi_2(\omega, A) := \bigcup_{\phi_0 \in \mathbb{R}^d} \Xi_2(\omega, A, \phi_0)$. □

It follows from Definition 8.4 that a function $t : \mathbb{Z} \to SO(3)$ is a UPF over ϕ_0 iff a $\nu \in [0, 1)$ exists such that either (8.9) or (8.10) holds for all $n \in \mathbb{Z}$.

Of course any UPR is uniquely determined by the corresponding UPF but the converse is not true, i.e., different UPF's can have the same UPR. It is also clear that $\Xi_2(\omega, A, \cdot)$ is 2π-periodic.

Remarks:

(1) Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$ and let t be a UPF of (ω, A) over some $\phi_0 \in \mathbb{R}^d$. Let ν denote the UPR of t and let j be an integer. From Remark 2 of Section 8.2 we know that the function $t' : \mathbb{Z} \to SO(3)$, defined by $t'(n) := t(n+j)$, is an SPF over $\phi_0 + 2\pi j\omega$ and that its differential phase function λ' is given by $\lambda'(n) := \lambda(n+j) = \nu$, where λ is the differential phase function of t. Thus λ' has the constant value ν whence t' is a UPF over $\phi_0 + 2\pi j\omega$ with UPR ν. It also follows from Remark 2 of Section 8.2 that t' is ω–quasiperiodic if t is ω–quasiperiodic. Thus, for every integer j, $\Xi_2(\omega, A, \phi_0 + 2\pi j\omega) = \Xi_2(\omega, A, \phi_0)$.

(2) Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$ and $\phi_0 \in \mathbb{R}^d$. By Remark 1 of Section 8.2 we know that $\Psi_{\omega,A}(\cdot;\phi_0)$ is an SPF over ϕ_0 with the differential phase function $\lambda(n) = 0$. Thus $\Psi_{\omega,A}(\cdot;\phi_0)$ is an UPF over ϕ_0 with UPR 0. □

Theorem 8.5 *a) Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$. If $\nu \in \Xi_2(\omega, A, \phi_0)$ for some $\phi_0 \in \mathbb{R}^d$ then every spin trajectory of (ω, A) over ϕ_0 is (ω,ν)–quasiperiodic.*

b) Let $(\omega, A) \in \mathcal{ACB}(d,\omega)$ and $(\omega, A') := R_{d,\omega}(T;\omega, A) \in \mathcal{AT}(d,\omega)$ with $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. Then for an arbitrary $\phi_0 \in \mathbb{R}^d$ the function $t : \mathbb{Z} \to SO(3)$, defined by $t(n) := T(\phi_0 + 2\pi n\omega)$, is an ω–quasiperiodic UPF over ϕ_0 with UPR $\nu = PH(A')$.

c) Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$ be a and let $(1,\omega)$ be nonresonant. Let (ω, A) have an ω–quasiperiodic UPF t over some $\phi_0 \in \mathbb{R}^d$ with UPR ν. Then a unique $T \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^{3\times 3})$ exists such that, for all integers n, $t(n) = T(\phi_0 + 2\pi n\omega)$. Moreover $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. Furthermore, $(\omega, A) \in \mathcal{ACB}(d,\omega)$ and $(\omega, A') := R_{d,\omega}(T;\omega, A) \in \mathcal{AT}(d,\omega)$ with $PH(A') = \nu \in \Xi_1(\omega, A)$.

Proof of Theorem 8.5: See Section F.15. □

As mentioned in Section 7.6, I am mainly interested in spin-orbit tori that are almost coboundaries whence, by Theorem 8.5b, they have ω–quasiperiodic UPF's. However, as mentioned after Theorem 8.3, there are spin-orbit tori off orbital resonance which have no ω–quasiperiodic SPF whence they have no ω–quasiperiodic UPF.

Theorem 8.5a enables to do spectral analysis of spin trajectories as follows. In fact if $\nu \in \Xi_2(\omega, A, \phi_0)$ and S is a spin trajectory of (ω, A) over ϕ_0 then, by Lemma D.4d and Remark 1 in Section D.3, the spectrum of each component S_i of S is a subset of $Y_{(\omega,\nu)}$ (the spectrum of a complex valued function on $\mathbb{Z}$ is defined in Section D.3).

It is enlightening and easy to obtain a connection between Floquet theory and UPF's as follows. I say that $(\omega, A) \in \mathcal{SOT}(d)$ satisfies the generalized Floquet Theorem over $\phi_0 \in \mathbb{R}^d$ if a quasiperiodic $SO(3)$–valued function p and a real 3×3 matrix B exist such that $p(0) = I_{3\times 3}$ and such that, for all integers n, $\Psi_{\omega,A}(n; \phi_0) = p(n)\exp(nB)$. In fact it follows from Definition 8.4 that if t is an ω–quasiperiodic UPF over ϕ_0 with UPR ν then the generalized Floquet Theorem holds over ϕ_0 since one can define p and B by $p(n) := t(n)t^T(0), B := 2\pi\nu t(0)\mathcal{J}t^T(0)$. In particular one concludes from Theorem 8.5b that if (ω, A) is an almost coboundary then the generalized Floquet Theorem is satisfied over every $\phi_0 \in \mathbb{R}^d$.

The following theorem (Theorem 8.6) reveals the structure of the sets $\Xi_2(\omega, A, \phi_0)$ (and this in turn will reveal, in the next section, the structure of the sets $\Xi_1(\omega, A)$). To prepare for the following theorem let $(\omega, A) \in \mathcal{SOT}(d, \omega)$ and let $\phi_0 \in \mathbb{R}^d$.

I first recall from Definition D.1 that, for $\omega \in \mathbb{R}^d$, Y_ω is defined by $Y_\omega := \{m^T\omega + n : m \in \mathbb{Z}^d, n \in \mathbb{Z}\}$. For the following theorem I need the equivalence relation $\sim_\omega$ on $[0, 1)$ by which elements $\nu_1, \nu_2 \in [0, 1)$ are equivalent iff there exist $(\varepsilon, y) \in \{1, -1\} \times Y_\omega$ such that $\nu_2 = \varepsilon\nu_1 + y$. The equivalence class of a $\nu \in [0, 1)$ is denoted by $[\nu]_\omega$. Clearly

$$[\nu]_\omega = \{(\varepsilon\nu + y) \in [0, 1) : \varepsilon \in \{1, -1\}, y \in Y_\omega\}$$
$$= \{\lfloor \varepsilon\nu + y \rfloor : \varepsilon \in \{1, -1\}, y \in Y_\omega\} = \{\lfloor \varepsilon\nu + j^T\omega \rfloor : \varepsilon \in \{1, -1\}, j \in \mathbb{Z}^d\} \,. \quad (8.11)$$

To get a feel for the equivalence relation $\sim_\omega$ I now show that if ν is in $\Xi_2(\omega, A, \phi_0)$ then

$$[\nu]_\omega \subset \Xi_2(\omega, A, \phi_0) \ . \tag{8.12}$$

In fact if $\nu \in \Xi_2(\omega, A, \phi_0)$ then by Definition 8.4 an ω–quasiperiodic UPF t exists over ϕ_0 which has UPR ν. I pick a $y \in Y_\omega$ and define the function $t' : \mathbb{Z} \to SO(3)$ by $t'(n) := t(n)\exp(-\mathcal{J}2\pi ny)$. Clearly t' is an ω–quasiperiodic function. Furthermore for $n \in \mathbb{Z}$ we have, by (8.9),

$$\begin{aligned}
&t'^T(n+1)A(\phi_0 + 2\pi n\omega)t'(n) \\
&= \exp(\mathcal{J}2\pi(n+1)y)t^T(n+1)A(\phi_0 + 2\pi n\omega)t(n)\exp(-\mathcal{J}2\pi ny) \\
&= \exp(\mathcal{J}2\pi(n+1)y)\exp(2\pi\nu\mathcal{J})\exp(-\mathcal{J}2\pi ny) = \exp(\mathcal{J}2\pi(\nu + y)) \ .
\end{aligned}$$

Thus t' is an ω–quasiperiodic UPF over ϕ_0 with UPR $\lfloor \nu + y \rfloor$. I define the function $t'' : \mathbb{Z} \to SO(3)$ by $t''(n) := t(n)\exp(\mathcal{J}2\pi ny)\mathcal{J}'$, where $\mathcal{J}'$ is given by (7.20). Clearly t'' is an ω–quasiperiodic function. Furthermore for $n \in \mathbb{Z}$ we have by (8.9)

$$\begin{aligned}
&t''^T(n+1)A(\phi_0 + 2\pi n\omega)t''(n) \\
&= \mathcal{J}'\exp(-\mathcal{J}2\pi(n+1)y)t^T(n+1)A(\phi_0 + 2\pi n\omega)t(n)\exp(\mathcal{J}2\pi ny)\mathcal{J}' \\
&= \mathcal{J}'\exp(-\mathcal{J}2\pi(n+1)y)\exp(2\pi\nu\mathcal{J})\exp(\mathcal{J}2\pi ny)\mathcal{J}' = \mathcal{J}'\exp(\mathcal{J}2\pi(\nu - y))\mathcal{J}' \\
&= \exp(\mathcal{J}'\mathcal{J}\mathcal{J}'2\pi(\nu - y)) = \exp(-\mathcal{J}2\pi(\nu - y)) = \exp(\mathcal{J}2\pi(-\nu + y)) \ ,
\end{aligned}$$

where in the fifth equality I used (7.21). Thus t'' is an ω–quasiperiodic UPF over ϕ_0 with UPR $\lfloor -\nu + y \rfloor$. I have therefore shown that if $\nu \in \Xi_2(\omega, A, \phi_0)$ and $\varepsilon \in \{1, -1\}, y \in Y_\omega$ then $\lfloor \varepsilon\nu + y \rfloor \in \Xi_2(\omega, A, \phi_0)$ so that, by (8.11), the inclusion (8.12) holds, as was to be proven. While obtaining (8.12) was elementary, the following theorem strengthens this inclusion to an equality. Since the proof of Theorem 8.6 involves rather sophisticated properties of quasiperiodic functions, this indicates that (8.13) is a much deeper property than (8.12).

Theorem 8.6 *(Structure of $\Xi_2(\omega, A, \phi_0)$) Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$ and let $\phi_0 \in \mathbb{R}^d$. If $\nu \in \Xi_2(\omega, A, \phi_0)$ then*

$$\Xi_2(\omega, A, \phi_0) = [\nu]_\omega \ . \tag{8.13}$$

Proof of Theorem 8.6: See Section F.16. □

8.4 Introducing spin tune and spin-orbit resonance of second kind

In this work the main purpose of UPF's and UPR's is to enhance the knowledge of the spin tunes and spin-orbit resonances of first kind. The following theorem gives a first glance at the relation between spin tunes of first kind and UPR's, in particular between $\Xi_1(\omega, A)$ and $\Xi_2(\omega, A, \phi_0)$.

Theorem 8.7 *a) Let (ω, A) be a spin-orbit torus. If $\nu \in \Xi_1(\omega, A)$ then $[\nu]_\omega \subset \Xi_1(\omega, A)$. Moreover, if $y \in ([0,1) \cap Y_\omega)$ then $[y]_\omega = [0,1) \cap Y_\omega$. Furthermore either $([0,1) \cap Y_\omega) \subset \Xi_1(\omega, A)$ or $\Xi_1(\omega, A) \cap Y_\omega = \emptyset$.*

b) Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$. Then for all $\phi_0 \in \mathbb{R}^d$

$$\Xi_1(\omega, A) \subset \Xi_2(\omega, A, \phi_0) \ . \tag{8.14}$$

Moreover, if $\Xi_1(\omega, A)$ is nonempty, then, for all $\phi_0 \in \mathbb{R}^d$, $\Xi_1(\omega, A) = \Xi_2(\omega, A, \phi_0)$.

c) Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$ and let $(1, \omega)$ be nonresonant. Then, for all $\phi_0 \in \mathbb{R}^d$, $\Xi_1(\omega, A) = \Xi_2(\omega, A, \phi_0)$.

d) Let $(\omega, A), (\omega, A') \in \mathcal{SOT}(d, \omega)$ with $(\omega, A) \sim_{d,\omega} (\omega, A')$ and let $\phi_0 \in \mathbb{R}^d$. Then $\Xi_2(\omega, A', \phi_0) = \Xi_2(\omega, A, \phi_0)$.

Proof of Theorem 8.7: See Section F.17. □

In the case of most practical interest, i.e., when (ω, A) is an almost coboundary, the sets $\Xi_1(\omega, A)$ and $\Xi_2(\omega, A, \phi_0)$ are equal by Theorem 8.7b. The following definition of spin tune of second kind transfers the spin tune definition in [BEH04] from the flow formalism to the map formalism.

Definition 8.8 *(Spin tune of second kind, spin-orbit resonance of second kind) Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$. Then (ω, A) is said to be 'well–tuned' if all $\Xi_2(\omega, A, \phi_0)$ are nonempty and equal, where ϕ_0 varies over $\mathbb{R}^d$. Otherwise (ω, A) is said to be 'ill–tuned'. Of course, if (ω, A) is well–tuned, then, due to Definition 8.4 all $\Xi_2(\omega, A, \phi_0)$ are equal to $\Xi_2(\omega, A)$, where again ϕ_0 varies over $\mathbb{R}^d$. For a well–tuned spin-orbit torus I call the elements of $\Xi_2(\omega, A)$ 'spin tunes of second kind'.*

If the spin-orbit torus is well–tuned then it is said to be 'on spin-orbit resonance of second kind' if 0 is a spin tune of second kind and it is said to be 'off spin-orbit resonance of second kind' if 0 is not a spin tune of second kind. □

Proposition 8.9 *a) Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$. If $(\omega, A) \in \mathcal{ACB}(d, \omega)$ then (ω, A) is well–tuned and the spin tunes of first and second kind are the same. If $\nu \in \Xi_1(\omega, A)$ then $\Xi_1(\omega, A) = [\nu]_\omega$. If (ω, A) is well–tuned and if ν is a spin tune of second kind then, for all $\phi_0 \in \mathbb{R}^d$, $\Xi_2(\omega, A) = \Xi_2(\omega, A, \phi_0) = [\nu]_\omega$.*

b) Let $(\omega, A), (\omega, A') \in \mathcal{SOT}(d, \omega)$ and $(\omega, A) \in \mathcal{ACB}(d, \omega)$. Then either $\Xi_1(\omega, A) \cap \Xi_1(\omega, A') = \emptyset$ or $\Xi_1(\omega, A) = \Xi_1(\omega, A')$. In the former case $(\omega, A) \not\sim_{d,\omega} (\omega, A')$ and in the latter case $(\omega, A) \sim_{d,\omega} (\omega, A')$, $(\omega, A') \in \mathcal{ACB}(d, \omega)$.

c) If (ω, A) is a spin-orbit torus and if $(1, \omega)$ is nonresonant then the following hold. The spin-orbit torus (ω, A) is well–tuned iff $(\omega, A) \in \mathcal{ACB}(d, \omega)$. If (ω, A) is well–tuned then $\Xi_1(\omega, A) = \Xi_2(\omega, A)$.

d) For every spin-orbit torus (ω, A) the following hold. If ν is a spin tune of second *kind of* (ω, A) *then each spin trajectory of* (ω, A) *is* (ω, ν)*–quasiperiodic. If* ν *is a spin tune of first kind of* (ω, A) *then each spin trajectory of* (ω, A) *is* (ω, ν)*–quasiperiodic.*

e) A $(\omega, A) \in \mathcal{SOT}(d, \omega)$ *is well–tuned iff the* $\Xi_2(\omega, A, \phi_0)$ *have a common element when* ϕ_0 *varies over* $\mathbb{R}^d$.

f) If $(\omega, A) \in \mathcal{SOT}(d, \omega)$ *then the following hold. The set* $\Xi_1(\omega, A)$ *and the sets* $\Xi_2(\omega, A, \phi_0)$*, where* ϕ_0 *varies over* $\mathbb{R}^d$*, have countably many elements. The spin-orbit torus is ill–tuned if* $\Xi_2(\omega, A)$ *has uncountably many elements.*

g) If $(\omega, A), (\omega, A') \in \mathcal{SOT}(d, \omega)$ *with* $(\omega, A) \sim_{d,\omega} (\omega, A')$ *then the following hold. Either both spin-orbit tori* $(\omega, A), (\omega, A')$ *are well–tuned or both of them are ill-tuned. Moreover if the spin-orbit tori* $(\omega, A), (\omega, A')$ *are well-tuned then they have the same spin tunes of second kind.*

Proof of Proposition 8.9: See Section F.18. □

Remark:

(1) An important conclusion from Proposition 8.9a is that two almost coboundaries $(\omega, A), (\omega, A') \in \mathcal{ACB}(d, \omega)$ need not belong to the same $R_{d,\omega}$-orbit, as follows. In fact, picking $\nu \in \Xi_1(\omega, A), \nu' \in \Xi_1(\omega, A')$ such that $[\nu]_\omega \neq [\nu']_\omega$, we have, by Proposition 8.9a, that $\Xi_1(\omega, A') = [\nu']_\omega \neq [\nu]_\omega = \Xi_1(\omega, A)$ whence, by Proposition 7.12a, $(\omega, A) \not\sim_{d,\omega} (\omega, A')$.

I now address the topic of spin-orbit resonances of first and second kind.

Proposition 8.10 *a) If a spin-orbit torus is on spin-orbit resonance of first kind then it is on spin-orbit resonance of second kind. If a spin-orbit torus is off spin-orbit resonance of first kind then it is off spin-orbit resonance of second kind.*

b) Let (ω, A) *be a spin-orbit torus. Then* (ω, A) *is on spin-orbit resonance of second kind iff all of its spin trajectories are* ω*–quasiperiodic.*

c) A $(\omega, A) \in \mathcal{SOT}(d,\omega)$ is on spin-orbit resonance of first kind iff $\Xi_1(\omega, A) = [0,1) \cap Y_\omega$. Furthermore a $(\omega, A) \in \mathcal{SOT}(d,\omega)$ is on spin-orbit resonance of first kind iff (ω, A) has a spin tune ν of first kind such that $m \in \mathbb{Z}^d, n \in \mathbb{Z}$ exist with

$$\nu = m^T\omega + n \,. \tag{8.15}$$

d) A $(\omega, A) \in \mathcal{SOT}(d,\omega)$ is on spin-orbit resonance of second kind iff, for all $\phi_0 \in \mathbb{R}^d$, $\Xi_2(\omega, A, \phi_0) = [0,1) \cap Y_\omega$. Furthermore a $(\omega, A) \in \mathcal{SOT}(d,\omega)$ is on spin-orbit resonance of second kind iff it has a spin tune ν of second kind such that $m \in \mathbb{Z}^d, n \in \mathbb{Z}$ which satisfy (8.15).

e) If $(\omega, A), (\omega, A') \in \mathcal{SOT}(d,\omega)$ are on spin-orbit resonance of first kind, then $(\omega, A) \sim_{d,\omega} (\omega, A')$.

f) If $(\omega, A), (\omega, A') \in \mathcal{SOT}(d,\omega)$ with $(\omega, A) \sim_{d,\omega} (\omega, A')$ then the following hold. Either both of $(\omega, A), (\omega, A')$ are on spin-orbit resonance of second kind or neither of them. Furthermore either both of them are off spin-orbit resonance of second kind or neither of them.

g) (Yokoya's uniqueness theorem revisited) Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$ and let $(1,\omega)$ be nonresonant. Let (ω, A) have an ISF $\mathcal{S}_G$ and an ISF which is different from $\mathcal{S}_G$ and $-\mathcal{S}_G$. Then (ω, A) is on spin-orbit resonance of second kind.

Proof of Proposition 8.10: See Section F.19. □

8.5 The SPRINT theorem and a corresponding spin tune algorithm

I now resume the theme of Section 7.7 and pose a question about the circumstances for which a weakly trivial spin-orbit torus is an almost coboundary. As a matter of fact, as in Section 7.7, I confine to the case off orbital resonance for which Theorem 8.11 answers the question. On the basis of Theorem 8.11 I then prove the 'SPRINT Theorem' (Corollary 8.12) and demonstrate its practical importance by outlining, after Corollary 8.12, an algorithm, used in the code SPRINT, to compute spin tunes of first and second kind.

Theorem 8.11 *Let* $(1,\omega)$ *be nonresonant and* $(\omega, A_1) \in \mathcal{WT}(d,\omega)$*. Thus, by Proposition 7.5a, eq. (7.30) holds for* $\phi \in \mathbb{R}^d$*, where* $M_1 := Ind_2(A_1)$*,* $f_1 := PHF(A_1)$*. Then, abbreviating the zeroth Fourier coefficient of* f_1 *by* $f_{1,0} := (1/2\pi)^d \int_0^{2\pi} \cdots \int_0^{2\pi} f_1(\phi) d\phi_1 \cdots d\phi_d$ *and defining* $\tilde{f}_1 := f_1 - f_{1,0} \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$*, the following hold:*

a) $(\omega, A_1) \in \mathcal{ACB}(d,\omega)$ *iff the following conditions are satisfied:* $M_1 = 0$ *and a* $g \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$ *exists such that (7.34) is true for all* $\phi \in \mathbb{R}^d$*.*

b) Let $M_1 = 0$ *and let* $g \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$ *exist such that (7.34) holds for all* $\phi \in \mathbb{R}^d$ *(thus, by Theorem 8.11a,* $(\omega, A_1) \in \mathcal{ACB}(d,\omega)$*). Then picking a* $N \in \mathbb{Z}^d$ *and defining* $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$ *by (7.29), the following hold. The spin-orbit torus* $(\omega, A_2) := R_{d,\omega}(T;\omega, A_1)$ *is almost trivial and, for* $\phi \in \mathbb{R}^d$*, we have*

$$A_2(\phi) = \exp(\mathcal{J} 2\pi\nu_2) \ , \tag{8.16}$$

where $\nu_2 := \lfloor f_{1,0} - N^T\omega \rfloor$*. Moreover* $\nu_2 \in \Xi_1(\omega, A_1)$*. The spin-orbit torus* $(\omega, A_3) := R_{d,\omega}(T\mathcal{J}';\omega, A_1)$ *is almost trivial and, for* $\phi \in \mathbb{R}^d$*, we have*

$$A_3(\phi) = \exp(\mathcal{J} 2\pi\nu_3) \ , \tag{8.17}$$

where $\nu_3 := \lfloor -f_{1,0} + N^T\omega \rfloor$*. Moreover* $\nu_3 \in \Xi_1(\omega, A_1)$*.*

c) Let $(\omega, A_1) \in \mathcal{ACB}(d,\omega)$*. Then* (ω, A_1) *is well-tuned and*

$$[\lfloor f_{1,0} \rfloor]_\omega = \Xi_1(\omega, A_1) = \Xi_2(\omega, A_1) \ . \tag{8.18}$$

Proof of Theorem 8.11: See Section F.20. □

Remark:

(1) Clearly, those spin-orbit tori in Theorem 8.11a, with $M_1 \neq 0$, are not almost coboundaries. Another consequence of Theorem 8.11a is the following. Let $(1, \omega)$ be nonresonant and let $(\omega, A_1), (\omega, A_2) \in \mathcal{WT}(d, \omega)$ such that $M_1, M_2 \neq 0$ and $M_1^2 - M_2^2 \neq 0$ where $M_i := Ind_2(A_i)$ $(i = 1, 2)$. Thus, by Theorem 7.14c, one observes that $(\omega, A_1) \not\sim_{d,\omega} (\omega, A_2)$. Moreover, by Theorem 8.11a, $(\omega, A_1), (\omega, A_2)$ are not almost coboundaries whence $\Xi_1(\omega, A_1) = \Xi_1(\omega, A_2) = \emptyset$. Therefore $(\omega, A_1), (\omega, A_2)$ provide an example of two spin-orbit tori in the same $\mathcal{SOT}(d, \omega)$ and with identical Ξ_1 but which are not on the same $R_{d,\omega}$-orbit. Thus this example shows that, in general, the converse of the first claim in Proposition 7.12a is not true. □

The following corollary to Theorem 8.11 I call the 'SPRINT Theorem' since it presents the facts used by the code SPRINT for the numerical calculation of spin tunes (of first and second kind) via stroboscopic averaging (for details on this code, see the remarks after Corollary 8.12). Note that the notation $A_1, M_1, f_1, f_{1,0}$ used in Corollary 8.12 serves to facilitate the comparison with Theorem 8.11.

Corollary 8.12 *(The SPRINT Theorem) Let $(\omega, A) \in \mathcal{ACB}(d, \omega)$ and let $(1, \omega)$ be nonresonant. Let us choose a $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ such that $(\omega, A_1) := R_{d,\omega}(T; \omega, A) \in \mathcal{WT}(d, \omega)$. Thus, by Proposition 7.5a, eq. (7.30) holds for $\phi \in \mathbb{R}^d$, where $M_1 := Ind_2(A_1)$, $f_1 := PHF(A_1)$. Abbreviating the zeroth Fourier coefficient of f_1 by $f_{1,0} := (1/2\pi)^d \int_0^{2\pi} \cdots \int_0^{2\pi} f_1(\phi) d\phi_1 \cdots d\phi_d$, the following hold:*

a) The spin-orbit tori (ω, A) and (ω, A_1) are well-tuned and their spin tunes of first and second kind satisfy

$$[\lfloor f_{1,0} \rfloor]_\omega = \Xi_1(\omega, A) = \Xi_2(\omega, A) = \Xi_1(\omega, A_1) = \Xi_2(\omega, A_1) \, . \tag{8.19}$$

b) We have $M_1 = 0$ *and, for* $\phi \in \mathbb{R}^d, n = 1, 2, ...,$

$$\Psi_{\omega,A_1}(n;\phi) = \exp\Big(\mathcal{J} 2\pi \sum_{j=0}^{n-1} f_1(\phi + 2\pi j\omega) \Big) . \tag{8.20}$$

Moreover, the zeroth Fourier coefficient of f_1 *reads as*

$$f_{1,0} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f_1(2\pi j\omega) . \tag{8.21}$$

c) The function $t : \mathbb{Z} \to SO(3)$*, defined by* $t(n) := T(2\pi n\omega)$*, is an* ω*-quasiperiodic SPF of* (ω, A) *over* $0 \in \mathbb{R}^d$ *and for* $n = 1, 2, ...$ *we have*

$$\Psi_{\omega,A}(n;0) = t(n) \exp\Big(\mathcal{J} 2\pi \sum_{j=0}^{n-1} f_1(2\pi j\omega) \Big) t^T(0) . \tag{8.22}$$

The function $S : \mathbb{Z} \to \mathbb{S}^2$*, defined by* $S(n) := \Psi_{\omega,A}(n;0)t(0)e^1$ *is a spin trajectory of* (ω, A) *over* $0 \in \mathbb{R}^d$*. Moreover for* $n = 1, 2, ...,$

$$\exp\Big(i 2\pi \sum_{j=0}^{n-1} f_1(2\pi j\omega) \Big) = (e^1 + ie^2)^T t^T(n) S(n) . \tag{8.23}$$

where, as usual, i *denotes the complex root of* -1 *lying in the upper complex plane.*

Proof of Corollary 8.12: See Section F.21. □

Corollary 8.12 is of practical interest for the numerical calculation of spin tunes (of first and second kind) via stroboscopic averaging in the code SPRINT [EPAC98, BHV98, Hof, Vo, BHV00, BEH00]. Note that SPRINT also employs a second method, which is due to Yokoya [Yo2] and different from stroboscopic averaging, but which is of no relevance for the point I want to make here. Thus in the following paragraph I sketch, by using the notation of Corollary 8.12, that particular algorithm in SPRINT which computes, via stroboscopic averaging, spin tunes of first and second kind. Note that SPRINT performs this algorithm not just for a single spin-orbit torus but for a whole family of spin-orbit tori (which constitute the spin-orbit system to be dealt with in a storage ring). This important circumstance, which is explained in Remark 2 of this section, is essential for putting the algorithm into perspective.

Now I outline the algorithm as it is used, up to some modifications which do not matter here, by the code SPRINT. Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$ be an almost coboundary and let it be off orbital resonance, i.e., let $(1, \omega)$ be nonresonant. As a first step the algorithm computes an ISF $\mathcal{S}_G$ of (ω, A) via the technique of stroboscopic averaging, which is a certain way of summing tracking data. As a matter of fact, the algorithm merely computes $\mathcal{S}_G$ at the points $\phi = 0$ and $\phi = 2\pi N\omega$ for some sufficiently large positive integer N, i.e., it computes the points $G(0)$ and $G(2\pi N\omega)$ in $\mathbb{S}^2$. From that, by a simple orthonormalization procedure, the algorithm computes a $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ whose third column is G. In fact, the algorithm merely computes T at the points $\phi = 0$ and $\phi = 2\pi N\omega$, i.e., computes the points $T(0) = t(0)$ and $T(2\pi N\omega) = t(N)$ in $SO(3)$. Note incidentally that, by Theorem 7.9, one has $R_{d,\omega}(T; \omega, A) \in \mathcal{WT}(d, \omega)$. So, let us abbreviate $(\omega, A_1) := R_{d,\omega}(T; \omega, A) \in \mathcal{WT}(d, \omega)$ because we are in the situation of Corollary 8.12. On the other hand the algorithm computes in a recursive way, via spin tracking, the points $S(1), ..., S(N)$ in $\mathbb{S}^2$ where $S(n) := \Psi_{\omega,A}(n; 0)t(0)e^1$. Now Corollary 8.12 comes into play since the algorithm uses the data $t(N), S(N)$ to compute a spin tune as follows. If N is sufficiently large (order of magnitude $N = 100000$), then, by Corollary 8.12b, we have

$$N f_{1,0} \approx \sum_{j=0}^{N-1} f_1(2\pi j\omega) ,$$

whence by Corollary 8.12c,

$$\exp(i2\pi N\lfloor f_{1,0}\rfloor) = \exp(i2\pi N f_{1,0}) \approx \exp\left(i2\pi \sum_{j=0}^{N-1} f_1(2\pi j\omega)\right)$$
$$= (e^1 + ie^2)^T t^T(N) S(N) .$$

Thus for large N a (unique) $\nu \in [0,1)$ exists such that

$$\exp(i2\pi N\nu) = (e^1 + ie^2)^T t^T(N) S(N) \ , \tag{8.24}$$

$$\lfloor f_{1,0} \rfloor \approx \nu \ . \tag{8.25}$$

To summarize: Solving (8.24) for $\nu \in [0,1)$ the algorithm obtains an approximation of $\lfloor f_{1,0} \rfloor$. However, by Corollary 8.12a, $\lfloor f_{1,0} \rfloor$ is a spin tune of first and second kind of (ω, A). Thus ν is an approximation of a spin tune of first and second kind of (ω, A) which completes my outline of the algorithm.

In retrospect one sees that the algorithm, being a blend of concepts and facts established in Chapters 7 and 8, computes $t(N), S(N)$ and applies (8.24). The computation of $t(N), S(N)$ is done by tracking, i.e., by solving the equations of motion (6.1),(6.2) in a recursive way.

Remark:

(2) We recall from the Introduction (see Section 5.1) that, in the situation of a storage ring, one is not only faced with a single spin-orbit torus but with a continuous family of spin-orbit tori labelled by an action-parameter J, i.e., with a spin-orbit system. Then the spin tune $\lfloor f_{1,0} \rfloor$ unfolds into a family of spin tunes paramterized by J. This function $\lfloor f_{1,0} \rfloor$ of J is called the 'amplitude dependent spin tune (ADST)' and experience shows that it is piecewise continuous in J. The piecewise continuity in J is due to the continuity of ω in J and to the fact that T is constructed in a way such that it depends piecewise continuously on the parameter J. The latter is achieved, thanks to the stroboscopic averaging technique, by constructing the above mentioned ISF $\mathcal{S}_G$ (whose generator G is the third column of T) such that G is a piecewise continuous function of the parameter J and by performing the orthonormalization procedure, which leads from G to T, in a piecewise continuous way.

Of course, the code SPRINT has to discretize the continuous J-values into a grid, and, once having choosen this grid sufficiently dense, SPRINT nicely exhibits the piecewise continuous dependence of $\lfloor f_{1,0} \rfloor$ on J. □

A completely different method of computing spin tunes is based on the spectral analysis of spin trajectories which is briefly outlined in Section 8.3. This method is outlined in even greater detail, for the flow formalism, in [BEH04].

8.6 The impact of Homotopy Theory on spin tunes of first kind

In this section I state and prove Theorem 8.15. Parts c) and d) of this theorem display how Homotopy Theory has an impact on the individual values of the spin tunes of first kind. In fact, in the situation of Theorems 8.15c,d, $\Xi_1(\omega, A)$ partitions into sets in a way, such each of these sets is associated with a certain subset of $[\mathbb{R}^d, SO(3)]_{2\pi}$. For more details and the practical implications of this, see the remarks after Theorem 8.15. Recall that $[\mathbb{R}^d, SO(3)]_{2\pi}$ is defined in Definition C.19.

Definition 8.13 *Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$ and $s \in \{1, -1\}^d$. Then $\Xi_1^s(\omega, A)$ is defined by*

$$\Xi_1^s(\omega, A) := \{PH(A') : (\omega, A') = R_{d,\omega}(T; \omega, A) \in \mathcal{AT}(d, \omega),$$
$$T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3)), Ind_{3,d}(T) = s\} \; .$$

Clearly for every $(\omega, A) \in \mathcal{SOT}(d, \omega)$ we have

$$\Xi_1(\omega, A) = \bigcup_{s \in \{1,-1\}^d} \Xi_1^s(\omega, A) \; . \tag{8.26}$$

With $\chi \in \mathbb{R}^k, s \in \{1,-1\}^k$ I define

$$Y_\chi^s := \{m^T\chi + n : m \in \mathbb{Z}^k, n \in \mathbb{Z}, s = ((-1)^{m_1}, ..., (-1)^{m_k})^T\} \subset Y_\chi \ ,$$
$$Y_\chi^{half} := \{\frac{m^T\chi + n}{2} : n \in \mathbb{Z}, m \in \mathbb{Z}^k, ((-1)^{m_1}, ..., (-1)^{m_k}) \neq (1, ..., 1)\} \ ,$$

where Y_χ is given by Definition D.1. □

Proposition 8.14 *If $(\omega, A) \in \mathcal{WCB}(d,\omega)$ and $s \in \{1,-1\}^d$ then there exists $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ with $SO(3)$-index s such that $R_{d,\omega}(T;\omega,A) \in \mathcal{WT}(d,\omega)$. If $(\omega, A) \in \mathcal{ACB}(d,\omega)$ then, for every $t \in \{1,-1\}^d$, $\Xi_1^t(\omega,A)$ is nonempty.*

Proof of Proposition 8.14: See Section F.22. □

If $\Xi_1(\omega,A)$ is nonempty then, by Proposition 8.14, each $\Xi_1^s(\omega,A)$ is nonempty which raises the option to see some structure in $\Xi_1^s(\omega,A)$ leading to the question of whether the $\Xi_1^s(\omega,A)$ overlap or don't, i.e., the question of whether the union on the rhs of (8.26) is disjoint or not. Theorem 8.15 gives conditions under which the $\Xi_1^s(\omega,A)$ indeed don't overlap. For the implications of this, see the remarks after Theorem 8.15.

Theorem 8.15 *Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$ and let $(1,\omega)$ be nonresonant. Then the following hold.*

a) Let $(\omega, A) \in \mathcal{ACB}(d,\omega)$ and let $T_1, T_2 \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ such that $(\omega, A_i) := R_{d,\omega}(T_i;\omega,A) \in \mathcal{AT}(d,\omega)$ where $i = 1,2$. Abbreviating $\nu_i := PH(A_i)$, where $i = 1,2$, and $s := Ind_{3,d}(T_1^T T_2)$ then either $(\nu_1 - \nu_2) \in Y_\omega^s$ or $(\nu_1 + \nu_2) \in Y_\omega^s$.

b) Let $(\omega, A) \in \mathcal{ACB}(d,\omega)$. If one picks, by using Proposition 8.14, a ν in $\Xi_1^{(1,...,1)}(\omega,A)$ then one obtains, for every $s \in \{1,-1\}^d$,

$$\Xi_1^s(\omega,A) \subset \{\varepsilon\nu + y : y \in Y_\omega^s, \varepsilon \in \{1,-1\}\} \ . \tag{8.27}$$

c) If $\Xi_1(\omega, A) \cap Y^{half}_\omega = \emptyset$ *and* $s, t \in \{1, -1\}^d$ *with* $s \neq t$ *then* $\Xi^s_1(\omega, A) \cap \Xi^t_1(\omega, A) = \emptyset$.

d) Let (ω, A) *have an ISF* $\mathcal{S}_G$ *and let it also have an ISF which is different from* $\mathcal{S}_G$ *and* $-\mathcal{S}_G$. *Then* $\Xi_1(\omega, A) \neq \emptyset$ *and, for* $s \neq t$, $\Xi^s_1(\omega, A) \cap \Xi^t_1(\omega, A) = \emptyset$.

e) Either $\Xi_1(\omega, A) \subset Y^{half}_\omega$ *or* $\Xi_1(\omega, A) \cap Y^{half}_\omega = \emptyset$.

Remark: The burden of the proof of Theorem 8.15 is on the proof of Theorem 8.15a.

Proof of Theorem 8.15: See Section F.23. □

Since Theorems 8.15c,d give conditions under which the $\Xi^s_1(\omega, A)$ don't overlap they display at the same time how Homotopy Theory impacts the spin tunes of first kind, as follows. Let $(\omega, A) \in \mathcal{ACB}(d, \omega)$ and $s^1 \neq s^2$ such that $\Xi^{s^1}_1(\omega, A) \cap \Xi^{s^2}_1(\omega, A) = \emptyset$. If $\nu_i \in \Xi^{s^i}_1(\omega, A)$ then, by Definition 8.13, a $T_i \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ exists with $Ind_{3,d}(T_i) = s^i$ and such that $(\omega, A_i) := R_{d,\omega}(T_i; \omega, A) \in \mathcal{AT}(d, \omega)$ and $\nu_i = PH(A_i)$ where $i = 1, 2$. Since $s^1 \neq s^2$ we have $Ind_{3,d}(T_1) \neq Ind_{3,d}(T_2)$ whence, by Proposition C.18e, $T_1 \not\simeq^{2\pi}_{SO(3)} T_2$, i.e., T_1, T_2 are not 2π-homotopic w.r.t. $SO(3)$.

I now discuss some aspects of the situation, in which the $\Xi^s_1(\omega, A)$ don't overlap, that are not only of theoretical but also of practical interest. Let $(\omega, A) \in \mathcal{ACB}(d, \omega)$ such that the $\Xi^s_1(\omega, A)$ don't overlap. Then the elements of $\Xi^{(1,...,1)}_1(\omega, A)$ are rather exceptional as follows. I recall from Definition 8.13 that for each element ν of $\Xi^{(1,...,1)}_1(\omega, A)$ a $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ exists with $Ind_{3,d}(T) = (1, ..., 1)^T$ and such that $(\omega, A') := R_{d,\omega}(T; \omega, A) \in \mathcal{AT}(d, \omega)$ and $\nu = PH(A')$. Note that, by Definitions C.12,C.14, every lifting of T w.r.t. $(\mathbb{S}^3, p_2, SO(3))$ is a function $\tilde{T} \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^3)$, i.e., is 2π-periodic. Thus in computer codes which compute T in the quaternion formalism, i.e., which deal with $\tilde{T}$, the elements of $\Xi^{(1,...,1)}_1(\omega, A)$ require a 2π-periodic $\tilde{T}$ whereas each element of $\Xi_1(\omega, A) \setminus \Xi^{(1,...,1)^T}_1(\omega, A)$ requires a $\tilde{T}$ which is not 2π-periodic. In other words, the spin tunes of first kind which are associated with 2π-periodic $\tilde{T}'s$, are rather exceptional. This phenomenon, which occurs in a similar way also in the spinor formalism (the latter formalism is mentioned in Section 5.3), was observed in [Hof, Section 4.1],[Yo2] and accordingly the present section is inspired by these two works.

8.7 Further properties of invariant spin fields

Lemma 8.16 *Let $G \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$ be of class C^1 and let $\omega \in \mathbb{R}^d$. Then a $(\omega, A) \in \mathcal{SOT}(d, \omega)$ exists which has an ISF $\mathcal{S}_G$ generated by G.*

Proof of Lemma 8.16: See Section F.24. □

I now resume the theme of Theorem 7.10.

Theorem 8.17 *Let ω be in $\mathbb{R}^d$ such that $(1, \omega)$ is nonresonant and $d \geq 2$. Then there exists a $(\omega, A) \in (\mathcal{SOT}(d, \omega) \setminus \mathcal{WCB}(d, \omega))$ which has an ISF $\mathcal{S}_G$. For every such spin-orbit torus, $\mathcal{S}_G$ and $-\mathcal{S}_G$ are the only ISF's.*

Proof of Theorem 8.17: See Section F.25. □

Chapter 9

Reconsidering the $\mathbb{Z}$-actions $L_{\omega,A}$ and $L_{\omega,A}^{(PF)}$

In this section I reconsider the $\mathbb{Z}$-actions $L_{\omega,A}$ and $L_{\omega,A}^{(PF)}$ introduced in Chapter 6.

9.1 Carving out the topological $\mathbb{Z}$-spaces $(\mathbb{R}^{d+3}, L_{\omega,A})$ as skew products of the topological $\mathbb{Z}$-spaces $(\mathbb{R}^d, L_\omega)$

Proposition 9.1 *Let (ω, A) be a d-dimensional spin-orbit torus. Then the function $h : \mathbb{R}^{d+3} \to \mathbb{R}^d$, defined, for $\phi \in \mathbb{R}^d, S \in \mathbb{R}^3$, by $h(\phi_1, ..., \phi_d, S) := (\phi_1, ..., \phi_d)^T$, is a continuous $\mathbb{Z}$-map from the topological $\mathbb{Z}$-space $(\mathbb{R}^{d+3}, L_{\omega,A})$ to the topological $\mathbb{Z}$-space $(\mathbb{R}^d, L_\omega)$. Moreover, the topological $\mathbb{Z}$-space $(\mathbb{R}^{d+3}, L_{\omega,A})$ is a skew product of the topological $\mathbb{Z}$-space $(\mathbb{R}^d, L_\omega)$.*

Proof of Proposition 9.1: See Section F.26. □

With Proposition 9.1 I can now put (6.6) into perspective. In fact, while in Section 6.1 I derived (6.6) from (6.1),(6.2) I now derive (6.6) in a different way. Since, by Proposition 9.1, $(\mathbb{R}^{d+3}, L_{\omega,A})$ is a skew product of $(\mathbb{R}^d, L_\omega)$ I can apply Remark 1 in Appendix B. According to that remark we get, for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d, S \in \mathbb{R}^3$,

$$L_{\omega,A}(n; \phi, S) = \begin{pmatrix} L_\omega(n; \phi) \\ L''(n; \phi, S) \end{pmatrix} , \tag{9.1}$$

where the function $L'' : \mathbb{Z} \times \mathbb{R}^{d+3} \to \mathbb{R}^3$ satisfies, for $m, n \in \mathbb{Z}, \phi \in \mathbb{R}^d, S \in \mathbb{R}^3$,

$$L''(n + m; \phi, S) = L''(n; \phi + 2\pi m\omega, L''(m; \phi, S)) , \tag{9.2}$$

where I also have used (6.14). Imposing the condition that $L''(n; \phi, S)$ is linear in S I get, for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d, S \in \mathbb{R}^3$,

$$L''(n; \phi, S) = L'''(n; \phi)S , \tag{9.3}$$

where L''' is a function from $\mathbb{Z} \times \mathbb{R}^d$ into the set of real 3×3 matrices. It follows from (9.2),(9.3) that, for $m, n \in \mathbb{Z}, \phi \in \mathbb{R}^d, S \in \mathbb{R}^3$,

$$L'''(n + m; \phi) = L'''(n; \phi + 2\pi m\omega)L'''(m; \phi) , \tag{9.4}$$

which is indeed (6.6) expressed in terms of L'''. We conclude that (6.6) follows from the facts that $(\mathbb{R}^{d+3}, L_{\omega,A})$ is a skew product of $(\mathbb{R}^d, L_\omega)$ and that the S-dependent components of $L_{\omega,A}(n; \phi, S)$ are linear in S.

9.2 Carving out the topological $\mathbb{Z}$-spaces $(\mathbb{R}^{d+3}, L_{\omega,A})$ as extensions of the topological $\mathbb{Z}$-spaces $(\mathbb{T}^d \times \mathbb{R}^3, L^{(T)}_{\omega,A})$

As mentioned in Section 6.1, the spin-orbit motion in $\mathbb{R}^{d+3}$ is closely related to an associated spin-orbit motion in $\mathbb{T}^d \times \mathbb{R}^3$ which is characterized by the $\mathbb{Z}$-action $L^{(T)}_{\omega,A}$

on $\mathbb{T}^d \times \mathbb{R}^3$ that is defined in Proposition 9.2b. In fact while the emphasis in the present work is on orbital motion in $\mathbb{R}^d$, a deeper study of spin-orbit tori will need a stronger focus on orbital motion in $\mathbb{T}^d$ and therefore the present section give a brief glimpse into this.

The d-torus $\mathbb{T}^d$ is defined by Definition C.2. Proposition 9.2, stated below, exhibits the relation between $L^{(T)}_{\omega,A}$ and $L_{\omega,A}$. But before I come to that I define the function $p_{5,d} : \mathbb{R}^{d+3} \to \mathbb{T}^d \times \mathbb{R}^3$, for $\phi \in \mathbb{R}^d, S \in \mathbb{R}^3$, by

$$p_{5,d}(\phi, S) := \begin{pmatrix} p_{4,d}(\phi) \\ S \end{pmatrix} = \begin{pmatrix} \exp(i\phi) \\ S \end{pmatrix} , \tag{9.5}$$

will turn out to be a $\mathbb{Z}$-map from $(\mathbb{R}^{d+3}, L_{\omega,A})$ to $(\mathbb{T}^d \times \mathbb{R}^3, L^{(T)}_{\omega,A})$. Note that, choosing the product topology on $\mathbb{T}^d \times \mathbb{R}^3$, we see that $p_{5,d}$ is a continuous. Moreover, $p_{5,d}$ is onto $\mathbb{T}^d \times \mathbb{R}^3$. One can even show that $(\mathbb{R}^{d+3}, p_{5,d}, \mathbb{T}^d \times \mathbb{R}^3)$ is a Hurewicz fibration (see Appendix C) but this property is not needed in this work.

If (ω, A) is a d-dimensional spin-orbit torus then $\Psi_{\omega,A}(n;\cdot) \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ whence it has a unique factor $\Psi'_{\omega,A}(n;\cdot) \in \mathcal{C}(\mathbb{T}^d, SO(3))$ w.r.t. $(\mathbb{R}^d, p_{4,d}, \mathbb{T}^d)$, i.e., $\Psi'_{\omega,A}(n;\cdot) = FAC_d(\Psi_{\omega,A}(n;\cdot); SO(3))$. In other words, $\Psi'_{\omega,A} : \mathbb{Z} \times \mathbb{T}^d \to SO(3)$ is the unique function such that for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d$,

$$\Psi_{\omega,A}(n;\phi) = \Psi'_{\omega,A}(n; p_{4,d}(\phi)) . \tag{9.6}$$

I can now state the proposition.

Proposition 9.2 *a) Let $\omega \in \mathbb{R}^d$ and let the function $L^{(T)}_{\omega} : \mathbb{Z} \times \mathbb{T}^d \to \mathbb{T}^d$ be defined, for $n \in \mathbb{Z}, z \in \mathbb{T}^d$, by*

$$L^{(T)}_{\omega}(n;z) := \Big(\exp(i2\pi n\omega_1)z_1, ..., \exp(i2\pi n\omega_d)z_d\Big)^T . \tag{9.7}$$

Then $L^{(T)}_{\omega}$ is a $\mathbb{Z}$-action on $\mathbb{T}^d$. Moreover $(\mathbb{T}^d, L^{(T)}_{\omega})$ is a topological $\mathbb{Z}$-space and $p_{4,d}$ is a continuous $\mathbb{Z}$-map from the topological $\mathbb{Z}$-space $(\mathbb{R}^d, L_{\omega})$ to the topological

$\mathbb{Z}$-space $(\mathbb{T}^d, L^{(T)}_\omega)$. Furthermore the topological $\mathbb{Z}$-space $(\mathbb{R}^d, L_\omega)$ is an extension of the topological $\mathbb{Z}$-space $(\mathbb{T}^d, L^{(T)}_\omega)$.

b) Let (ω, A) be a d-dimensional spin-orbit torus and let the function $L^{(T)}_{\omega,A} : \mathbb{Z}\times\mathbb{T}^d \times \mathbb{R}^3 \to \mathbb{T}^d \times \mathbb{R}^3$ be defined, for $n \in \mathbb{Z}, z \in \mathbb{T}^d, S \in \mathbb{R}^3$, by

$$L^{(T)}_{\omega,A}(n; z, S) := \begin{pmatrix} L^{(T)}_\omega(n; z) \\ \Psi'_{\omega,A}(n; z)S \end{pmatrix} , \tag{9.8}$$

where $\Psi'_{\omega,A}$ is defined by (9.6), i.e., $\Psi'_{\omega,A}(n; \cdot) = FAC_d(\Psi_{\omega,A}(n; \cdot); SO(3))$.

Then $L^{(T)}_{\omega,A}$ is a $\mathbb{Z}$-action on $\mathbb{T}^d \times \mathbb{R}^3$. Moreover $(\mathbb{T}^d \times \mathbb{R}^3, L^{(T)}_{\omega,A})$ is a topological $\mathbb{Z}$-space and $p_{5,d}$ is a continuous $\mathbb{Z}$-map from the topological $\mathbb{Z}$-space $(\mathbb{R}^{d+3}, L_{\omega,A})$ to the topological $\mathbb{Z}$-space $(\mathbb{T}^d \times \mathbb{R}^3, L^{(T)}_{\omega,A})$. Furthermore the topological $\mathbb{Z}$-space $(\mathbb{R}^{d+3}, L_{\omega,A})$ is an extension of the topological $\mathbb{Z}$-space $(\mathbb{T}^d \times \mathbb{R}^3, L^{(T)}_{\omega,A})$.

c) Let (ω, A) be a d-dimensional spin-orbit torus and let $(\mathbb{T}^d \times \mathbb{R}^3, L)$ be a topological $\mathbb{Z}$-space. If the function $p_{5,d}$ is a $\mathbb{Z}$-map from the topological $\mathbb{Z}$-space $(\mathbb{R}^{d+3}, L_{\omega,A})$ to the topological $\mathbb{Z}$-space $(\mathbb{T}^d \times \mathbb{R}^3, L)$, then $L = L^{(T)}_{\omega,A}$.

d) Let $(\mathbb{R}^{d+3}, L)$ be a topological $\mathbb{Z}$-space, let (ω, A) be a d-dimensional spin-orbit torus and let the function $p_{5,d}$ be a $\mathbb{Z}$-map from the topological $\mathbb{Z}$-space $(\mathbb{R}^{d+3}, L)$ to the topological $\mathbb{Z}$-space $(\mathbb{T}^d \times \mathbb{R}^3, L^{(T)}_{\omega,A})$. Then a $N \in \mathbb{Z}^d$ exists such that, for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d, S \in \mathbb{R}^3$,

$$L(n; \phi, S) = \begin{pmatrix} \phi + 2\pi n\omega + 2\pi nN \\ \Psi_{\omega,A}(n; \phi)S \end{pmatrix} . \tag{9.9}$$

Conversely, if N is an arbitrary element of $\mathbb{Z}^d$ and if a function $L : \mathbb{Z}\times\mathbb{R}^{d+3} \to \mathbb{R}^{d+3}$ is defined, for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d, S \in \mathbb{R}^3$, by (9.9), then $(\mathbb{R}^{d+3}, L)$ is a topological $\mathbb{Z}$-space and $p_{5,d}$ is a $\mathbb{Z}$-map from the topological $\mathbb{Z}$-space $(\mathbb{R}^{d+3}, L)$ to the topological $\mathbb{Z}$-space $(\mathbb{T}^d \times \mathbb{R}^3, L^{(T)}_{\omega,A})$ making the former an extension of the latter.

Proof of Proposition 9.2: See Section F.27. □

Due to (F.103) in the proof of Proposition 9.2b and due to Appendix B the function $\Psi'_{\omega,A}$ in Proposition 9.2b is a $SO(3)$-cocycle over the topological $\mathbb{Z}$-space $(\mathbb{T}^d, L_\omega^{(T)})$.

9.3 A principal $SO(3)$-bundle underlying $\mathcal{SOT}(d)$

The theory of spin-orbit tori developed so far in this work will in the present section be reconsidered in terms of the principal $SO(3)$-bundle $\lambda_{\mathcal{SOT}(d)}$, defined by (9.12). For every $(\omega, A) \in \mathcal{SOT}(d)$ we recall from Section 6.2 that $\Psi_{\omega,A}$ is a $SO(3)$-cocycle over the topological $\mathbb{Z}$-space $(\mathbb{R}^d, L_\omega)$. In Section 9.3.1 I will show that this allows me to encode (ω, A) into a group homomorphism, $\Phi_{\omega,A}$, from the group $\mathbb{Z}$ into the automorphism group $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$ of $\lambda_{\mathcal{SOT}(d)}$. This technique was apparently introduced, in the context of Dynamical Systems Theory, by Zimmer in the 1980's [Zi2] and further developed by Feres and coworkers in the 1990's [Fe, Section 6]. Thus for brevity I call this technique the 'Feres machinery'. The Feres machinery shows us in Sections 9.3.3 and 9.3.4 how, via $\Phi_{\omega,A}$, the associated bundle $\lambda_{\mathcal{SOT}(d)}[\mathbb{R}^3, L^{(3D)}]$, which is defined by (9.33), carries the two basic $\mathbb{Z}$-actions, $L_{\omega,A}$ and $L_{\omega,A}^{(PF)}$, of spin-orbit theory. I thus fulfill the motto, mentioned at the beginning of Chapter 9, of reconsidering $L_{\omega,A}$ and $L_{\omega,A}^{(PF)}$. Furthermore I prove in Section 9.3.5 a theorem, Theorem 9.5a, which is a special case of the reduction theorem which apparently was introduced by Zimmer. In particular our theorem shows the relation between invariant spin fields and invariant $SO_3(2)$-reductions of $\lambda_{\mathcal{SOT}(d)}$. Note that a reader who is interested in Section 9.3.5 can skip Sections 9.3.3 and 9.3.4. Clearly the present section widens the perspective since it demonstrates how the principal $SO(3)$-bundle $\lambda_{\mathcal{SOT}(d)}$ underlies the theory of spin-orbit tori.

The facts and features of principal bundles and their associated bundles which are needed here are presented in Appendix E where I follow the elegant treatment of Husemoller's book [Hus] which uses Cartan principal bundles (another textbook which uses Cartan principal bundles is [Mac]).

9.3.1 The principal $SO(3)$-bundle $\lambda_{\mathcal{SOT}(d)}$

The principal $SO(3)$-bundle $\lambda_{\mathcal{SOT}(d)}$ I introduce in this section is a product principal bundle and its underlying bundle is defined by

$$\xi_{\mathcal{SOT}(d)}^{(1)} := (\mathbb{R}^d \times SO(3), p_{\mathcal{SOT}(d)}^{(1)}, \mathbb{R}^d) \ , \tag{9.10}$$

where the function $p_{\mathcal{SOT}(d)}^{(1)} : \mathbb{R}^d \times SO(3) \to \mathbb{R}^d$ is the projection onto the first component, i.e., $p_{\mathcal{SOT}(d)}^{(1)}(\phi, R) := \phi$ for $\phi \in \mathbb{R}^d, R \in SO(3)$. Clearly, by Definition C.1, $\xi_{\mathcal{SOT}(d)}^{(1)}$ is a bundle and, since $p_{\mathcal{SOT}(d)}^{(1)}$ is onto $\mathbb{R}^d$, it is a fiber structure. Of course $\xi_{\mathcal{SOT}(d)}^{(1)}$ is a product bundle. To 'unfold' the bundle $\xi_{\mathcal{SOT}(d)}^{(1)}$ into a principal bundle I define the right $SO(3)$-action $R_{\mathcal{SOT}(d)}^{(1)}$ on $\mathbb{R}^d \times SO(3)$ by

$$R_{\mathcal{SOT}(d)}^{(1)}(R'; \phi, R) := (\phi, RR') \ , \tag{9.11}$$

where $\phi \in \mathbb{R}^d, R, R' \in SO(3)$. Clearly $(\mathbb{R}^d \times SO(3), R_{\mathcal{SOT}(d)}^{(1)})$ is a topological right $SO(3)$-space. One thus arrives at the quadruple

$$\lambda_{\mathcal{SOT}(d)} := (\xi_{\mathcal{SOT}(d)}^{(1)}, R_{\mathcal{SOT}(d)}^{(1)}) = (\mathbb{R}^d \times SO(3), p_{\mathcal{SOT}(d)}^{(1)}, \mathbb{R}^d, R_{\mathcal{SOT}(d)}^{(1)}) \ . \tag{9.12}$$

In Section E.6.1 it is shown that the topological right $SO(3)$-space $(\mathbb{R}^d \times SO(3), R_{\mathcal{SOT}(d)}^{(1)})$ is principal and that $\lambda_{\mathcal{SOT}(d)}$ is a principal $SO(3)$-bundle. Note that $\lambda_{\mathcal{SOT}(d)}^{(1)}$ is called a 'product principal $SO(3)$-bundle'.

Following Section E.6.1, I denote the set of morphisms from $\xi_{\mathcal{SOT}(d)}^{(1)}$ to itself in the category *Bun* of bundles by $\mathfrak{Mor}_{Bun}(\xi_{\mathcal{SOT}(d)}^{(1)})$. Note that, by definition, $\mathfrak{Mor}_{Bun}(\xi_{\mathcal{SOT}(d)}^{(1)})$ consists of the pairs $(\varphi, \bar{\varphi})$ for which $\varphi \in \mathcal{C}(\mathbb{R}^d \times SO(3), \mathbb{R}^d \times SO(3))$ and $\bar{\varphi} \in \mathcal{C}(\mathbb{R}^d, \mathbb{R}^d)$ such that

$$\bar{\varphi} \circ p_{\mathcal{SOT}(d)}^{(1)} = p_{\mathcal{SOT}(d)}^{(1)} \circ \varphi \ . \tag{9.13}$$

Following Section E.6.1, I denote the set of morphisms from $\lambda_{\mathcal{SOT}(d)}$ to itself in the category $Bun(SO(3))$ of principal $SO(3)$-bundles by $\mathfrak{Mor}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$. Note that, by definition, $\mathfrak{Mor}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$ consists of the pairs $(\varphi, \bar{\varphi})$ in $\mathfrak{Mor}_{Bun}(\xi^{(1)}_{\mathcal{SOT}(d)})$ for which φ is a $SO(3)$-map from the right G-space $(\mathbb{R}^d \times SO(3), R^{(1)}_{\mathcal{SOT}(d)})$ to itself. It follows from (E.79) that $\mathfrak{Mor}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$ has the following simple form:

$$\mathfrak{Mor}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)}) = \Bigg\{ (\varphi, \bar{\varphi}) \in \mathcal{C}(\mathbb{R}^d \times SO(3), \mathbb{R}^d \times SO(3)) \times \mathcal{C}(\mathbb{R}^d, \mathbb{R}^d) : \\ (\forall\, \phi \in \mathbb{R}^d, R \in SO(3)) \varphi(\phi, R) = \begin{pmatrix} \bar{\varphi}(\phi) \\ f(\phi) R \end{pmatrix}, f \in \mathcal{C}(\mathbb{R}^d, SO(3)) \Bigg\}. \quad (9.14)$$

Note that if $(\varphi, \bar{\varphi}) \in \mathfrak{Mor}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$ then by (9.14) the functions $\bar{\varphi}, f$ are uniquely determined by φ and φ is uniquely determined by $\bar{\varphi}, f$. Given $(\varphi_i, \bar{\varphi}_i) \in \mathfrak{Mor}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$ for $i = 1, 2$ and writing, by (9.14), $\varphi_i(\phi, R) = (\bar{\varphi}_i(\phi), f_i(\phi)R)$, the composition law of $Bun(SO(3))$ gives the morphism $(\varphi_2, \bar{\varphi}_2)(\varphi_1, \bar{\varphi}_1) = (\varphi_2 \circ \varphi_1, \bar{\varphi}_2 \circ \bar{\varphi}_1) \in \mathfrak{Mor}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$ where for $\phi \in \mathbb{R}^d, R \in SO(3)$

$$(\varphi_2 \circ \varphi_1)(\phi, R) = \varphi_2\Big(\bar{\varphi}_1(\phi), f_1(\phi)R \Big) = \begin{pmatrix} (\bar{\varphi}_2 \circ \bar{\varphi}_1)(\phi) \\ f_2(\bar{\varphi}_1(\phi)) f_1(\phi) R \end{pmatrix}. \quad (9.15)$$

Denoting by $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$ the set of isomorphisms in $\mathfrak{Mor}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$ it follows from (E.82) that

$$\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)}) = \Bigg\{ (\varphi, \bar{\varphi}) \in \mathcal{C}(\mathbb{R}^d \times SO(3), \mathbb{R}^d \times SO(3)) \times HOMEO(\mathbb{R}^d, \mathbb{R}^d) : \\ (\forall\, \phi \in \mathbb{R}^d, R \in SO(3)) \varphi(\phi, R) = \begin{pmatrix} \bar{\varphi}(\phi) \\ f(\phi) R \end{pmatrix}, f \in \mathcal{C}(\mathbb{R}^d, SO(3)) \Bigg\}, \quad (9.16)$$

where $HOMEO(\mathbb{R}^d, \mathbb{R}^d)$ denotes the set of homeomorphisms from $\mathbb{R}^d$ onto itself. Note that, for every category, isomorphisms from an object to itself are called 'automorphisms', which explains the notation $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$. Note that $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$ has a canonical group structure where the multiplication is given by the composition law of $Bun(SO(3))$.

Following Section E.6.5 I now encode the spin-orbit tori in $\mathcal{SOT}(d)$ into subgroups of $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$. Recalling Section 6.2, we have the function $\rho_{\mathcal{SOT}(d)}$: $\mathcal{SOT}(d) \to COC(\mathbb{R}^d, \mathbb{Z}, SO(3))$, which is defined for $(\omega, A) \in \mathcal{SOT}(d)$ by (6.15). Since $\rho_{\mathcal{SOT}(d)}$ is an injection it allows to encode spin-orbit tori into cocycles. Moreover, recalling Section E.4, I denote by $HOM_{\mathbb{Z}}(\lambda_{\mathcal{SOT}(d)})$ the set of group homomorphisms from $\mathbb{Z}$ into $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$ so Section E.6.5 gives us an injection $\rho_{\mathbb{R}^d,\mathbb{Z},SO(3)} : COC(\mathbb{R}^d, \mathbb{Z}, SO(3)) \to HOM_{\mathbb{Z}}(\lambda_{\mathcal{SOT}(d)})$ which is defined for $(l, f) \in COC(\mathbb{R}^d, \mathbb{Z}, SO(3))$ by

$$\rho_{\mathbb{R}^d,\mathbb{Z},SO(3)}(l, f) := \Phi \; , \tag{9.17}$$

where, for $n \in \mathbb{Z}$,

$$\Phi(n) := (\varphi(n; \cdot), l(n; \cdot)) \; , \tag{9.18}$$

and where, for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d, R \in SO(3)$,

$$\varphi(n; \phi, R) := \begin{pmatrix} l(n; \phi) \\ f(n; \phi)R \end{pmatrix} . \tag{9.19}$$

Note that the injection $\rho_{\mathbb{R}^d,\mathbb{Z},SO(3)}$ is a special case of a more general construction which is outlined in Remark 1 of Section E.6.5 and which is based on the cross sections of the bundle $\xi^{(1)}_{\mathcal{SOT}(d)}$. It follows from (6.15), (9.17),(9.18) (9.19) that for $(\omega, A) \in \mathcal{SOT}(d)$

$$(\rho_{\mathbb{R}^d,\mathbb{Z},SO(3)} \circ \rho_{\mathcal{SOT}(d)})(\omega, A) = \rho_{\mathbb{R}^d,\mathbb{Z},SO(3)}(L_\omega, \Psi_{\omega,A}) = \Phi_{\omega,A} \; , \tag{9.20}$$

where, for $n \in \mathbb{Z}$,

$$\Phi_{\omega,A}(n) := (\varphi_{\omega,A}(n; \cdot), L_\omega(n; \cdot)) \; , \tag{9.21}$$

and where for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d, R \in SO(3)$

$$\varphi_{\omega,A}(n; \phi, R) := \begin{pmatrix} L_\omega(n; \phi) \\ \Psi_{\omega,A}(n; \phi)R \end{pmatrix} = \begin{pmatrix} \phi + 2\pi n\omega \\ \Psi_{\omega,A}(n; \phi)R \end{pmatrix} . \tag{9.22}$$

Since $\rho_{\mathcal{SOT}(d)}$ and $\rho_{\mathbb{R}^d,\mathbb{Z},SO(3)}$ are one-one, it follows from (9.20) that every spin-orbit torus $(\omega, A) \in \mathcal{SOT}(d)$ is uniquely characterized by the group homomorphism $\Phi_{\omega,A}$ whence (ω, A) is encoded in the subgroup $\Phi_{\omega,A}(\mathbb{Z})$ of $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$. I call the group homomorphisms $\Phi_{\omega,A}$ 'tied' to $\mathcal{SOT}(d)$.

Equipping $\mathbb{Z}$ with the discrete topology one concludes from Section E.6.5 that $\rho_{\mathbb{R}^d,\mathbb{Z},SO(3)}$ is a bijection onto $HOM_{\mathbb{Z}}(\lambda_{\mathcal{SOT}(d)})$. Thus, given a $\Phi \in HOM_{\mathbb{Z}}(\lambda_{\mathcal{SOT}(d)})$ and since $\rho_{\mathbb{R}^d,\mathbb{Z},SO(3)}$ is a bijection onto $HOM_{\mathbb{Z}}(\lambda_{\mathcal{SOT}(d)})$, eq. (9.17) holds where $(l, f) \in COC(\mathbb{R}^d, \mathbb{Z}, SO(3))$ is defined by $(l, f) := \rho^{-1}_{\mathbb{R}^d,\mathbb{Z},SO(3)}(\Phi)$. It is easy to see by (9.17),(9.18) (9.19) that Φ is tied to $\mathcal{SOT}(d)$ iff $l(1;\cdot)$ is a translation on $\mathbb{R}^d$ and $f(1;\phi)$ is 2π-periodic in ϕ. Thus not every group homomorphism in $HOM_{\mathbb{Z}}(\lambda_{\mathcal{SOT}(d)})$ is tied to $\mathcal{SOT}(d)$.

It is also worthwile to note that since, for $(\omega, A) \in \mathcal{SOT}(d)$, the function $\Phi_{\omega,A}$ is a group homomorphism it follows from the composition law of $Bun(SO(3))$ and (9.21) that $\varphi_{\omega,A}$ is a $\mathbb{Z}$-action on $\mathbb{R}^d \times SO(3)$.

To discuss $R_{d,\omega}$ in the context of $\lambda_{\mathcal{SOT}(d)}$, let $(\omega, A), (\omega, A') \in \mathcal{SOT}(d), T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ and $R_{d,\omega}(T;\omega, A) = (\omega, A')$. Thus by Theorem 7.3a we have, for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d$,

$$\Psi_{\omega,A'}(n;\phi) = T^T(L_\omega(n;\phi))\Psi_{\omega,A}(n;\phi)T(\phi) \; . \tag{9.23}$$

It follows from (9.20) that

$$(\rho_{\mathbb{R}^d,\mathbb{Z},SO(3)} \circ \rho_{\mathcal{SOT}(d)})(\omega, A') = \rho_{\mathbb{R}^d,\mathbb{Z},SO(3)}(L_\omega, \Psi_{\omega,A'}) = \Phi_{\omega,A'} \; , \tag{9.24}$$

where, for $n \in \mathbb{Z}$,

$$\Phi_{\omega,A'}(n) = (\varphi_{\omega,A'}(n;\cdot), L_\omega(n;\cdot)) \; , \tag{9.25}$$

and where for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d, R \in SO(3)$

$$\varphi_{\omega,A'}(n;\phi, R) := \begin{pmatrix} L_\omega(n;\phi) \\ \Psi_{\omega,A'}(n;\phi)R \end{pmatrix} = \begin{pmatrix} L_\omega(n;\phi) \\ T^T(L_\omega(n;\phi))\Psi_{\omega,A}(n;\phi)T(\phi)R \end{pmatrix} , \tag{9.26}$$

where in the second equality I used (9.23). I define $\varphi_T \in \mathcal{C}(\mathbb{R}^d \times SO(3), \mathbb{R}^d \times SO(3)$ for $\phi \in \mathbb{R}^d, R \in SO(3)$ by

$$\varphi_T(\phi, R) := \begin{pmatrix} \phi \\ T(\phi)R \end{pmatrix} . \tag{9.27}$$

Using (E.12),(E.141) the gauge group of $\lambda_{\mathcal{SOT}(d)}$ reads as

$$\begin{aligned}
&\mathfrak{Gau}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)}) = \\
&\quad \{\varphi \in \mathcal{C}(\mathbb{R}^d \times SO(3), \mathbb{R}^d \times SO(3)) : (\varphi, id_{\mathbb{R}^d}) \in \mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})\} \\
&\qquad = \{\varphi \in \mathcal{C}(\mathbb{R}^d \times SO(3), \mathbb{R}^d \times SO(3)) : \\
&\qquad\quad (\forall\, \phi \in \mathbb{R}^d, R \in SO(3))\varphi(\phi, R) = \begin{pmatrix} \phi \\ f(\phi)R \end{pmatrix}, f \in \mathcal{C}(\mathbb{R}^d, SO(3))\} \, ,
\end{aligned} \tag{9.28}$$

whence $\varphi_T \in \mathfrak{Gau}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$ and $\Phi_T := (\varphi_T, id_{\mathbb{R}^d}) \in \mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$. I define $\Phi' \in HOM_{\mathbb{Z}}(\lambda_{\mathcal{SOT}(d)})$ for $n \in \mathbb{Z}$ by

$$\begin{aligned}
\Phi'(n) &:= \Phi_T^{-1}\Phi_{\omega,A}(n)\Phi_T = (\varphi_T, id_{\mathbb{R}^d})^{-1}(\varphi_{\omega,A}(n;\cdot), L_\omega(n;\cdot))(\varphi_T, id_{\mathbb{R}^d}) \\
&= (\varphi_T^{-1} \circ \varphi_{\omega,A}(n;\cdot) \circ \varphi_T, L_\omega(n;\cdot)) \, ,
\end{aligned} \tag{9.29}$$

where I also used (9.21). One concludes from (9.22),(9.29), (E.146) that for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d, R \in SO(3)$

$$\begin{aligned}
(\Phi'(n))(\phi, R) &= \left(\begin{pmatrix} L_\omega(n;\phi) \\ T^T(L_\omega(n;\phi))\Psi_{\omega,A}(n;\phi)T(\phi)R \end{pmatrix}, L_\omega(n;\phi)\right) \\
&= \left(\begin{pmatrix} L_\omega(n;\phi) \\ \Psi_{\omega,A'}(n;\phi)R \end{pmatrix}, L_\omega(n;\phi)\right) .
\end{aligned} \tag{9.30}$$

One concludes from (9.25),(9.26),(9.30) that $\Phi_{\omega,A'} = \Phi'$ whence I have shown that the transformation via $R_{d,\omega}(T;\cdot)$ corresponds in $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$ to a conjugation by Φ_T. In other words, on the level of $\lambda_{\mathcal{SOT}(d)}$, the gauge group $\mathfrak{Gau}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$ takes over the job from the group $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$.

9.3.2 The bundle $\lambda_{\mathcal{SOT}(d)}[\mathbb{R}^3, L^{(3D)}]$ associated with $\lambda_{\mathcal{SOT}(d)}$

In this section I introduce the bundle $\lambda_{\mathcal{SOT}(d)}[\mathbb{R}^3, L^{(3D)}]$ which in the ensuing sections will be the substratum by which $\lambda_{\mathcal{SOT}(d)}$ carries the $\mathbb{Z}$-actions $L_{\omega,A}$ and $L^{(PF)}_{\omega,A}$. I define the topological left $SO(3)$-space $(\mathbb{R}^3, L^{(3D)})$ where the function $L^{(3D)} : SO(3)\times\mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$L^{(3D)}(R;S) := RS\ , \tag{9.31}$$

with $S \in \mathbb{R}^3, R \in SO(3)$ and where RS is the matrix product of R and S. Following the standard technique of constructing associated bundles, which is outlined in Section E.2 and, for the case of product principal bundles, in Section E.6.2, one defines the function $R^{(2)}_{\mathcal{SOT}(d)} : SO(3) \times \mathbb{R}^d \times SO(3) \times \mathbb{R}^3 \to \mathbb{R}^d \times SO(3) \times \mathbb{R}^3$ for $\phi \in \mathbb{R}^d, R, R' \in SO(3), S \in \mathbb{R}^3$, by

$$R^{(2)}_{\mathcal{SOT}(d)}(R';\phi,R,S) := \begin{pmatrix} R^{(1)}_{\mathcal{SOT}(d)}(R';\phi,R) \\ L^{(3D)}(R'^{-1};S) \end{pmatrix} = \begin{pmatrix} \phi \\ RR' \\ R'^{-1}S \end{pmatrix}\ , \tag{9.32}$$

and observes that $(\mathbb{R}^d \times SO(3) \times \mathbb{R}^3, R^{(2)}_{\mathcal{SOT}(d)})$ is a topological right $SO(3)$-space. Denoting the orbit space of $(\mathbb{R}^d \times SO(3) \times \mathbb{R}^3, R^{(2)}_{\mathcal{SOT}(d)})$ by $E^{(3)}_{\mathcal{SOT}(d)}$, i.e., in the notation of Appendix B, $E^{(3)}_{\mathcal{SOT}(d)} := (\mathbb{R}^d \times SO(3) \times \mathbb{R}^3)/R^{(2)}_{\mathcal{SOT}(d)}$ and the canonical surjection: $\mathbb{R}^d \times SO(3) \times \mathbb{R}^3 \to E^{(3)}_{\mathcal{SOT}(d)}$ by $p^{(2)}_{\mathcal{SOT}(d)}$, one obtains the bundle:

$$\lambda_{\mathcal{SOT}(d)}[\mathbb{R}^3, L^{(3D)}] =: \xi^{(3)}_{\mathcal{SOT}(d)} = (E^{(3)}_{\mathcal{SOT}(d)}, p^{(3)}_{\mathcal{SOT}(d)}, \mathbb{R}^d)\ , \tag{9.33}$$

where the continuous function $p^{(3)}_{\mathcal{SOT}(d)} : E^{(3)}_{\mathcal{SOT}(d)} \to \mathbb{R}^d$ is the unique function: $E^{(3)}_{\mathcal{SOT}(d)} \to \mathbb{R}^d$ which satisfies

$$p^{(3)}_{\mathcal{SOT}(d)} \circ p^{(2)}_{\mathcal{SOT}(d)} = p^{(1)}_{\mathcal{SOT}(d)}\ . \tag{9.34}$$

One calls $\xi^{(3)}_{\mathcal{SOT}(d)}$ the ‘bundle associated with $\lambda_{\mathcal{SOT}(d)}$ via the topological left $SO(3)$-space $(\mathbb{R}^3, L^{(3D)})$’. Note again that the above properties of the associated bundle follow from Sections E.2 and E.6.2.

9.3.3 How $\lambda_{\mathcal{SOT}(d)}$ carries the $\mathbb{Z}$-action $L_{\omega,A}$

I now have all ingredients at my disposal to apply the Feres machinery. As outlined in Sections E.3 and E.6.4, this machinery provides us with a canonical left $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$-action, $L^{(1)}_{\mathcal{SOT}(d)}$, on $E^{(3)}_{\mathcal{SOT}(d)}$ and this will allow us in the present section to recover $L_{\omega,A}$. Specializing (E.41) to the present case it is shown in Section E.3.1 that the function $L^{(1)}_{\mathcal{SOT}(d)} : \mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)}) \times E^{(3)}_{\mathcal{SOT}(d)} \to E^{(3)}_{\mathcal{SOT}(d)}$ which is defined for $(\varphi, \bar{\varphi}) \in \mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$ and $\phi \in \mathbb{R}^d, R \in SO(3), S \in \mathbb{R}^3$ by

$$L^{(1)}_{\mathcal{SOT}(d)}(\varphi, \bar{\varphi}; p^{(2)}_{\mathcal{SOT}(d)}(\phi, R, S)) := p^{(2)}_{\mathcal{SOT}(d)}(\varphi(\phi, R), S) , \tag{9.35}$$

is a left $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$-action on $E^{(3)}_{\mathcal{SOT}(d)}$ whence $(E^{(3)}_{\mathcal{SOT}(d)}, L^{(1)}_{\mathcal{SOT}(d)})$ is a left $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$-space. Note that by Section E.3.1 $L^{(1)}_{\mathcal{SOT}(d)}(\varphi, \bar{\varphi}; \cdot)$ is a homeomorphism onto $E^{(3)}_{\mathcal{SOT}(d)}$. With now showing that the bundle $\xi^{(3)}_{\mathcal{SOT}(d)}$ is trivial I construct a left $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$-space which is conjugate to $(E^{(3)}_{\mathcal{SOT}(d)}, L^{(1)}_{\mathcal{SOT}(d)})$. Specializing (E.84) to the present case I define the function $r^{(1)}_{\mathcal{SOT}(d)} : \mathbb{R}^d \times SO(3) \times \mathbb{R}^3 \to \mathbb{R}^{d+3}$ for $\phi \in \mathbb{R}^d, R \in SO(3), S \in \mathbb{R}^3$ by

$$r^{(1)}_{\mathcal{SOT}(d)}(\phi, R, S) := \begin{pmatrix} \phi \\ L^{(3D)}(R; S) \end{pmatrix} = \begin{pmatrix} \phi \\ RS \end{pmatrix} \tag{9.36}$$

and conclude by Section E.6.2 that there exists a unique function $r^{(2)}_{\mathcal{SOT}(d)} : E^{(3)}_{\mathcal{SOT}(d)} \to \mathbb{R}^{d+3}$ such that

$$r^{(2)}_{\mathcal{SOT}(d)} \circ p^{(2)}_{\mathcal{SOT}(d)} = r^{(1)}_{\mathcal{SOT}(d)} \tag{9.37}$$

and that $r^{(2)}_{\mathcal{SOT}(d)}$ is a homeomorphism onto $\mathbb{R}^{d+3}$. Defining the bundle

$$\xi^{(4)}_{\mathcal{SOT}(d)} = (\mathbb{R}^{d+3}, p^{(4)}_{\mathcal{SOT}(d)}, \mathbb{R}^d) , \tag{9.38}$$

where $p^{(4)}_{\mathcal{SOT}(d)}(\phi, S) := \phi$, we know from Section E.6.2 that $(r^{(2)}_{\mathcal{SOT}(d)}, id_{\mathbb{R}^d})$ is an isomorphism from $\xi^{(3)}_{\mathcal{SOT}(d)}$ to $\xi^{(4)}_{\mathcal{SOT}(d)}$ in the category Bun of bundles. Thus the bundle $\xi^{(3)}_{\mathcal{SOT}(d)}$ is trivial. Specializing (E.102) to the present case I define the function $\tilde{L}^{(1)}_{\mathcal{SOT}(d)} : \mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)}) \times \mathbb{R}^{d+3} \to \mathbb{R}^{d+3}$ for $(\varphi, \bar{\varphi}) \in \mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$ and $\phi \in \mathbb{R}^d, S \in \mathbb{R}^3$ by

$$\tilde{L}^{(1)}_{\mathcal{SOT}(d)}(\varphi, \bar{\varphi}; \phi, S) := r^{(2)}_{\mathcal{SOT}(d)}(L^{(1)}_{\mathcal{SOT}(d)}(\varphi, \bar{\varphi}; (r^{(2)}_{\mathcal{SOT}(d)})^{-1}(\phi, S))) \ , \quad (9.39)$$

whence

$$\tilde{L}^{(1)}_{\mathcal{SOT}(d)}(\varphi, \bar{\varphi}; \cdot) \circ r^{(2)}_{\mathcal{SOT}(d)} = r^{(2)}_{\mathcal{SOT}(d)} \circ L^{(1)}_{\mathcal{SOT}(d)}(\varphi, \bar{\varphi}; \cdot) \ . \quad (9.40)$$

Since $L^{(1)}_{\mathcal{SOT}(d)}$ is a left $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$-action on $E^{(3)}_{\mathcal{SOT}(d)}$ and $r^{(2)}_{\mathcal{SOT}(d)}$ is a bijection onto $\mathbb{R}^{d+3}$ it follows from (9.40) that $\tilde{L}^{(1)}_{\mathcal{SOT}(d)}$ is a left $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$-action on $\mathbb{R}^{d+3}$ and that the left $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$-spaces $(E^{(3)}_{\mathcal{SOT}(d)}, L^{(1)}_{\mathcal{SOT}(d)})$, $(\mathbb{R}^{d+3}, \tilde{L}^{(1)}_{\mathcal{SOT}(d)})$ are conjugate. Note also that since $L^{(1)}_{\mathcal{SOT}(d)}(\varphi, \bar{\varphi}; \cdot)$ is a homeomorphism onto $E^{(3)}_{\mathcal{SOT}(d)}$ and $r^{(2)}_{\mathcal{SOT}(d)}$ is a homeomorphism onto $\mathbb{R}^{d+3}$, it follows from (9.40) that $\tilde{L}^{(1)}_{\mathcal{SOT}(d)}(\varphi, \bar{\varphi}; \cdot)$ is a homeomorphism onto $\mathbb{R}^{d+3}$. In fact we will now see that $\tilde{L}^{(1)}_{\mathcal{SOT}(d)}$ has a very simple structure. Specializing (E.104) to the present case I obtain for $(\varphi, \bar{\varphi}) \in \mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$ and $\phi \in \mathbb{R}^d, R \in SO(3), S \in \mathbb{R}^3$

$$\tilde{L}^{(1)}_{\mathcal{SOT}(d)}(\varphi, \bar{\varphi}; \phi, S) = r^{(1)}_{\mathcal{SOT}(d)}(\varphi(\phi, R), L^{(3D)}(R^{-1}; S)) = r^{(1)}_{\mathcal{SOT}(d)}(\varphi(\phi, R), R^{-1}S) \ . \quad (9.41)$$

Of course if $(\varphi, \bar{\varphi}) \in \mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$ then by (9.16) we have for $\phi \in \mathbb{R}^d, R \in SO(3)$

$$\varphi(\phi, R) = \begin{pmatrix} \bar{\varphi}(\phi) \\ f(\phi)R \end{pmatrix} , \quad (9.42)$$

where $f \in \mathcal{C}(\mathbb{R}^d, SO(3))$. Thus by (9.36),(9.41) I obtain for
$(\varphi, \bar{\varphi}) \in \mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$ and $\phi \in \mathbb{R}^d, R \in SO(3), S \in \mathbb{R}^3$ the simple formula

$$\tilde{L}^{(1)}_{\mathcal{SOT}(d)}(\varphi, \bar{\varphi}; \phi, S) = r^{(1)}_{\mathcal{SOT}(d)}(\bar{\varphi}(\phi), f(\phi)R, R^{-1}S) = \begin{pmatrix} \bar{\varphi}(\phi) \\ f(\phi)S \end{pmatrix} . \quad (9.43)$$

Note also that (9.43) confirms our assertion that $\tilde{L}^{(1)}_{\mathcal{SOT}(d)}(\varphi,\bar{\varphi};\cdot)$ is a homeomorphism onto $\mathbb{R}^{d+3}$. To bring spin-orbit theory into the picture I now pick a spin-orbit torus $(\omega,A)\in\mathcal{SOT}(d)$ and conclude from (9.21),(9.41) that for $n\in\mathbb{Z},\phi\in\mathbb{R}^d,S\in\mathbb{R}^3$

$$\begin{aligned}\tilde{L}^{(1)}_{\mathcal{SOT}(d)}(\Phi_{\omega,A}(n);\phi,S) &= \tilde{L}^{(1)}_{\mathcal{SOT}(d)}(\varphi_{\omega,A}(n;\cdot),L_{\omega,A}(n;\cdot);\phi,S)\\ &= r^{(1)}_{\mathcal{SOT}(d)}(\varphi_{\omega,A}(n;\phi,R),R^{-1}S)\ ,\end{aligned}\tag{9.44}$$

where $\varphi_{\omega,A}$ is given by (9.22). It follows from (6.9),(9.22),(9.36), (9.44) the remarkable result that for $n\in\mathbb{Z},\phi\in\mathbb{R}^d,S\in\mathbb{R}^3$

$$\begin{aligned}&\tilde{L}^{(1)}_{\mathcal{SOT}(d)}(\Phi_{\omega,A}(n);\phi,S) = r^{(1)}_{\mathcal{SOT}(d)}(\varphi_{\omega,A}(n;\phi,R),R^{-1}S)\\ &= r^{(1)}_{\mathcal{SOT}(d)}(\phi+2\pi n\omega,\Psi_{\omega,A}(n;\phi)R,R^{-1}S) = \begin{pmatrix}\phi+2\pi n\omega\\ \Psi_{\omega,A}(n;\phi)S\end{pmatrix}\\ &= L_{\omega,A}(n;\phi,S)\ .\end{aligned}\tag{9.45}$$

Having thus recovered $L_{\omega,A}$ I put this into perspective by defining the function $\hat{L}_{\omega,A}:\mathbb{Z}\times E^{(3)}_{\mathcal{SOT}(d)}\to E^{(3)}_{\mathcal{SOT}(d)}$ for $n\in\mathbb{Z},z\in E^{(3)}_{\mathcal{SOT}(d)}$ by

$$\hat{L}_{\omega,A}(n;z) := L^{(1)}_{\mathcal{SOT}(d)}(\Phi_{\omega,A}(n);z)\ .\tag{9.46}$$

Since $L^{(1)}_{\mathcal{SOT}(d)}$ is a left $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$-action on $E^{(3)}_{\mathcal{SOT}(d)}$ and since $\Phi_{\omega,A}$ is a group homomorphism into $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$ it follows from (9.46) that $\hat{L}_{\omega,A}$ is a $\mathbb{Z}$-action on $E^{(3)}_{\mathcal{SOT}(d)}$. Since $L^{(1)}_{\mathcal{SOT}(d)}(\Phi_{\omega,A}(n);\cdot)$ is continuous, it follows from (9.46) that $\hat{L}_{\omega,A}(n;\cdot)$ is continuous whence $(E^{(3)}_{\mathcal{SOT}(d)},\hat{L}_{\omega,A})$ is a topological $\mathbb{Z}$-space. Furthermore one concludes from (9.40),(9.45),(9.46) that for $n\in\mathbb{Z}$

$$\begin{aligned}L_{\omega,A}(n;\cdot)\circ r^{(2)}_{\mathcal{SOT}(d)} &= \tilde{L}^{(1)}_{\mathcal{SOT}(d)}(\Phi_{\omega,A}(n);\cdot)\circ r^{(2)}_{\mathcal{SOT}(d)} = r^{(2)}_{\mathcal{SOT}(d)}\circ L^{(1)}_{\mathcal{SOT}(d)}(\Phi_{\omega,A}(n);\cdot)\\ &= r^{(2)}_{\mathcal{SOT}(d)}\circ\hat{L}_{\omega,A}(n;\cdot)\ .\end{aligned}\tag{9.47}$$

In other words, since $r^{(2)}_{\mathcal{SOT}(d)}\in HOMEO(E^{(3)}_{\mathcal{SOT}(d)},\mathbb{R}^{d+3})$, (9.47) tells us that the topological $\mathbb{Z}$-spaces $(E^{(3)}_{\mathcal{SOT}(d)},\hat{L}_{\omega,A})$ and $(\mathbb{R}^{d+3},L_{\omega,A})$ are conjugate. This fact demonstrates how $\lambda_{\mathcal{SOT}(d)}$ carries $L_{\omega,A}$ in a canonical way and it thus establishes $\lambda_{\mathcal{SOT}(d)}$

as an appropriate principal bundle. Note also that specializing (E.40) to the present case one observes, for every integer n,

$$(\hat{L}_{\omega,A}(n;\cdot), L_\omega(n;\cdot)) \in \mathfrak{Mor}_{Bun}(\xi^{(3)}_{\mathcal{SOT}(d)})$$

and, by Remark 1 in Section E.3.1, obtain that $(\hat{L}_{\omega,A}(n;\cdot), L_\omega(n;\cdot))$ is a fibre morphism on $\xi^{(3)}_{\mathcal{SOT}(d)}$ so that (9.47) reveals a close relationship between spin-orbit trajectories and the fibre morphisms on the associated bundle.

9.3.4 How $\lambda_{\mathcal{SOT}(d)}$ carries the $\mathbb{Z}$-action $L^{(PF)}_{\omega,A}$

In the previous section I employed the canonical left $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$-action $L^{(1)}_{\mathcal{SOT}(d)}$ and in the present section I build up on that. In fact, as outlined in detail in Sections E.3.2 and E.6.4, the Feres machinery provides us with a canonical left $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$-action, $L^{(2)}_{\mathcal{SOT}(d)}$, on the set $\Gamma(\xi^{(3)})$ of cross sections of the associated bundle and it will allow me in the present section to recover $L^{(PF)}_{\omega,A}$. Specializing (E.46) to the present case it is shown in Section E.3.2 that the function $L^{(2)}_{\mathcal{SOT}(d)}$: $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})\times\Gamma(\xi^{(3)}) \to \Gamma(\xi^{(3)})$ defined for $(\varphi,\bar{\varphi}) \in \mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$ and $\sigma \in \Gamma(\xi^{(3)}), \phi \in \mathbb{R}^d$ by

$$\Big(L^{(2)}_{\mathcal{SOT}(d)}(\varphi,\bar{\varphi};\sigma)\Big)(\phi) = L^{(1)}_{\mathcal{SOT}(d)}(\varphi,\bar{\varphi};\sigma(\bar{\varphi}^{-1}(\phi)))\ , \tag{9.48}$$

is a left $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$-action on $\Gamma(\xi^{(3)})$ whence $(\Gamma(\xi^{(3)}), L^{(2)}_{\mathcal{SOT}(d)})$ is a left $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$-space. Clearly $L^{(2)}_{\mathcal{SOT}(d)}$ builds up on $L^{(1)}_{\mathcal{SOT}(d)}$. Specializing (E.107) to the present case I define the function $r^{(3)}_{\mathcal{SOT}(d)} : \Gamma(\xi^{(3)}) \to \Gamma(\xi^{(4)})$ for $\sigma \in \Gamma(\xi^{(3)})$ by

$$r^{(3)}_{\mathcal{SOT}(d)}(\sigma) := r^{(2)}_{\mathcal{SOT}(d)} \circ \sigma\ . \tag{9.49}$$

It is shown in Section E.6.4 that $r^{(3)}_{\mathcal{SOT}(d)}$ is a bijection onto $\Gamma(\xi^{(4)})$. Specializing (E.110) to the present case I define the function $\tilde{L}^{(2)}_{\mathcal{SOT}(d)} : \mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)}) \times$

$\Gamma(\xi^{(4)}) \to \Gamma(\xi^{(4)})$ for $(\varphi,\bar{\varphi}) \in \mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$ and $\sigma \in \Gamma(\xi^{(4)})$ by

$$\tilde{L}^{(2)}_{\mathcal{SOT}(d)}(\varphi,\bar{\varphi};\sigma) := r^{(3)}_{\mathcal{SOT}(d)}(L^{(2)}_{\mathcal{SOT}(d)}(\varphi,\bar{\varphi};(r^{(3)}_{\mathcal{SOT}(d)})^{-1}(\sigma)))\ , \tag{9.50}$$

whence in analogy with (E.111)

$$\tilde{L}^{(2)}_{\mathcal{SOT}(d)}(\varphi,\bar{\varphi};\cdot)\circ r^{(3)}_{\mathcal{SOT}(d)} = r^{(3)}_{\mathcal{SOT}(d)} \circ L^{(2)}_{\mathcal{SOT}(d)}(\varphi,\bar{\varphi};\cdot)\ . \tag{9.51}$$

Since $L^{(2)}_{\mathcal{SOT}(d)}$ is a left $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$-action on $\Gamma(\xi^{(3)})$ and $r^{(3)}_{\mathcal{SOT}(d)}$ is a bijection onto $\Gamma(\xi^{(4)})$ it follows from (9.51) that $\tilde{L}^{(2)}_{\mathcal{SOT}(d)}$ is a left $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$-action on $\Gamma(\xi^{(4)})$ and that the left $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$-spaces $(\Gamma(\xi^{(3)}), L^{(2)}_{\mathcal{SOT}(d)})$, $(\Gamma(\xi^{(4)}), \tilde{L}^{(2)}_{\mathcal{SOT}(d)})$ are conjugate. We will now see that $\tilde{L}^{(2)}_{\mathcal{SOT}(d)}$ has a very simple structure. In fact specializing (E.113) to the present case one obtains for $(\varphi,\bar{\varphi}) \in \mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$ and $\sigma \in \Gamma(\xi^{(4)}), \phi \in \mathbb{R}^d$

$$\Big(\tilde{L}^{(2)}_{\mathcal{SOT}(d)}(\varphi,\bar{\varphi};\sigma)\Big)(\phi) = \tilde{L}^{(1)}_{\mathcal{SOT}(d)}(\varphi,\bar{\varphi};\sigma(\bar{\varphi}^{-1}(\phi)))\ . \tag{9.52}$$

Recalling Definition C.1 we have for $\sigma \in \Gamma(\xi^{(4)})$ that $p^{(4)}_{\mathcal{SOT}(d)} \circ \sigma = id_{\mathbb{R}^d}$ whence for $\phi \in \mathbb{R}^d$ we have

$$\sigma(\phi) = \begin{pmatrix} \phi \\ \hat{\sigma}(\phi) \end{pmatrix}\ , \tag{9.53}$$

where $\hat{\sigma} \in \mathcal{C}(\mathbb{R}^d,\mathbb{R}^3)$. I thus obtain by specializing (E.114) to the present case the simple formula

$$\begin{aligned}\Big(\tilde{L}^{(2)}_{\mathcal{SOT}(d)}(\varphi,\bar{\varphi};\sigma)\Big)(\phi) &= \Big(\phi, L^{(3D)}(f(\bar{\varphi}^{-1}(\phi));\hat{\sigma}(\bar{\varphi}^{-1}(\phi)))\Big) \\ &= \begin{pmatrix} \phi \\ f(\bar{\varphi}^{-1}(\phi))\hat{\sigma}(\bar{\varphi}^{-1}(\phi)) \end{pmatrix}\ ,\end{aligned} \tag{9.54}$$

where $f \in \mathcal{C}(\mathbb{R}^d, SO(3))$ is determined from φ by (9.42). To bring spin-orbit theory into the picture I now pick a spin-orbit torus $(\omega, A) \in \mathcal{SOT}(d)$ and define the function $\hat{L}^{(PF)}_{\omega,A} : \mathbb{Z} \times \Gamma(\xi^{(4)}) \to \Gamma(\xi^{(4)})$ for $n \in \mathbb{Z}, \sigma \in \Gamma(\xi^{(4)})$ by

$$\hat{L}^{(PF)}_{\omega,A}(n;\sigma) := \tilde{L}^{(2)}_{\mathcal{SOT}(d)}(\Phi_{\omega,A}(n);\sigma)\ . \tag{9.55}$$

Since $\tilde{L}^{(2)}_{\mathcal{SOT}(d)}$ is a left $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$-action on $\Gamma(\xi^{(4)})$ and since $\Phi_{\omega,A}$ is a group homomorphism into $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$ it follows from (9.55) that $\hat{L}^{(PF)}_{\omega,A}$ is a $\mathbb{Z}$-action on $\Gamma(\xi^{(4)})$ whence $(\Gamma(\xi^{(4)}), \hat{L}^{(PF)}_{\omega,A})$ is a $\mathbb{Z}$-space. We conclude from (9.21),(9.22), (9.45),(9.52),(9.53), (9.55) that for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d, \sigma \in \Gamma(\xi^{(4)})$

$$
\begin{aligned}
&(\hat{L}^{(PF)}_{\omega,A}(n;\sigma))(\phi) = (\tilde{L}^{(2)}_{\mathcal{SOT}(d)}(\Phi_{\omega,A}(n);\sigma))(\phi) \\
&= (\tilde{L}^{(2)}_{\mathcal{SOT}(d)}(\varphi_{\omega,A}(n;\cdot), L_{\omega,A}(n;\cdot);\sigma))(\phi) \\
&= \tilde{L}^{(1)}_{\mathcal{SOT}(d)}(\varphi_{\omega,A}(n;\cdot), L_{\omega,A}(n;\cdot);\sigma(L_{\omega,A}(-n;\phi))) \\
&= \tilde{L}^{(1)}_{\mathcal{SOT}(d)}(\varphi_{\omega,A}(n;\cdot), L_{\omega,A}(n;\cdot);L_{\omega,A}(-n;\phi), \hat{\sigma}(L_{\omega,A}(-n;\phi))) \\
&= \tilde{L}^{(1)}_{\mathcal{SOT}(d)}(\varphi_{\omega,A}(n;\cdot), L_{\omega,A}(n;\cdot);\phi - 2\pi n\omega, \hat{\sigma}(\phi - 2\pi n\omega)) \\
&= \tilde{L}^{(1)}_{\mathcal{SOT}(d)}(\Phi_{\omega,A}(n);\phi - 2\pi n\omega, \hat{\sigma}(\phi - 2\pi n\omega)) \\
&= \begin{pmatrix} \phi \\ \Psi_{\omega,A}(n;\phi - 2\pi n\omega)\hat{\sigma}(\phi - 2\pi n\omega) \end{pmatrix} .
\end{aligned}
\tag{9.56}
$$

Since by (9.53) the first component of no $\sigma \in \Gamma(\xi^{(4)})$ carries any information about σ it is not a surprise that the $\mathbb{Z}$-space $(\Gamma(\xi^{(4)}), \hat{L}^{(PF)}_{\omega,A})$ is conjugate to a $\mathbb{Z}$-space which does not carry the redundant first component of (9.53). In fact I define the function $r^{(4)}_{\mathcal{SOT}(d)} : \mathcal{C}(\mathbb{R}^d, \mathbb{R}^3) \to \Gamma(\xi^{(4)}_{\mathcal{SOT}(d)})$ for $G \in \mathcal{C}(\mathbb{R}^d, \mathbb{R}^3)$ and $\phi \in \mathbb{R}^d$ by

$$
(r^{(4)}_{\mathcal{SOT}(d)}(G))(\phi) := \begin{pmatrix} \phi \\ G(\phi) \end{pmatrix} .
\tag{9.57}
$$

Note that $r^{(4)}_{\mathcal{SOT}(d)}$ is a bijection onto $\Gamma(\xi^{(4)}_{\mathcal{SOT}(d)})$. For $\sigma = r^{(4)}_{\mathcal{SOT}(d)}(G)$ we have by (9.53), (9.57) that $G = \hat{\sigma}$ whence one concludes from (9.56),(9.57) that for $G \in \mathcal{C}(\mathbb{R}^d, \mathbb{R}^3)$ and $n \in \mathbb{Z}, \phi \in \mathbb{R}^d$

$$
\begin{aligned}
&\left(\hat{L}^{(PF)}_{\omega,A}(n; r^{(4)}_{\mathcal{SOT}(d)}(G))\right)(\phi) = \begin{pmatrix} \phi \\ \Psi_{\omega,A}(n;\phi - 2\pi n\omega)G(\phi - 2\pi n\omega) \end{pmatrix} \\
&= \left(r^{(4)}_{\mathcal{SOT}(d)}\left(\Psi_{\omega,A}(n;\cdot - 2\pi n\omega)G(\cdot - 2\pi n\omega)\right)\right)(\phi) ,
\end{aligned}
$$

so that by (9.50),(9.55) for $G \in \mathcal{C}(\mathbb{R}^d, \mathbb{R}^3)$ and $n \in \mathbb{Z}$

$$\begin{aligned} &r^{(4)}_{\mathcal{SOT}(d)}\left(\Psi_{\omega,A}(n;\cdot - 2\pi n\omega)G(\cdot - 2\pi n\omega)\right) = \hat{L}^{(PF)}_{\omega,A}(n; r^{(4)}_{\mathcal{SOT}(d)}(G)) \\ &= \tilde{L}^{(2)}_{\mathcal{SOT}(d)}(\Phi_{\omega,A}(n); r^{(4)}_{\mathcal{SOT}(d)}(G)) \\ &= r^{(3)}_{\mathcal{SOT}(d)}\left(L^{(2)}_{\mathcal{SOT}(d)}(\Phi_{\omega,A}(n); (r^{(3)}_{\mathcal{SOT}(d)})^{-1}(r^{(4)}_{\mathcal{SOT}(d)}(G)))\right) . \end{aligned} \tag{9.58}$$

Defining the function $r^{(5)}_{\mathcal{SOT}(d)} : \mathcal{C}(\mathbb{R}^d, \mathbb{R}^3) \to \Gamma(\xi^{(3)}_{\mathcal{SOT}(d)})$ by $r^{(5)}_{\mathcal{SOT}(d)} := (r^{(3)}_{\mathcal{SOT}(d)})^{-1} \circ r^{(4)}_{\mathcal{SOT}(d)}$ one observes that $r^{(5)}_{\mathcal{SOT}(d)}$ is a bijection onto $\Gamma(\xi^{(3)}_{\mathcal{SOT}(d)})$ and that by (9.58) for $G \in \mathcal{C}(\mathbb{R}^d, \mathbb{R}^3)$ and $n \in \mathbb{Z}$

$$\begin{aligned} &\Psi_{\omega,A}(n;\cdot - 2\pi n\omega)G(\cdot - 2\pi n\omega) \\ &= (r^{(4)}_{\mathcal{SOT}(d)})^{-1}\left(r^{(3)}_{\mathcal{SOT}(d)}\left(L^{(2)}_{\mathcal{SOT}(d)}(\Phi_{\omega,A}(n); (r^{(3)}_{\mathcal{SOT}(d)})^{-1}(r^{(4)}_{\mathcal{SOT}(d)}(G)))\right)\right) \\ &= (r^{(5)}_{\mathcal{SOT}(d)})^{-1}\left(L^{(2)}_{\mathcal{SOT}(d)}(\Phi_{\omega,A}(n); r^{(5)}_{\mathcal{SOT}(d)}(G))\right) . \end{aligned} \tag{9.59}$$

By (6.20) we have for $G \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$ and $n \in \mathbb{Z}$ that $\Psi_{\omega,A}(n;\cdot - 2\pi n\omega)G(\cdot - 2\pi n\omega) = L^{(PF)}_{\omega,A}(n; G)$ whence by (9.59) we obtain the remarkable result that for $G \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$ and $n \in \mathbb{Z}$

$$L^{(PF)}_{\omega,A}(n; G) = (r^{(5)}_{\mathcal{SOT}(d)})^{-1}\left(L^{(2)}_{\mathcal{SOT}(d)}(\Phi_{\omega,A}(n); r^{(5)}_{\mathcal{SOT}(d)}(G))\right) , \tag{9.60}$$

which tells us how $\lambda_{\mathcal{SOT}(d)}$ carries $L^{(PF)}_{\omega,A}$ in a canonical way. In particular since $L^{(2)}_{\mathcal{SOT}(d)}$ acts on $\Gamma(\xi^{(3)}_{\mathcal{SOT}(d)})$ we see in (9.60) a close relationship between polarization fields and cross sections of the associated bundle.

9.3.5 Reducing the structure group $SO(3)$

The most important objectives of the Feres machinery are the reduction theorems and the rigidity theorems [Fe] and in this section I will be concerned with the former (the latter are beyond the scope of this work). The reduction theorems deal, in our context, with the reduction of the structure group $SO(3)$ of $\lambda_{\mathcal{SOT}(d)}$ to a closed subgroup of $SO(3)$ and its impact on the dynamics, i.e., on $\mathcal{SOT}(d)$. This leads to Theorem 9.5.

Let H be a closed topological subgroup of $SO(3)$. Recalling Section E.5, a principal H-bundle, $\hat{\lambda}$, is called a 'H-reduction of $\lambda_{\mathcal{SOT}(d)}$' if the total space of $\hat{\lambda}$ is a closed subset $\hat{E}$ of the total space $\mathbb{R}^d \times SO(3)$ of $\lambda_{\mathcal{SOT}(d)}$ and if $\hat{\lambda}$ has the form

$$\hat{\lambda} = (\hat{E}, p^{(1)}_{\mathcal{SOT}(d)}\Big|\hat{E}, \mathbb{R}^d, R^{(1)}_{\mathcal{SOT}(d)}\Big|(H \times \hat{E})) . \tag{9.61}$$

Note that two H-reductions of $\lambda_{\mathcal{SOT}(d)}$ are different iff their total spaces are different. The set of all H-reductions of $\lambda_{\mathcal{SOT}(d)}$ is denoted by $RED_H(\lambda_{\mathcal{SOT}(d)})$. The condition that $\hat{\lambda}$ is a principal H-bundle is a strong restriction on the possible forms of $\hat{E}$ and the following proposition gives an account of this.

Proposition 9.3 *Let H be a closed topological subgroup of $SO(3)$.*
If $f \in \mathcal{C}(\mathbb{R}^d, SO(3)/H)$ then $\check{E}_{f,H}$, defined by

$$\check{E}_{f,H} := \{(\phi, R) \in \mathbb{R}^d \times SO(3) : f(\phi) = RH\} , \tag{9.62}$$

is a closed subspace of $\mathbb{R}^d \times SO(3)$ where $RH := \{RR' : R' \in H\}$ and where the space $SO(3)/H$ is defined in Section E.5. Moreover, if $f \in \mathcal{C}(\mathbb{R}^d, SO(3)/H)$ then the quadruple:

$$\widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)},H}(f) := (\check{E}_{f,H}, p^{(1)}_{\mathcal{SOT}(d)}\Big|\check{E}_{f,H}, \mathbb{R}^d, R^{(1)}_{\mathcal{SOT}(d)}\Big|(H \times \check{E}_{f,H})) , \tag{9.63}$$

is a H-reduction of $\lambda_{\mathcal{SOT}(d)}$. Furthermore $\widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)},H}$ is a bijection from $\mathcal{C}(\mathbb{R}^d, SO(3)/H)$ onto $RED_H(\lambda_{\mathcal{SOT}(d)})$. In particular, every H-reduction of $\lambda_{\mathcal{SOT}(d)}$ is of the form (9.63).

Proof of Proposition 9.3: See Section F.28. □

While Proposition 9.3 states a one-one correspondence between $RED_H(\lambda_{\mathcal{SOT}(d)})$ and $\mathcal{C}(\mathbb{R}^d, SO(3)/H)$ there is also a one-one correspondence between $RED_H(\lambda_{\mathcal{SOT}(d)})$ and the cross sections of the associated bundle $\lambda_{\mathcal{SOT}(d)}[SO(3)/H, L_{SO(3)/H}]$ where the left $SO(3)$-action $L_{SO(3)/H}$ is defined by (E.62). In fact it follows from Theorem E.3b in Section E.6.6 that the function $MAIN_{\lambda_{\mathcal{SOT}(d)},H} : \Gamma(\lambda_{\mathcal{SOT}(d)}[SO(3)/H, L_{SO(3)/H}]) \to RED_H(\lambda_{\mathcal{SOT}(d)})$, which is defined by (E.162), is a bijection onto $RED_H(\lambda_{\mathcal{SOT}(d)})$. However I here do not need $MAIN_{\lambda_{\mathcal{SOT}(d)},H}$ but rather focus on $\widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)},H}$.

The following proposition builds up on the fact that $SO_3(2)$ is a closed topological subgroup of $SO(3)$ (see Definition C.2).

Proposition 9.4 *a) The function $F : SO(3)/SO_3(2) \to \mathbb{S}^2$, defined for $R \in SO(3)$ by*

$$F(RSO_3(2)) := L^{(3D)}(R; e^3) = Re^3 \;, \tag{9.64}$$

is a homeomorphism onto $\mathbb{S}^2$ where $RSO_3(2)) := \{RR' : R' \in SO_3(2))\}$ and where $L^{(3D)}$ is defined by (9.31). Moreover for $S \in \mathbb{S}^2, R, R' \in SO(3)$

$$F(L_{SO(3)/SO_3(2)}(R'; RSO_3(2))) = L^{(3D)}(R'; F(RSO_3(2))) \;, \tag{9.65}$$

$$F^{-1}(L^{(3D)}(R; S)) = L_{SO(3)/SO_3(2)}(R; F^{-1}(S)) \;, \tag{9.66}$$

where $L_{SO(3)/SO_3(2)}$ is defined by (E.62).

b) For every $f \in \mathcal{C}(\mathbb{R}^d, SO(3)/SO_3(2))$ we have

$$\check{E}_{f,SO_3(2)} = \{(\phi, R) \in \mathbb{R}^d \times SO(3) : (F \circ f)(\phi) = Re^3\} \;. \tag{9.67}$$

The function $\overline{MAIN}_{\lambda_{\mathcal{SOT}(d)},SO_3(2)} : \mathcal{C}(\mathbb{R}^d, \mathbb{S}^2) \to RED_{SO_3(2)}(\lambda_{\mathcal{SOT}(d)})$, defined, for $G \in \mathcal{C}(\mathbb{R}^d, \mathbb{S}^2)$, by

$$\overline{MAIN}_{\lambda_{\mathcal{SOT}(d)},SO_3(2)}(G) := \widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)},SO_3(2)}(F^{-1} \circ G) \;, \tag{9.68}$$

is a bijection onto $RED_{SO_3(2)}(\lambda_{\mathcal{SOT}(d)})$ where $\widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)},SO_3(2)}$ is defined by (9.63).

Proof of Proposition 9.4: See Section F.29. □

We recall from Proposition 9.3 that $\widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)},H}$ is a bijection from $\mathcal{C}(\mathbb{R}^d, SO(3)/H)$ onto $RED_H(\lambda_{\mathcal{SOT}(d)})$ whence every H-reduction of $\lambda_{\mathcal{SOT}(d)}$ is of the form $\widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)},H}(f)$. I now define $RED_{H,per}(\lambda_{\mathcal{SOT}(d)})$ by

$$RED_{H,per}(\lambda_{\mathcal{SOT}(d)}) := \{\widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)},H}(f) : f \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3)/H)\} \,. \quad (9.69)$$

If $(\varphi, \bar{\varphi}) \in \mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$ and if $f \in \mathcal{C}(\mathbb{R}^d, SO(3)/H)$ then, recalling Section E.5, $\widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)},H}(f)$ is called 'invariant under $(\varphi, \bar{\varphi})$' if the total space, $\check{E}_{f,H}$, of $\widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)},H}(f)$ is invariant under φ, i.e., $\varphi(\check{E}_{f,H}) = \check{E}_{f,H}$ where $\check{E}_{f,H}$ is defined by (9.62). Furthermore if $(\omega, A) \in \mathcal{SOT}(d)$ and $f \in \mathcal{C}(\mathbb{R}^d, SO(3)/H)$ then $\widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)},H}(f)$ is called 'invariant under the group $\Phi_{\omega,A}(\mathbb{Z})$' if it is invariant under each $\Phi_{\omega,A}(n)$. Recalling from Section 9.3.1 that $\varphi_{\omega,A}$ is a $\mathbb{Z}$-action on $\mathbb{R}^d \times SO(3)$, one here observes that the restriction of $\varphi_{\omega,A}$ to $\mathbb{Z} \times \check{E}_{f,H}$ is a $\mathbb{Z}$-action if $\widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)},H}(f)$ is invariant under $\Phi_{\omega,A}(\mathbb{Z})$.

Of course, by the special structure of the group $\mathbb{Z}$ and since $\Phi_{\omega,A}$ is a group homomorphism, $\widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)},H}(f)$ is invariant under the group $\Phi_{\omega,A}(\mathbb{Z})$ iff it is invariant under $\Phi_{\omega,A}(1)$, i.e., iff $\varphi_{\omega,A}(1; \check{E}_{f,H}) = \check{E}_{f,H}$ where $\varphi_{\omega,A}$ is defined by (9.22).

Part a) of the following theorem is a special case of Zimmer's reduction theorem [Fe].

Theorem 9.5 *Let $(\omega, A) \in \mathcal{SOT}(d)$. Then the following hold.*

a) Let H be a closed topological subgroup of $SO(3)$ and let $f \in \mathcal{C}(\mathbb{R}^d, SO(3)/H)$. Then the H-reduction $\widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)},H}(f)$ of $\lambda_{\mathcal{SOT}(d)}$ is invariant under the group $\Phi_{\omega,A}(\mathbb{Z})$ iff, for every $\phi \in \mathbb{R}^d$,

$$f(L_\omega(1; \phi)) = L_{SO(3)/H}(A(\phi); f(\phi)) \,, \quad (9.70)$$

where $L_{SO(3)/H}$ is defined by (E.62).

b) Let $G \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$. Then the $SO_3(2)$-reduction $\widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)},SO_3(2)}(F^{-1} \circ G)$ of $\lambda_{\mathcal{SOT}(d)}$ is invariant under $\Phi_{\omega,A}(\mathbb{Z})$ iff $\mathcal{S}_G$ is an invariant spin field of (ω, A). In particular (ω, A) has an invariant spin field iff $\lambda_{\mathcal{SOT}(d)}$ has a 2π-periodic $SO_3(2)$-reduction which is invariant under $\Phi_{\omega,A}(\mathbb{Z})$.

Proof of Theorem 9.5: See Section F.30. □

Note by (9.63),(9.67) and Theorem 9.5b that if $(\omega, A) \in \mathcal{SOT}(d)$ and $\mathcal{S}_G$ is an invariant spin field of (ω, A) then the total space of the invariant $SO_3(2)$-reduction $\widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)},SO_3(2)}(F^{-1} \circ G)$ of $\lambda_{\mathcal{SOT}(d)}$ has the form

$$\check{E}_{F^{-1}\circ G,SO_3(2)} = \{(\phi, R) \in \mathbb{R}^d \times SO(3) : G(\phi) = Re^3\} . \tag{9.71}$$

Thus (9.71) represents the invariant spin field $\mathcal{S}_G$ by a subset of $\mathbb{R}^d \times SO(3)$, i.e., we have a 'geometrization' of invariant spin fields. Another aspect of Theorem 9.5b is that the existence of an invariant spin field of (ω, A) is a symmetry property of (ω, A).

One more aspect of Theorem 9.5 is the following. While, by Theorem 9.5b, invariant spin fields are linked to 2π-periodic invariant $SO_3(2)$-reductions of $\lambda_{\mathcal{SOT}(d)}$, it is easy to show, by Theorem 9.5a, that spin-orbit resonances of first kind are linked to 2π-periodic invariant H-reductions of $\lambda_{\mathcal{SOT}(d)}$ where H is the trivial subgroup of $SO(3)$. Thus the existence of spin tunes of first kind of (ω, A) is a symmetry property of (ω, A).

9.3.6 Closing remarks on $\lambda_{\mathcal{SOT}(d)}$

I have now completed my coverage of principal bundles since my only objective in this regard was to show how the principal $SO(3)$-bundle $\lambda_{\mathcal{SOT}(d)}$ underlies the theory of $\mathcal{SOT}(d)$.

Following the Feres machinery one could extend my study. However this would go beyond the scope of the present work. So I just mention four points. Firstly, by using the linearity of $L^{(3D)}(R;S)$ in S, one can extend the structure group from $SO(3)$ to $GL(3)$ and study, by a 'prolongation' of the principal $SO(3)$-bundle $\lambda_{\mathcal{SOT}(d)}$ to a principal $GL(3)$-bundle, the $\mathbb{Z}$-actions $L_{\omega,A}$ and $L^{(PF)}_{\omega,A}$ in terms of vector bundle techniques ($GL(n)$ denotes the group of real nonsingular $n \times n$–matrices). Secondly, one can go beyond Theorem 9.5 to study invariant H-reductions of $\lambda_{\mathcal{SOT}(d)}$ in a more general way by asking what closed subgroups H of $SO(3)$ allow for 2π-periodic H-reductions which are invariant under a given spin-orbit torus in $\mathcal{SOT}(d)$. For such a study the 'algebraic hull' is an important tool which was introduced by Zimmer in the 1980's. Thirdly one can apply rigidity theorems which allow to discuss properties which are stable (='rigid') under the extension of the group $\mathbb{Z}$ of the evolution variable. Fourthly, the choice of $\lambda_{\mathcal{SOT}(d)}$ is not unique. For example an alternative choice is to employ $\mathbb{T}^d$ rather than $\mathbb{R}^d$ in the definition of the total resp. base space of the principal $SO(3)$-bundle. In fact this alternative choice is very convenient when one would go deeper into the matter of spin-orbit tori but for the purposes of the present work the choice of $\lambda_{\mathcal{SOT}(d)}$ is sufficient and leads to analogous results as if one would use $\mathbb{T}^d$ instead of $\mathbb{R}^d$.

Chapter 10

Summary of spin-orbit tori and outlook

As pointed out in the Introduction, the second part of this thesis studies spin-orbit tori in terms of the map formalism equations of motion (6.1),(6.2) which plays a central role in the mathematical study of polarized beams in storage rings.

From a technical point of view a distinguishing feature of the present work is to formulate all concepts and properties in mathematical terms. Accordingly the mathematical notion of spin-orbit torus is introduced and a number of properties of spin-orbit tori are derived. Most of my definitions that are related to spin-orbit tori are distilled from established concepts in Polarized Beam Physics which are then translated into the language of Mathematics. The subsets $\mathcal{CB}(d,\omega) \subset \mathcal{ACB}(d,\omega) \subset \mathcal{WCB}(d,\omega)$ of the set $\mathcal{SOT}$ of spin-orbit tori have been introduced and discussed in some detail. I noted that spin-orbit tori (ω, A) of interest are almost coboundaries, i.e., are in $\mathcal{ACB}(d,\omega)$ and they have the form $A(\phi) = T^T(\phi + 2\pi\omega)\exp(\mathcal{J}2\pi\nu)T(\phi)$.

To my knowledge the results of the thesis are either new (e.g., Theorem 9.5b about the impact of Principal Bundle Theory on invariant spin fields) or were never formulated in mathematically precise terms whence were never rigorously proved before (e.g., Corollary 8.12 aka the SPRINT Theorem). Note that some results (e.g., Yokoya's uniqueness theorem 7.13) were rigorously proved before for the flow formalism (see [BEH04]).

I have gathered quite a bit of insight into the invariant spin field (as well as into the spin tune) which is central for Polarized Beam Physics, as explained in Section 7.6. From Section 6.3 we know that an invariant spin field is tied with the equation $G(\phi) = A(\phi - 2\pi\omega)G(\phi - 2\pi\omega)$. I formulated the ISF conjecture which states that if (ω, A) is off orbital resonance, i.e., $(1, \omega)$ nonresonant, then an invariant spin field exists. Theorem 7.9 states that if (ω, A) is a weak coboundary, then an invariant spin field exists. Theorem 7.10a states that if $\mathcal{S}_G$ is an invariant spin field and if G is 2π-nullhomotopic then (ω, A) is a weak coboundary. Theorem 8.17 states that there are spin-orbit tori which have an invariant spin field and which are not weak coboundaries. Finally Theorem 9.5b shows that the existence of an invariant spin field of (ω, A) is a symmetry property of (ω, A). In fact Theorem 9.5b ties the existence of an invariant spin field to an $SO_3(2)$-reduction of the principal $SO(3)$-bundle $\lambda_{\mathcal{SOT}(d)}$.

It is also worthwile to mention that the machinery of Chapter 9 can be applied to any linear n-dimensional nonautonomous ODE $\dot{y} = Y(t)y$ since the standard procedure of making it autonomous, encodes the ODE into a $GL(n)$-cocycle over the time translations whence encodes it into a principal $GL(n)$-bundle with base space $\mathbb{R}$. This will be addressed in a future publication of the author.

For a detailed outline of this work see Section 5.2. Avenues for further work are of course plentiful. In addition to those mentioned in Section 5.3, one topic of further studies could be the continuation of the work of Section 9.3. In fact, as outlined in Section 9.3.6, there are further applications of the principal $SO(3)$-bundle $\lambda_{\mathcal{SOT}(d)}$ in waiting which will shed further light into the matter of spin-orbit tori.

Appendices

Appendix A

A.1 Details on the self field

Maxwell's equations (3.10) imply

$$\Box \mathbf{E} = \mathbf{S}^{el} , \qquad \Box \mathbf{B} = \mathbf{S}^{mag} , \tag{A.1}$$

where

$$\mathbf{S}^{el} = (S_Z^{el}, S_X^{el}, S_Y^{el})^T := Z_0(c\nabla_{\bar{\mathbf{R}}}\bar{\rho} + \partial_u \bar{\mathbf{J}}) = Z_0 \begin{pmatrix} c\partial_Z \bar{\rho} + \partial_u \bar{J}_Z \\ c\partial_X \bar{\rho} + \partial_u \bar{J}_X \\ c\partial_Y \bar{\rho} + \partial_u \bar{J}_Y \end{pmatrix} , \tag{A.2}$$

$$\mathbf{S}^{mag} = (S_Z^{mag}, S_X^{mag}, S_Y^{mag})^T := -\mu_0 \nabla_{\bar{\mathbf{R}}} \times \bar{\mathbf{J}} = -\frac{Z_0}{c} \begin{pmatrix} \partial_X \bar{J}_Y - \partial_Y \bar{J}_X \\ \partial_Y \bar{J}_Z - \partial_Z \bar{J}_Y \\ \partial_Z \bar{J}_X - \partial_X \bar{J}_Z \end{pmatrix} . \tag{A.3}$$

In the nonshielding scenario we obtain from (3.16),(A.1) that

$$\begin{aligned} \mathbf{E}(\bar{\mathbf{R}}, u) = \mathbf{E}^{nsh}(\bar{\mathbf{R}}, u) &:= -\int_{\mathbb{R}^4} d\bar{\mathbf{R}}' du' G(\bar{\mathbf{R}} - \bar{\mathbf{R}}', u - u') 1_{[u_0,\infty)}(u') \mathbf{S}^{el}(\bar{\mathbf{R}}', u') , \\ \mathbf{B}(\bar{\mathbf{R}}, u) = \mathbf{B}^{nsh}(\bar{\mathbf{R}}, u) &:= -\int_{\mathbb{R}^4} d\bar{\mathbf{R}}' du' G(\bar{\mathbf{R}} - \bar{\mathbf{R}}', u - u') 1_{[u_0,\infty)}(u') \mathbf{S}^{mag}(\bar{\mathbf{R}}', u') , \end{aligned} \tag{A.4}$$

where

$$G(\bar{\mathbf{R}}, u) := \frac{1}{4\pi|\bar{\mathbf{R}}|}\delta(u - |\bar{\mathbf{R}}|) \; . \tag{A.5}$$

In the shielding scenario we obtain from (3.16),(3.22),(3.23), (A.1) that for $Y \in [-g, g]$

$$\begin{aligned}
E_Z(\bar{\mathbf{R}}, u) = E_Z^{sh}(\bar{\mathbf{R}}, u) &:= -\int_{\mathbb{R}^4} d\bar{\mathbf{R}}' du' G_D(\bar{\mathbf{R}}, u, \bar{\mathbf{R}}', u') 1_{[u_0,\infty)}(u') S_Z^{el}(\bar{\mathbf{R}}', u') \; , \\
E_X(\bar{\mathbf{R}}, u) = E_X^{sh}(\bar{\mathbf{R}}, u) &:= -\int_{\mathbb{R}^4} d\bar{\mathbf{R}}' du' G_D(\bar{\mathbf{R}}, u, \bar{\mathbf{R}}', u') 1_{[u_0,\infty)}(u') S_X^{el}(\bar{\mathbf{R}}', u') \; , \\
E_Y(\bar{\mathbf{R}}, u) = E_Y^{sh}(\bar{\mathbf{R}}, u) &:= -\int_{\mathbb{R}^4} d\bar{\mathbf{R}}' du' G_N(\bar{\mathbf{R}}, u, \bar{\mathbf{R}}', u') 1_{[u_0,\infty)}(u') S_Y^{el}(\bar{\mathbf{R}}', u') \; , \\
B_Z(\bar{\mathbf{R}}, u) = B_Z^{sh}(\bar{\mathbf{R}}, u) &:= -\int_{\mathbb{R}^4} d\bar{\mathbf{R}}' du' G_N(\bar{\mathbf{R}}, u, \bar{\mathbf{R}}', u') 1_{[u_0,\infty)}(u') S_Z^{mag}(\bar{\mathbf{R}}', u') \; , \\
B_X(\bar{\mathbf{R}}, u) = B_X^{sh}(\bar{\mathbf{R}}, u) &:= -\int_{\mathbb{R}^4} d\bar{\mathbf{R}}' du' G_N(\bar{\mathbf{R}}, u, \bar{\mathbf{R}}', u') 1_{[u_0,\infty)}(u') S_X^{mag}(\bar{\mathbf{R}}', u') \; , \\
B_Y(\bar{\mathbf{R}}, u) = B_Y^{sh}(\bar{\mathbf{R}}, u) &:= -\int_{\mathbb{R}^4} d\bar{\mathbf{R}}' du' G_D(\bar{\mathbf{R}}, u, \bar{\mathbf{R}}', u') 1_{[u_0,\infty)}(u') S_Y^{mag}(\bar{\mathbf{R}}', u') \; ,
\end{aligned} \tag{A.6}$$

where

$$\begin{aligned}
G_D(\bar{\mathbf{R}}, u, \bar{\mathbf{R}}', u') &:= \sum_{k\in\mathbb{Z}} (-1)^k G(Z - Z', X - X', Y - (-1)^k Y' - 2kg, u - u') \; , \\
G_N(\bar{\mathbf{R}}, u, \bar{\mathbf{R}}', u') &:= \sum_{k\in\mathbb{Z}} G(Z - Z', X - X', Y - (-1)^k Y' - 2kg, u - u') \; .
\end{aligned} \tag{A.7}$$

Note that G is a fundamental solution of the wave equation without shielding, i.e.,

$$\Box G(\bar{\mathbf{R}}, u) = -\delta(Z)\delta(X)\delta(Y)\delta(u) \; . \tag{A.8}$$

See, for example, [Ja]. Note also that one can construct G_D, G_N by the method of image charges.

From now one we confine in this section to the sheet beam whence we have by

(3.40), (A.2),(A.3)

$$
\begin{aligned}
S_Z^{el}(\bar{\mathbf{R}},u) &= Z_0[c\partial_Z\bar{\rho}(\bar{\mathbf{R}},u)+\partial_u\bar{J}_Z(\bar{\mathbf{R}},u)] \\
&= Z_0\delta(Y)[c\partial_Z\rho_L(\mathbf{R},u)+\partial_u J_{L,Z}(\mathbf{R},u)] =: \delta(Y)S_{L,Z}^{el}(\mathbf{R},u)\ , \\
S_X^{el}(\bar{\mathbf{R}},u) &= Z_0[c\partial_X\bar{\rho}(\bar{\mathbf{R}},u)+\partial_u\bar{J}_X(\bar{\mathbf{R}},u)] \\
&= Z_0\delta(Y)[c\partial_X\rho_L(\mathbf{R},u)+\partial_u J_{L,X}(\mathbf{R},u)] =: \delta(Y)S_{L,X}^{el}(\mathbf{R},u)\ , \\
S_Y^{el}(\bar{\mathbf{R}},u) &= Z_0[c\partial_Y\bar{\rho}(\bar{\mathbf{R}},u)+\partial_u\bar{J}_Y(\bar{\mathbf{R}},u)] \\
&= Z_0c\rho_L(\mathbf{R},u)\frac{d}{dY}\delta(Y) =: S_{L,Y}^{el}(\mathbf{R},u)\frac{d}{dY}\delta(Y)\ , \\
S_Z^{mag}(\bar{\mathbf{R}},u) &= \frac{Z_0}{c}\partial_Y\bar{J}_X(\bar{\mathbf{R}},u) = \frac{Z_0}{c}J_{L,X}(\mathbf{R},u)\frac{d}{dY}\delta(Y) \\
&=: S_{L,Z}^{mag}(\mathbf{R},u)\frac{d}{dY}\delta(Y)\ , \\
S_X^{mag}(\bar{\mathbf{R}},u) &= -\frac{Z_0}{c}\partial_Y\bar{J}_Z(\bar{\mathbf{R}},u) = -\frac{Z_0}{c}J_{L,Z}(\mathbf{R},u)\frac{d}{dY}\delta(Y) \\
&=: S_{L,X}^{mag}(\mathbf{R},u)\frac{d}{dY}\delta(Y)\ , \\
S_Y^{mag}(\bar{\mathbf{R}},u) &= -\frac{Z_0}{c}(\partial_Z\bar{J}_X(\bar{\mathbf{R}},u)-\partial_X\bar{J}_Z(\bar{\mathbf{R}},u)) \\
&= -\frac{Z_0}{c}\delta(Y)(\partial_Z J_{L,X}(\mathbf{R},u)-\partial_X J_{L,Z}(\mathbf{R},u)) =: \delta(Y)S_{L,Y}^{mag}(\mathbf{R},u)\ .
\end{aligned}
\tag{A.9}
$$

Note that by (3.37), (A.9)

$$
\mathbf{S} = (S_{L,Z}^{el}, S_{L,X}^{el}, S_{L,Y}^{mag})^T\ . \tag{A.10}
$$

In the nonshielding scenario we obtain from (A.4),(A.5),(A.9) that

$$
\begin{aligned}
E_Z^{nsh}(\bar{\mathbf{R}},u) &= -\int_{\mathbb{R}^4} d\bar{\mathbf{R}}' du' G(\mathbf{R}-\mathbf{R}', Y-Y', u-u') 1_{[u_0,\infty)}(u') S_Z^{el}(\bar{\mathbf{R}}', u') \\
&= -\int_{\mathbb{R}^4} d\bar{\mathbf{R}}' du' G(\mathbf{R}-\mathbf{R}', Y-Y', u-u') 1_{[u_0,\infty)}(u') \delta(Y') S_{L,Z}^{el}(\mathbf{R}', u') \\
&= -\int_{\mathbb{R}^3} d\mathbf{R}' du' 1_{[u_0,\infty)}(u') G(\mathbf{R}-\mathbf{R}', Y, u-u') S_{L,Z}^{el}(\mathbf{R}', u') \\
&= -\frac{1}{4\pi}\int_{\mathbb{R}^3} d\mathbf{R}' du' 1_{[u_0,\infty)}(u') \frac{\delta(u-u'-\sqrt{|\mathbf{R}-\mathbf{R}'|^2+Y^2})}{\sqrt{|\mathbf{R}-\mathbf{R}'|^2+Y^2}} S_{L,Z}^{el}(\mathbf{R}', u') \\
&= -\frac{1}{4\pi}\int_{\mathbb{R}^2} d\mathbf{R}' 1_{[u_0,\infty)}(u-\sqrt{|\mathbf{R}-\mathbf{R}'|^2+Y^2}) \frac{S_{L,Z}^{el}(\mathbf{R}', u-\sqrt{|\mathbf{R}-\mathbf{R}'|^2+Y^2})}{\sqrt{|\mathbf{R}-\mathbf{R}'|^2+Y^2}} \ ,
\end{aligned}
\tag{A.11}
$$

and analogously

$$
\begin{aligned}
&E_X^{nsh}(\bar{\mathbf{R}},u) \\
&\quad = -\frac{1}{4\pi}\int_{\mathbb{R}^2} d\mathbf{R}' 1_{[u_0,\infty)}(u-\sqrt{|\mathbf{R}-\mathbf{R}'|^2+Y^2}) \frac{S_{L,X}^{el}(\mathbf{R}', u-\sqrt{|\mathbf{R}-\mathbf{R}'|^2+Y^2})}{\sqrt{|\mathbf{R}-\mathbf{R}'|^2+Y^2}} \ , \\
&B_Y^{nsh}(\bar{\mathbf{R}},u) \\
&\quad = -\frac{1}{4\pi}\int_{\mathbb{R}^2} d\mathbf{R}' 1_{[u_0,\infty)}(u-\sqrt{|\mathbf{R}-\mathbf{R}'|^2+Y^2}) \frac{S_{L,Y}^{mag}(\mathbf{R}', u-\sqrt{|\mathbf{R}-\mathbf{R}'|^2+Y^2})}{\sqrt{|\mathbf{R}-\mathbf{R}'|^2+Y^2}} \ .
\end{aligned}
\tag{A.12}
$$

Note by (A.11),(A.12) that $E_Z^{nsh}(\bar{\mathbf{R}},u), E_X^{nsh}(\bar{\mathbf{R}},u), B_Y^{nsh}(\bar{\mathbf{R}},u)$ are even in Y. Abbreviating

$$
\mathcal{F}^{nsh} := (E_Z^{nsh}, E_X^{nsh}, B_Y^{nsh})^T \ , \qquad \mathcal{F}_L^{nsh}(\mathbf{R},u) := \mathcal{F}^{nsh}(\mathbf{R},0,u) \ , \tag{A.13}
$$

we obtain from (A.10),(A.11),(A.12) that

$$\begin{aligned}
\mathcal{F}^{nsh}(\mathbf{R},Y,u) &= (E_Z^{nsh}(\mathbf{R},Y,u), E_X^{nsh}(\mathbf{R},Y,u), B_Y^{nsh}(\mathbf{R},Y,u))^T \\
&= -\frac{1}{4\pi}\int_{\mathbb{R}^2} d\mathbf{R}' \frac{1_{[u_0,\infty)}(u-\sqrt{|\mathbf{R}-\mathbf{R}'|^2+Y^2})}{\sqrt{|\mathbf{R}-\mathbf{R}'|^2+Y^2}} \\
&\quad \cdot \begin{pmatrix} S_{L,Z}^{el}(\mathbf{R}', u-\sqrt{|\mathbf{R}-\mathbf{R}'|^2+Y^2}) \\ S_{L,X}^{el}(\mathbf{R}', u-\sqrt{|\mathbf{R}-\mathbf{R}'|^2+Y^2}) \\ S_{L,Y}^{mag}(\mathbf{R}', u-\sqrt{|\mathbf{R}-\mathbf{R}'|^2+Y^2}) \end{pmatrix} \\
&= -\frac{1}{4\pi}\int_{\mathbb{R}^2} d\mathbf{R}' 1_{[u_0,\infty)}(u-\sqrt{|\mathbf{R}-\mathbf{R}'|^2+Y^2}) \frac{\mathbf{S}(\mathbf{R}', u-\sqrt{|\mathbf{R}-\mathbf{R}'|^2+Y^2})}{\sqrt{|\mathbf{R}-\mathbf{R}'|^2+Y^2}} \ ,
\end{aligned} \tag{A.14}$$

whence, by (A.13),

$$\mathcal{F}_L^{nsh}(\mathbf{R},u) = -\frac{1}{4\pi}\int_{\mathbb{R}^2} d\mathbf{R}' 1_{[u_0,\infty)}(u-|\mathbf{R}-\mathbf{R}'|)\frac{\mathbf{S}(\mathbf{R}', u-|\mathbf{R}-\mathbf{R}'|)}{|\mathbf{R}-\mathbf{R}'|} \ . \tag{A.15}$$

Also in the nonshielding scenario we obtain from (A.4),(A.9) that

$$\begin{aligned}
E_Y^{nsh}(\bar{\mathbf{R}},u) &= -\int_{\mathbb{R}^4} d\bar{\mathbf{R}}' du' G(\mathbf{R}-\mathbf{R}', Y-Y', u-u') 1_{[u_0,\infty)}(u') S_Y^{el}(\bar{\mathbf{R}}', u') \\
&= -\int_{\mathbb{R}^4} d\bar{\mathbf{R}}' du' G(\mathbf{R}-\mathbf{R}', Y-Y', u-u') 1_{[u_0,\infty)}(u') S_{L,Y}^{el}(\mathbf{R}', u') \frac{d}{dY'}\delta(Y') \ , \\
B_Z^{nsh}(\bar{\mathbf{R}},u) &= -\int_{\mathbb{R}^4} d\bar{\mathbf{R}}' du' G(\mathbf{R}-\mathbf{R}', Y-Y', u-u') 1_{[u_0,\infty)}(u') S_Z^{mag}(\bar{\mathbf{R}}', u') \\
&= -\int_{\mathbb{R}^4} d\bar{\mathbf{R}}' du' G(\mathbf{R}-\mathbf{R}', Y-Y', u-u') 1_{[u_0,\infty)}(u') S_{L,Z}^{mag}(\mathbf{R}', u') \frac{d}{dY'}\delta(Y') \ , \\
B_X^{nsh}(\bar{\mathbf{R}},u) &= -\int_{\mathbb{R}^4} d\bar{\mathbf{R}}' du' G(\mathbf{R}-\mathbf{R}', Y-Y', u-u') 1_{[u_0,\infty)}(u') S_X^{mag}(\bar{\mathbf{R}}', u') \\
&= -\int_{\mathbb{R}^4} d\bar{\mathbf{R}}' du' G(\mathbf{R}-\mathbf{R}', Y-Y', u-u') 1_{[u_0,\infty)}(u') S_{L,X}^{mag}(\mathbf{R}', u') \frac{d}{dY'}\delta(Y') \ .
\end{aligned} \tag{A.16}$$

Note by (A.5) that $G(\bar{\mathbf{R}},u)$ is even in Y whence, by (A.16), $E_Y^{nsh}(\bar{\mathbf{R}},u)$, $B_Z^{nsh}(\bar{\mathbf{R}},u), B_X^{nsh}(\bar{\mathbf{R}},u)$ are odd in Y.

In the shielding scenario we obtain from (A.6),(A.7),(A.9) that

$$
\begin{aligned}
E_Z^{sh}(\bar{\mathbf{R}},u) &= -\int_{\mathbb{R}^4} d\bar{\mathbf{R}}' du' G_D(\bar{\mathbf{R}},u,\bar{\mathbf{R}}',u') 1_{[u_0,\infty)}(u') S_Z^{el}(\bar{\mathbf{R}}',u') \\
&= -\int_{\mathbb{R}^4} d\bar{\mathbf{R}}' du' G_D(\mathbf{R},Y,u,\mathbf{R}',Y',u') 1_{[u_0,\infty)}(u') \delta(Y') S_{L,Z}^{el}(\mathbf{R}',u') \\
&= -\int_{\mathbb{R}^3} d\mathbf{R}' du' 1_{[u_0,\infty)}(u') G_D(\mathbf{R},Y,u,\mathbf{R}',0,u') S_{L,Z}^{el}(\mathbf{R}',u') \\
&= -\sum_{k\in\mathbb{Z}} (-1)^k \int_{\mathbb{R}^3} d\mathbf{R}' du' 1_{[u_0,\infty)}(u') G(\mathbf{R}-\mathbf{R}', Y-2kg, u-u') S_{L,Z}^{el}(\mathbf{R}',u') \ ,
\end{aligned}
$$

whence, by (A.11),

$$
\begin{aligned}
E_Z^{sh}(\bar{\mathbf{R}},u) &= \sum_{k\in\mathbb{Z}} (-1)^k E_Z^{nsh}(\mathbf{R},Y-2kg,u) \\
&= -\frac{1}{4\pi} \sum_{k\in\mathbb{Z}} (-1)^k \int_{\mathbb{R}^2} d\mathbf{R}' 1_{[u_0,\infty)}(u - \sqrt{|\mathbf{R}-\mathbf{R}'|^2 + (Y-2kg)^2}) \\
&\quad \cdot \frac{S_{L,Z}^{el}(\mathbf{R}', u - \sqrt{|\mathbf{R}-\mathbf{R}'|^2 + (Y-2kg)^2})}{\sqrt{|\mathbf{R}-\mathbf{R}'|^2 + (Y-2kg)^2}} \ ,
\end{aligned}
\tag{A.17}
$$

and analogously

$$
\begin{aligned}
E_X^{sh}(\bar{\mathbf{R}},u) &= \sum_{k\in\mathbb{Z}} (-1)^k E_X^{nsh}(\mathbf{R},Y-2kg,u) \\
&= -\frac{1}{4\pi} \sum_{k\in\mathbb{Z}} (-1)^k \int_{\mathbb{R}^2} d\mathbf{R}' 1_{[u_0,\infty)}(u - \sqrt{|\mathbf{R}-\mathbf{R}'|^2 + (Y-2kg)^2}) \\
&\quad \cdot \frac{S_{L,X}^{el}(\mathbf{R}', u - \sqrt{|\mathbf{R}-\mathbf{R}'|^2 + (Y-2kg)^2})}{\sqrt{|\mathbf{R}-\mathbf{R}'|^2 + (Y-2kg)^2}} \ , \\
B_Y^{sh}(\bar{\mathbf{R}},u) &= \sum_{k\in\mathbb{Z}} (-1)^k B_Y^{nsh}(\mathbf{R},Y-2kg,u) \\
&= -\frac{1}{4\pi} \sum_{k\in\mathbb{Z}} (-1)^k \int_{\mathbb{R}^2} d\mathbf{R}' 1_{[u_0,\infty)}(u - \sqrt{|\mathbf{R}-\mathbf{R}'|^2 + (Y-2kg)^2}) \\
&\quad \cdot \frac{S_{L,Y}^{mag}(\mathbf{R}', u - \sqrt{|\mathbf{R}-\mathbf{R}'|^2 + (Y-2kg)^2})}{\sqrt{|\mathbf{R}-\mathbf{R}'|^2 + (Y-2kg)^2}} \ .
\end{aligned}
\tag{A.18}
$$

Since $E_Z^{nsh}(\bar{\mathbf{R}},u)$, $E_X^{nsh}(\bar{\mathbf{R}},u)$, $B_Y^{nsh}(\bar{\mathbf{R}},u)$ are even in Y, it follows from (A.17),(A.18) that $E_Z^{sh}(\bar{\mathbf{R}}, u), E_X^{sh}(\bar{\mathbf{R}}, u), B_Y^{sh}(\bar{\mathbf{R}}, u)$ are even in Y. Abbreviating

$$\mathcal{F}^{sh} := (E_Z^{sh}, E_X^{sh}, B_Y^{sh})^T \ , \qquad \mathcal{F}_L^{sh}(\mathbf{R}, u) := \mathcal{F}^{sh}(\mathbf{R}, 0, u) \ , \tag{A.19}$$

we obtain from (A.13),(A.14),(A.17), (A.18) that

$$\begin{aligned}
&\mathcal{F}^{sh}(\mathbf{R}, Y, u) = (E_Z^{sh}(\mathbf{R}, Y, u), E_X^{sh}(\mathbf{R}, Y, u), B_Y^{sh}(\mathbf{R}, Y, u))^T \\
&= \sum_{k\in\mathbb{Z}} (-1)^k \Big(E_Z^{nsh}(\mathbf{R}, Y - 2kg, u), E_X^{nsh}(\mathbf{R}, Y - 2kg, u), B_Y^{nsh}(\mathbf{R}, Y - 2kg, u) \Big)^T \\
&= \sum_{k\in\mathbb{Z}} (-1)^k \mathcal{F}^{nsh}(\mathbf{R}, Y - 2kg, u) \\
&= -\frac{1}{4\pi} \sum_{k\in\mathbb{Z}} (-1)^k \int_{\mathbb{R}^2} d\mathbf{R}' 1_{[u_0,\infty)}(u - \sqrt{|\mathbf{R} - \mathbf{R}'|^2 + (Y - 2kg)^2}) \\
&\qquad \cdot \frac{\mathbf{S}(\mathbf{R}', u - \sqrt{|\mathbf{R} - \mathbf{R}'|^2 + (Y - 2kg)^2})}{\sqrt{|\mathbf{R} - \mathbf{R}'|^2 + (Y - 2kg)^2}} \ .
\end{aligned} \tag{A.20}$$

Since $\mathcal{F}^{nsh}(\mathbf{R}, Y, u)$ is even in Y and since, by (A.20), $\mathcal{F}^{sh}(\mathbf{R}, Y, u) = \sum_{k\in\mathbb{Z}} (-1)^k \mathcal{F}^{nsh}(\mathbf{R}, Y - 2kg, u)$, it follows that $\mathcal{F}^{sh}$ satisfies the Dirichlet boundary condition (3.43). It also follows from (A.19),(A.20) that

$$\begin{aligned}
\mathcal{F}_L^{sh}(\mathbf{R}, u) &= \sum_{k\in\mathbb{Z}} (-1)^k \mathcal{F}^{nsh}(\mathbf{R}, -2kg, u) = \sum_{k\in\mathbb{Z}} (-1)^k \mathcal{F}^{nsh}(\mathbf{R}, 2kg, u) \\
&= -\frac{1}{4\pi} \sum_{k\in\mathbb{Z}} (-1)^k \int_{\mathbb{R}^2} d\mathbf{R}' 1_{[u_0,\infty)}(u - \sqrt{|\mathbf{R} - \mathbf{R}'|^2 + (2kg)^2}) \\
&\qquad \cdot \frac{\mathbf{S}(\mathbf{R}', u - \sqrt{|\mathbf{R} - \mathbf{R}'|^2 + (2kg)^2})}{\sqrt{|\mathbf{R} - \mathbf{R}'|^2 + (2kg)^2}} \ .
\end{aligned} \tag{A.21}$$

Also we obtain from (A.6),(A.7), (A.9),(A.16) that

$$
\begin{aligned}
&E_Y^{sh}(\bar{\mathbf{R}},u) = -\int_{\mathbb{R}^4} d\bar{\mathbf{R}}' du' G_N(\bar{\mathbf{R}},u,\bar{\mathbf{R}}',u') 1_{[u_0,\infty)}(u') S_Y^{el}(\bar{\mathbf{R}}',u') \\
&= -\int_{\mathbb{R}^4} d\bar{\mathbf{R}}' du' G_N(\bar{\mathbf{R}},u,\bar{\mathbf{R}}',u') 1_{[u_0,\infty)}(u') S_{L,Y}^{el}(\mathbf{R}',u') \frac{d}{dY'}\delta(Y') \\
&= -\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^4} d\bar{\mathbf{R}}' du' G(\mathbf{R}-\mathbf{R}', Y-(-1)^k Y' - 2kg, u-u') 1_{[u_0,\infty)}(u') \\
&\qquad \cdot S_{L,Y}^{el}(\mathbf{R}',u') \frac{d}{dY'}\delta(Y') \\
&= -\sum_{k\in\mathbb{Z}} (-1)^k \int_{\mathbb{R}^4} d\bar{\mathbf{R}}' du' G(\mathbf{R}-\mathbf{R}', Y - Y' - 2kg, u-u') 1_{[u_0,\infty)}(u') \\
&\qquad \cdot S_{L,Y}^{el}(\mathbf{R}',u') \frac{d}{dY'}\delta(Y') = \sum_{k\in\mathbb{Z}} (-1)^k E_Y^{nsh}(\mathbf{R}, Y-2kg, u) \; ,
\end{aligned}
\tag{A.22}
$$

and analogously

$$
\begin{aligned}
B_Z^{sh}(\bar{\mathbf{R}},u) &= -\sum_{k\in\mathbb{Z}} (-1)^k B_Z^{nsh}(\mathbf{R}, Y-2kg, u) \; , \\
B_X^{sh}(\bar{\mathbf{R}},u) &= -\sum_{k\in\mathbb{Z}} (-1)^k B_X^{nsh}(\mathbf{R}, Y-2kg, u) \; .
\end{aligned}
\tag{A.23}
$$

Since $E_Y^{nsh}(\bar{\mathbf{R}},u), B_Z^{nsh}(\bar{\mathbf{R}},u), B_X^{nsh}(\bar{\mathbf{R}},u)$ are odd in Y it follows from (A.22),(A.23) that $E_Y^{sh}(\bar{\mathbf{R}},u), B_Z^{sh}(\bar{\mathbf{R}},u), B_X^{sh}(\bar{\mathbf{R}},u)$ are odd in Y and satisfy the Neumann boundary condition (3.23).

We conclude that, in both scenarios, $E_Y(\bar{\mathbf{R}},u), B_Z(\bar{\mathbf{R}},u)$, $B_X(\bar{\mathbf{R}},u)$ are odd in Y and $E_Z(\bar{\mathbf{R}},u), E_X(\bar{\mathbf{R}},u), B_Y(\bar{\mathbf{R}},u)$ are even in Y.

As explained in Section 3.1 it is useful to rewrite the field integral in (A.14) by applying a string of substitutions. To do so we first write (A.14) in the form

$$
\mathcal{F}^{nsh}(\mathbf{R},Y,u) = \int_{\mathbb{R}^2} d\mathbf{R}' \frac{F(\mathbf{R}', u - \sqrt{|\mathbf{R}-\mathbf{R}'|^2 + Y^2})}{\sqrt{|\mathbf{R}-\mathbf{R}'|^2 + Y^2}} \; , \tag{A.24}
$$

where

$$
F(\mathbf{R},u) := -\frac{1}{4\pi} 1_{[u_0,\infty)}(u) \mathbf{S}(\mathbf{R},u) \; . \tag{A.25}
$$

Using polar coordinates we obtain from (A.24)

$$\begin{aligned}\mathcal{F}^{nsh}(\mathbf{R},Y,u) &= \int_{\mathbb{R}^2} d\mathbf{R}'' \frac{F(\mathbf{R}+\mathbf{R}'', u-\sqrt{|\mathbf{R}''|^2+Y^2})}{\sqrt{|\mathbf{R}''|^2+Y^2}} \\ &= \int_0^\infty d\chi\chi \int_{-\pi}^{\pi} d\theta \frac{F(\mathbf{R}+\chi\mathbf{e}(\theta), u-\sqrt{\chi^2+Y^2})}{\sqrt{\chi^2+Y^2}} \;, \end{aligned} \tag{A.26}$$

where $\mathbf{e}(\theta) := (\cos(\theta), \sin(\theta))^T$. Performing in (A.26) the substitution $\chi \to \xi := \sqrt{\chi^2+Y^2}$ we obtain

$$\begin{aligned}\mathcal{F}^{nsh}(\mathbf{R},Y,u) &= \int_{|Y|}^\infty d\xi\xi \int_{-\pi}^{\pi} d\theta \frac{F(\mathbf{R}+\sqrt{\xi^2-Y^2}\mathbf{e}(\theta), u-\xi)}{\xi} \\ &= \int_{|Y|}^\infty d\xi \int_{-\pi}^{\pi} d\theta F(\mathbf{R}+\sqrt{\xi^2-Y^2}\mathbf{e}(\theta), u-\xi) \;. \end{aligned} \tag{A.27}$$

Performing in (A.27) the substitution $\xi \to v := u - \xi$ we obtain

$$\begin{aligned}\mathcal{F}^{nsh}(\mathbf{R},Y,u) &= -\int_{u-|Y|}^{-\infty} dv \int_{-\pi}^{\pi} d\theta F(\mathbf{R}+\sqrt{(u-v)^2-Y^2}\mathbf{e}(\theta), v) \;. \\ &= \int_{-\infty}^{u-|Y|} dv \int_{-\pi}^{\pi} d\theta F(\mathbf{R}+\sqrt{(u-v)^2-Y^2}\mathbf{e}(\theta), v) \;, \end{aligned} \tag{A.28}$$

whence by (A.25)

$$\mathcal{F}^{nsh}(\mathbf{R},Y,u) = -\frac{1}{4\pi}\int_{-\infty}^{u-|Y|} dv 1_{[u_0,\infty)}(v) \int_{-\pi}^{\pi} d\theta \mathbf{S}(\mathbf{R}+\sqrt{(u-v)^2-Y^2}\mathbf{e}(\theta), v) \;. \tag{A.29}$$

A.2 Derivation of the 4D Vlasov equation

For the sheet beam the 6D Vlasov equation (3.6) reads by (3.2),(3.8),(3.9), (3.24) as

$$
\begin{aligned}
0 &= \partial_u \bar{f}(\bar{\mathbf{R}},\bar{\mathbf{P}};u) + \dot{\bar{\mathbf{R}}}\cdot\nabla_{\bar{\mathbf{R}}}\bar{f}(\bar{\mathbf{R}},\bar{\mathbf{P}};u) + \dot{\bar{\mathbf{P}}}\cdot\nabla_{\bar{\mathbf{P}}}\bar{f}(\bar{\mathbf{R}},\bar{\mathbf{P}};u) \\
&= \partial_u f_L(\mathbf{R},\mathbf{P};u)\delta(Y)\delta(P_Y) + \frac{P_Z}{mc\bar{\gamma}}\partial_Z f_L(\mathbf{R},\mathbf{P};u)\delta(Y)\delta(P_Y) \\
&\quad + \frac{P_X}{mc\bar{\gamma}}\partial_X f_L(\mathbf{R},\mathbf{P};u)\delta(Y)\delta(P_Y) + \frac{P_Y}{mc\bar{\gamma}} f_L(\mathbf{R},\mathbf{P};u)\delta(P_Y)\frac{d}{dY}\delta(Y) \\
&\quad + \dot{P}_Z\partial_{P_Z} f_L(\mathbf{R},\mathbf{P};u)\delta(Y)\delta(P_Y) + \dot{P}_X\partial_{P_X} f_L(\mathbf{R},\mathbf{P};u)\delta(Y)\delta(P_Y) \\
&\quad + \dot{P}_Y f_L(\mathbf{R},\mathbf{P};u)\delta(Y)\frac{d}{dP_Y}\delta(P_Y) \\
&= \partial_u f_L(\mathbf{R},\mathbf{P};u)\delta(Y)\delta(P_Y) + \frac{P_Z}{mc\bar{\gamma}}\partial_Z f_L(\mathbf{R},\mathbf{P};u)\delta(Y)\delta(P_Y) \\
&\quad + \frac{P_X}{mc\bar{\gamma}}\partial_X f_L(\mathbf{R},\mathbf{P};u)\delta(Y)\delta(P_Y) + \frac{P_Y}{mc\bar{\gamma}} f_L(\mathbf{R},\mathbf{P};u)\delta(P_Y)\frac{d}{dY}\delta(Y) \\
&\quad + \frac{q}{c}\bigg(E_Z(\bar{\mathbf{R}},u) + \frac{P_X}{m\bar{\gamma}}[B_Y(\bar{\mathbf{R}},u) + \bar{B}_{ext,Y}(\bar{\mathbf{R}})] \\
&\qquad - \frac{P_Y}{m\bar{\gamma}}[B_X(\bar{\mathbf{R}},u) + \bar{B}_{ext,X}(\bar{\mathbf{R}})]\bigg)\partial_{P_Z} f_L(\mathbf{R},\mathbf{P};u)\delta(Y)\delta(P_Y) \\
&\quad + \frac{q}{c}\bigg(E_X(\bar{\mathbf{R}},u) - \frac{P_Z}{m\bar{\gamma}}[B_Y(\bar{\mathbf{R}},u) + \bar{B}_{ext,Y}(\bar{\mathbf{R}})] \\
&\qquad + \frac{P_Y}{m\bar{\gamma}}[B_Z(\bar{\mathbf{R}},u) + \bar{B}_{ext,Z}(\bar{\mathbf{R}})]\bigg)\partial_{P_X} f_L(\mathbf{R},\mathbf{P};u)\delta(Y)\delta(P_Y) \\
&\quad + \frac{q}{c}\bigg(E_Y(\bar{\mathbf{R}},u) + \frac{P_Z}{m\bar{\gamma}}[B_X(\bar{\mathbf{R}},u) + \bar{B}_{ext,X}(\bar{\mathbf{R}})] \\
&\qquad - \frac{P_X}{m\bar{\gamma}}[B_Z(\bar{\mathbf{R}},u) + \bar{B}_{ext,Z}(\bar{\mathbf{R}})]\bigg) f_L(\mathbf{R},\mathbf{P};u)\delta(Y)\frac{d}{dP_Y}\delta(P_Y)\ . \qquad \text{(A.30)}
\end{aligned}
$$

Since, by Section A.1, $E_Y(\bar{\mathbf{R}}, u), B_Z(\bar{\mathbf{R}}, u), B_X(\bar{\mathbf{R}}, u)$ are odd in Y and due to (3.3), (3.5),(3.29) we have

$$
\begin{aligned}
&\frac{P_Z}{mc\bar{\gamma}}\delta(Y)\delta(P_Y) = \frac{P_Z}{mc\gamma}\delta(Y)\delta(P_Y)\ , \qquad \frac{P_X}{mc\bar{\gamma}}\delta(Y)\delta(P_Y) = \frac{P_X}{mc\gamma}\delta(Y)\delta(P_Y)\ , \\
&P_Y\delta(P_Y)\frac{d}{dY}\delta(Y) = 0\ , \\
&\bigg(E_Z(\bar{\mathbf{R}}, u) + \frac{P_X}{m\bar{\gamma}}[B_Y(\bar{\mathbf{R}}, u) + \bar{B}_{ext,Y}(\bar{\mathbf{R}})] \\
&\qquad - \frac{P_Y}{m\bar{\gamma}}[B_X(\bar{\mathbf{R}}, u) + \bar{B}_{ext,X}(\bar{\mathbf{R}})]\bigg)\delta(Y)\delta(P_Y) \\
&\qquad = \bigg(E_Z(\mathbf{R}, 0, u) + \frac{P_X}{m\gamma}[B_Y(\mathbf{R}, 0, u) + B_{ext}(Z)]\bigg)\delta(Y)\delta(P_Y)\ , \\
&\bigg(E_X(\bar{\mathbf{R}}, u) - \frac{P_Z}{m\bar{\gamma}}[B_Y(\bar{\mathbf{R}}, u) + \bar{B}_{ext,Y}(\bar{\mathbf{R}})] \\
&\qquad + \frac{P_Y}{m\bar{\gamma}}[B_Z(\bar{\mathbf{R}}, u) + \bar{B}_{ext,Z}(\bar{\mathbf{R}})]\bigg)\delta(Y)\delta(P_Y) \\
&\qquad = \bigg(E_X(\mathbf{R}, 0, u) - \frac{P_Z}{m\gamma}[B_Y(\mathbf{R}, 0, u) + B_{ext}(Z)]\bigg)\delta(Y)\delta(P_Y)\ , \\
&\bigg(E_Y(\bar{\mathbf{R}}, u) + \frac{P_Z}{m\bar{\gamma}}[B_X(\bar{\mathbf{R}}, u) + \bar{B}_{ext,X}(\bar{\mathbf{R}})] \\
&\qquad - \frac{P_X}{m\bar{\gamma}}[B_Z(\bar{\mathbf{R}}, u) + \bar{B}_{ext,Z}(\bar{\mathbf{R}})]\bigg)\delta(Y)\frac{d}{dP_Y}\delta(P_Y) = 0\ .
\end{aligned}
\tag{A.31}
$$

It follows from (3.29),(3.28),(3.31), (A.30),(A.31) that

$$
\begin{aligned}
0 &= \delta(Y)\delta(P_Y)\Bigg(\partial_u f_L(\mathbf{R},\mathbf{P};u) + \frac{P_Z}{mc\gamma}\partial_Z f_L(\mathbf{R},\mathbf{P};u) + \frac{P_X}{mc\gamma}\partial_X f_L(\mathbf{R},\mathbf{P};u) \\
&+\frac{q}{c}\Bigg(E_Z(\mathbf{R},0,u) + \frac{P_X}{m\gamma}[B_Y(\mathbf{R},0,u) + B_{ext}(Z)]\Bigg)\partial_{P_Z} f_L(\mathbf{R},\mathbf{P};u) \\
&+\frac{q}{c}\Bigg(E_X(\mathbf{R},0,u) - \frac{P_Z}{m\gamma}[B_Y(\mathbf{R},0,u) + B_{ext}(Z)]\Bigg)\partial_{P_X} f_L(\mathbf{R},\mathbf{P};u)\Bigg) \\
&= \delta(Y)\delta(P_Y)\Bigg(\partial_u f_L(\mathbf{R},\mathbf{P};u) + \frac{\mathbf{P}}{mc\gamma}\cdot\nabla_{\mathbf{R}} f_L(\mathbf{R},\mathbf{P};u) \\
&+\frac{q}{c}\Bigg(\mathbf{E}_\perp(\mathbf{R},u) + \frac{1}{m\gamma}(P_X,-P_Z)^T[B_\perp(\mathbf{R},u) + B_{ext}(Z)]\Bigg)\cdot\nabla_{\mathbf{P}} f_L(\mathbf{R},\mathbf{P};u) \\
&= \delta(Y)\delta(P_Y)\Bigg(\partial_u f_L(\mathbf{R},\mathbf{P};u) + \dot{\mathbf{R}}\cdot\nabla_{\mathbf{R}} f_L(\mathbf{R},\mathbf{P};u) + \dot{\mathbf{P}}\cdot\nabla_{\mathbf{P}} f_L(\mathbf{R},\mathbf{P};u)\Bigg)\,,
\end{aligned}
\tag{A.32}
$$

whence the 4D Vlasov equation (3.26) holds.

Since we assume that our 6D+3D Vlasov-Maxwell problem is well-posed in both scenarios (shielding and nonshielding) we thus conclude from Section A.1 and the present section that if $\bar{f}$ is initially of the sheet beam form (3.24) then $\bar{f}$ remains in this form and $\mathcal{F}_L$ satisfies (A.15) resp. (A.21).

As mentioned in Chapter 1, the first part of this thesis (consisting of Chapters 2-4 and Appendix A) does not aim at rigorousness. For example in the above derivation of (3.26) I used (A.31) which contains the term $E_Y(\bar{\mathbf{R}},u)\delta(Y)$ which, as a function of Y, is proportional to $\frac{Y}{|Y|}\delta(Y)$. A rigorous treatment therefore warrants to deal with $\frac{Y}{|Y|}\delta(Y)$ which however is not defined in the theory of Schwartz distributions. Nevertheless modern generalizations of Schwartz' distribution theory (see, e.g., [Hos]) cope with $\frac{Y}{|Y|}\delta(Y)$ which allows to study (A.31) rigorously.

A.3 Kernel density estimation

In this section I present some material on kernel density estimators (in Section A.3.7 I comment on practical aspects w.r.t. our code).

A.3.1 Generalities

Let $Y_1, ..., Y_N$ be $\mathbb{R}^d$-valued random vectors which are independent identically distributed with probability density, f, and let $Y := (Y_1, ..., Y_N)$. Let the 'kernel' be a function $K : \mathbb{R}^d \to [0, \infty)$ which is continuous, even, has finite second moments and satisfies

$$\int_{\mathbb{R}^d} dy K(y) = 1 \ . \tag{A.33}$$

For $H > 0$ we define $K_H : \mathbb{R}^d \to [0, \infty)$ for $y \in \mathbb{R}^d$ by

$$K_H(y) := \frac{1}{H^d} K(\frac{y}{H}) \ . \tag{A.34}$$

Clearly K_H is continuous, even, and satisfies

$$\int_{\mathbb{R}^d} dy K_H(y) = 1 \ . \tag{A.35}$$

Given a kernel K the density estimation gives a random variable $\hat{f}$ which is parametrized by $y \in \mathbb{R}^d$ and $H > 0$ and which is defined by

$$\hat{f}(y, H) = \hat{f}(y, H, Y) := \frac{1}{N} \sum_{j=1}^{N} K_H(y - Y_j) = \frac{1}{H^d N} \sum_{j=1}^{N} K(\frac{y - Y_j}{H}) \ . \tag{A.36}$$

The selection of the bandwith H will be discussed in later sections so it suffices here to say that we will deal with a $MISE$ driven bandwith selector. Note also that, by (A.33),(A.36),

$$\int_{\mathbb{R}^d} dy \hat{f}(y, H) = 1 \ . \tag{A.37}$$

A very common kernel in the univariate ($d = 1$) case is the 1D Epanechnikov kernel $K = K_{C0,1D}$ which is defined by

$$K_{C0,1D}(y) := \frac{3}{4}(1 - y^2)1_{[0,1]}(y^2) = \frac{3}{4}(1 - y^2)1_{[-1,1]}(y) \; . \tag{A.38}$$

Another kernel in the univariate case is $K = K_{C1,1D}$ which is defined by

$$K_{C1,1D}(y) := \frac{15}{16}(1 - y^2)^2 1_{[0,1]}(y^2) = \frac{15}{16}(1 - y^2)^2 1_{[-1,1]}(y) \; . \tag{A.39}$$

On the basis of (A.38),(A.39) one defines in the bivariate ($d = 2$) case the kernels $K = K_{C0,2D,P}$ and $K = K_{C1,2D,P}$ by

$$\begin{aligned} & K_{C0,2D,P}(y_1, y_2) := K_{C0,1D}(y_1)K_{C0,1D}(y_2) \\ & = \frac{9}{16}(1 - y_1^2)(1 - y_2^2)1_{[-1,1]}(y_1)1_{[-1,1]}(y_2) \\ & = \frac{9}{16}(1 - y_1^2)(1 - y_2^2)1_{[-1,1]\times[-1,1]}(y_1, y_2) \; , \end{aligned} \tag{A.40}$$

$$\begin{aligned} & K_{C1,2D,P}(y_1, y_2) := K_{C1,1D}(y_1)K_{C1,1D}(y_2) \\ & = \frac{225}{256}(1 - y_1^2)^2(1 - y_2^2)^2 1_{[-1,1]}(y_1)1_{[-1,1]}(y_2) \\ & = \frac{225}{256}(1 - y_1^2)^2(1 - y_2^2)^2 1_{[-1,1]\times[-1,1]}(y_1, y_2) \; . \end{aligned} \tag{A.41}$$

An important class of kernels is of the radial form

$$K(y) = \check{K}(y^T y) \; , \tag{A.42}$$

where $\check{K} : [0, \infty) \to [0, \infty)$ is continuous. Note that in the case (A.42) we have

$$1 = \int_{\mathbb{R}^d} dy K(y) = \begin{cases} \int_0^\infty \frac{dt\check{K}(t)}{\sqrt{t}} & \text{if } d = 1 \\ \pi \int_0^\infty dt\check{K}(t) & \text{if } d = 2 \end{cases} \; , \tag{A.43}$$

and, for $y \in \mathbb{R}^d$,

$$K_H(y) = \frac{1}{H^d}\check{K}(\frac{y^T y}{H^2}) \; . \tag{A.44}$$

Radial examples in the univariate case are $K_{C0,1D}$ (see (A.38)) and $K_{C1,1D}$ (see (A.39)) since

$$K_{C0,1D}(y) = \check{K}_{C0,1D}(y^2) \;, \qquad \check{K}_{C0,1D}(t) := \frac{3}{4}(1-t)1_{[0,1]}(t) \;, \tag{A.45}$$

$$K_{C1,1D}(y) = \check{K}_{C1,1D}(y^2) \;, \qquad \check{K}_{C1,1D}(t) := \frac{15}{16}(1-t)^2 1_{[0,1]}(t) \;. \tag{A.46}$$

A radial example in the univariate case with global support is the Gaussian kernel $K = K_{Gauss,1D}$ which reads as

$$K_{Gauss,1D}(y) := \check{K}_{Gauss,1D}(y^2) \;, \quad \check{K}_{Gauss,1D}(t) := (2\pi)^{-1/2} \exp(-t/2) \;. \tag{A.47}$$

Radial examples in the bivariate case are the kernels $K = K_{C0,2D,R}$ and $K = K_{C1,2D,R}$ which read as

$$K_{C0,2D,R}(y_1, y_2) := \check{K}_{C0,2D}(y_1^2 + y_2^2) \;, \qquad \check{K}_{C0,2D}(t) := \frac{2}{\pi}(1-t)1_{[0,1]}(t) \;, \tag{A.48}$$

$$K_{C1,2D,R}(y_1, y_2) := \check{K}_{C1,2D}(y_1^2 + y_2^2) \;, \qquad \check{K}_{C1,2D}(t) := \frac{3}{\pi}(1-t)^2 1_{[0,1]}(t) \;. \tag{A.49}$$

Note that the functions $K_{C0,1D}, K_{C0,2D,P}, K_{C0,2D,R}$ are of class C^0 but not of class C^1. In contrast the functions $K_{C1,1D}, K_{C1,2D,R}$ are of class C^1 but not of class C^2. A radial example in the bivariate case with global support is the Gaussian kernel $K = K_{Gauss,2D}$ which reads as

$$K_{Gauss,2D}(y) := \check{K}_{Gauss,2D}(y_1^2 + y_d^2) \;, \quad \check{K}_{Gauss,2D}(t) := (2\pi)^{-1} \exp(-t/2) \;. \tag{A.50}$$

In Section A.3.6 we will see that the Fourier transforms of kernels are of interest. We thus define for a real valued function g on $\mathbb{R}^d$ its Fourier transform by

$$\tilde{g}(y) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} dy' \exp(iy^T y') g(y) \;. \tag{A.51}$$

To give an example, we conclude from (A.38),(A.51) that

$$\begin{aligned}\tilde{K}_{C0,1D}(y) &= (2\pi)^{-1/2}\int_{\mathbb{R}} dy' \exp(iyy')K_{C0,1D}(y')\\ &= (2\pi)^{-1/2}\frac{3}{4}\int_{-1}^{1} dy' \exp(iyy')(1-y'^2)\\ &= \sqrt{\frac{9}{32\pi}}[1+\frac{\partial^2}{\partial y^2}]\int_{-1}^{1} dy' \exp(iyy') = \sqrt{\frac{9}{32\pi}}[1+\frac{\partial^2}{\partial y^2}](\frac{2}{y}\sin(y))\\ &= \sqrt{\frac{9}{2\pi}}\frac{\sin(y)-y\cos(y)}{y^3}\ ,\end{aligned} \tag{A.52}$$

whence by (A.40),(A.51)

$$\begin{aligned}\tilde{K}_{C0,2D}(y) &= (2\pi)^{-1}\int_{\mathbb{R}^2} dy' \exp(iy^Ty')K_{C0,2D}(y')\\ &= (2\pi)^{-1}\int_{\mathbb{R}^2} dy' \exp(iy_1y'_1)\exp(iy_2y'_2)K_{C0,1D}(y'_1)K_{C0,1D}(y'_2)\\ &= \tilde{K}_{C0,1D}(y_1)\tilde{K}_{C0,1D}(y_2)\\ &= \frac{9}{2\pi}\frac{\sin(y_1)-y_1\cos(y_1)}{y_1^3}\frac{\sin(y_2)-y_2\cos(y_2)}{y_2^3}\ .\end{aligned} \tag{A.53}$$

A.3.2 Algorithmic aspects of the kernel density estimator

In this section we outline two algorithms, A1 and A2, for computing $\hat{f}$ on a grid and by estimating their costs we show that for compact support kernels they are very efficient. We here restrict to the bivariate case where we define the grid points $y_{\alpha,\beta}$ by

$$y_{\alpha,\beta} := (z_\alpha, x_\beta)\ , \qquad z_\alpha := \frac{\alpha}{m}\ , \qquad x_\beta := \frac{\beta}{n}\ , \qquad (\alpha,\beta\in\mathbb{Z})\ , \tag{A.54}$$

where m, n are fixed positive integers characterizing the grid spacings. We also define the random variables $X_1, ..., X_N, Z_1, ..., Z_N$ by

$$Y_j =: (Z_j, X_j)\ . \tag{A.55}$$

Thus in this section we are interested in the values of $\hat{f}$ on the grid points $y_{\alpha,\beta}$. However in our applications we are faced with the situation where the Y_i are concentrated in $[0,1] \times [0,1]$ whence we are only interested in the values of $\hat{f}(y_{\alpha,\beta})$ when $\alpha = 0, ..., m, \beta = 0, ..., n$. Thus defining for $\alpha = 0, ..., m, \beta = 0, ..., n$

$$\hat{f}_{grid}(\alpha, \beta, H, Y) = \hat{f}_{grid}(\alpha, \beta) := \hat{f}(y_{\alpha,\beta}, H, Y) = \frac{1}{H^2 N} \sum_{j=1}^{N} K(\frac{y_{\alpha,\beta} - Y_j}{H}) \;, \quad (A.56)$$

we develop two algorithms, A1 and A2, to compute $\hat{f}_{grid}$. Note that algorithm A2, which for us is the one of practical interest, builds up on algorithm A1 so we will outline algorithm A1 first. We define B_j as that set of indices (α, β) for which $K((y_{\alpha,\beta} - Y_j)/H)$ is nonzero, i.e., for $j = 1, ..., N$ we define

$$B_j := \{(\alpha, \beta) \in \mathbb{Z}^2 : 0 \leq \alpha \leq m, 0 \leq \beta \leq n, K(\frac{y_{\alpha,\beta} - Y_j}{H}) \neq 0\} \;. \quad (A.57)$$

We now outline algorithm A1 which works for arbitrary kernels and which marches forward in j (where $j = 1, ..., N$). One first initializes the 2D array $\hat{f}_{grid}$ to zero. Then, for $j = 1$, one computes the set B_1 via (A.57) and then, for every $(\alpha, \beta) \in B_1$, one computes $(1/NH^2)K(\frac{y_{\alpha,\beta}-Y_1}{H})$ and adds it to the (α, β)-element of the array $\hat{f}_{grid}$. One then repeats this procedure for $j = 2$ and so on until one has completed with $j = N$. The resulting expression of $\hat{f}_{grid}$ obviously satisfies for $\alpha = 0, ..., m, \beta = 0, ..., n$

$$\hat{f}_{grid}(\alpha, \beta) = \frac{1}{NH^2} \sum_{j \in \{k \in \mathbb{Z}: 1 \leq k \leq N, (\alpha,\beta) \in B_k\}} K(\frac{y_{\alpha,\beta} - Y_j}{H}) \;, \quad (A.58)$$

whence, due to (A.57) $\hat{f}_{grid}$ has the desired form (A.56). The number of function evaluations (='computational cost'), C, of algorithm A1 is $C = \sum_{j=1}^{N} \#(B_j)$ with $\#(B_j)$ being the cardinality of the set B_j. Note that if K has global support (e.g., if K in is the Gaussian $K_{Gauss,2D}$ of (A.50)) then, by (A.57), $B_j = \{(\alpha, \beta) \in \mathbb{Z}^2 : 0 \leq \alpha \leq m, 0 \leq \beta \leq n\}$ whence $\#(B_j) = (m+1)(n+1)$ so that $C = N(m+1)(n+1) \approx Nmn$. In contrast, if K has compact support then $\#(B_j)$ can be notably smaller than

$(m+1)(n+1)$ and so the cost can be notably smaller than Nmn (we come back to this point after we have outlined algorithm A2).

The motivation for algorithm A2 is the simple observation that in general the B_j are subsets of $\mathbb{Z}^2$ which are not rectangular. This is an inconvenience of algorithm A1 and so algorithm A2 resolves this inconvenience by replacing the B_j by rectangular sets (the $\hat{B}_j$ defined below). To make algorithm A2 work we assume that the kernel function K has support in $[-1,1] \times [-1,1]$. Algorithm A2 is now easy to define: it is identical with algorithm A1 except that the B_j are replaced by the $\hat{B}_j$ which are defined as follows. We define for $j = 1, ..., N$ the square $S_j \subset \mathbb{R}^2$ by

$$S_j := [Z_j - H, Z_j + H] \times [X_j - H, X_j + H] \,, \tag{A.59}$$

and the rectangles $\check{B}_j, \hat{B}_j$ by

$$\check{B}_j := \{(\alpha, \beta) \in \mathbb{Z}^2 : y_{\alpha,\beta} \in S_j\} \,, \tag{A.60}$$

$$\hat{B}_j := \check{B}_j \cap \{(\alpha, \beta) \in \mathbb{Z}^2 : 0 \le \alpha \le m, 0 \le \beta \le n\} \,. \tag{A.61}$$

Note that by (A.54),(A.59), (A.60)

$$\begin{aligned}
\check{B}_j &= \{(\alpha, \beta) \in \mathbb{Z}^2 : Z_j - H \le z_\alpha \le Z_j + H, X_j - H \le x_\beta \le X_j + H\} \\
&= \{(\alpha, \beta) \in \mathbb{Z}^2 : m(Z_j - H) \le m z_\alpha \le m(Z_j + H), \\
&\qquad n(X_j - H) \le n x_\beta \le n(X_j + H)\} \\
&= \{(\alpha, \beta) \in \mathbb{Z}^2 : m(Z_j - H) \le \alpha \le m(Z_j + H), \\
&\qquad n(X_j - H) \le \beta \le n(X_j + H)\} \,.
\end{aligned} \tag{A.62}$$

Eq. (A.62) shows us that $\check{B}_j$ is a rectangle. Of course, by (A.60),(A.61), $\hat{B}_j$ is the intersection of the rectangle $\check{B}_j$ and the rectangle $\{(\alpha, \beta) \in \mathbb{Z}^2 : 0 \le \alpha \le m, 0 \le \beta \le n\}$ whence $\hat{B}_j$ is a rectangle. To show that algorithm A2 computes $\hat{f}_{grid}$ in agreement with (A.56), we conclude from (A.57) that if $(\alpha, \beta) \in B_j$ then $K(\frac{y_{\alpha,\beta} - Y_j}{H}) \ne 0$ so that, since K has support in $[-1,1] \times [-1,1]$ and due to (A.54), we obtain that

$-1 \leq \frac{z_\alpha - Z_j}{H} \leq 1, -1 \leq \frac{x_\beta - X_j}{H} \leq 1$ which implies by (A.62) that $(\alpha, \beta) \in \check{B}_j$. Thus we have shown that

$$B_j \subset \check{B}_j \ . \tag{A.63}$$

Since by (A.57) $B_j = B_j \cap \{(\alpha, \beta) \in \mathbb{Z}^2 : 0 \leq \alpha \leq m, 0 \leq \beta \leq n\}$, we conclude from (A.61),(A.63) that

$$B_j \subset \hat{B}_j \ . \tag{A.64}$$

Obviously the resulting expression of $\hat{f}_{grid}$ for algorithm A2 is given by

$$\hat{f}_{grid}(\alpha, \beta) = \frac{1}{NH^2} \sum_{j \in \{k \in \mathbb{Z}: 1 \leq k \leq N, (\alpha,\beta) \in \hat{B}_k\}} K(\frac{y_{\alpha,\beta} - Y_j}{H}) \ . \tag{A.65}$$

It follows from (A.57),(A.64) that

$$\sum_{j \in \{k \in \mathbb{Z}: 1 \leq k \leq N, (\alpha,\beta) \in B_k\}} K(\frac{y_{\alpha,\beta} - Y_j}{H}) = \sum_{j \in \{k \in \mathbb{Z}: 1 \leq k \leq N, (\alpha,\beta) \in \hat{B}_k\}} K(\frac{y_{\alpha,\beta} - Y_j}{H}) \ ,$$

whence algorithm A2 produces the same correct $\hat{f}_{grid}$ as algorithm A1 which completes the proof that the resulting expression, (A.65), of $\hat{f}_{grid}$ in algorithm A2 has the desired form (A.56).

The computational cost, $\hat{C}$, of algorithm A2 is $\hat{C} = \sum_{j=1}^{N} \#(\hat{B}_j)$. It is clear by (A.61),(A.62) that the cost $\hat{C}$ is independent of the kernel K. Recalling that the computational cost of algorithm A1 is $C = \sum_{j=1}^{N} \#(B_j)$ it follows from (A.64) that

$$C \leq \hat{C} \ . \tag{A.66}$$

If $H \ll 1$ (which is usually the case) then the average of $\#(\hat{B}_j)$ over j is approximately the average of $\#(\check{B}_j)$ over j whence $\hat{C} = \sum_{j=1}^{N} \#(\hat{B}_j) \approx \sum_{j=1}^{N} \#(\check{B}_j)$. To estimate the cost of algorithm A2 let $H \ll 1$. Then the average of $\#(\hat{B}_j)$ over j is approximately $4H^2mn$ whence $\hat{C} \approx 4H^2mnN$. In particular by (A.66) the cost of algorithm A1 satisfies $C \tilde{\leq} 4H^2mnN$. For example if $m = n = 100, H = 0.01$ then

$\hat{C} \approx 4N$. In contrast, in the global support case the cost of algorithm A1 would be $C \approx Nm^2 = 10^4 N$ which is about a factor 2500 higher than $\hat{C}$.

We now compare C and $\hat{C}$ for special kernels. Firstly we consider $K = K_{C0,2D,P}$ in which case (see (A.40)) the support of K is contained in $[-1,1] \times [-1,1]$ so that one can apply algorithm A2 (recall that algorithm A1 can be applied for any kernel). Furthermore by (A.40), (A.54),(A.55),(A.57), (A.61),(A.62)

$$
\begin{aligned}
B_j &= \{(\alpha,\beta) \in \mathbb{Z}^2 : 0 \le \alpha \le m, 0 \le \beta \le n, K_{C0,2D,P}(\frac{y_{\alpha,\beta} - Y_j}{H}) \ne 0\} \\
&= \{(\alpha,\beta) \in \mathbb{Z}^2 : 0 \le \alpha \le m, 0 \le \beta \le n, \\
&\quad (1 - (\frac{z_\alpha - Z_j}{H})^2 (1 - (\frac{x_\beta - X_j}{H})^2) 1_{[-1,1]\times[-1,1]}(\frac{y_{\alpha,\beta} - Y_j}{H}) \ne 0\} \\
&= \{(\alpha,\beta) \in \mathbb{Z}^2 : 0 \le \alpha \le m, 0 \le \beta \le n, (1 - (\frac{z_\alpha - Z_j}{H})^2)(1 - (\frac{x_\beta - X_j}{H})^2) \ne 0\} \\
&\cup \{(\alpha,\beta) \in \mathbb{Z}^2 : 0 \le \alpha \le m, 0 \le \beta \le n, -1 \le \frac{z_\alpha - Z_j}{H} \le 1, -1 \le \frac{x_\beta - X_j}{H} \le 1\} \\
&= \{(\alpha,\beta) \in \mathbb{Z}^2 : 0 \le \alpha \le m, 0 \le \beta \le n, (1 - (\frac{z_\alpha - Z_j}{H})^2)(1 - (\frac{x_\beta - X_j}{H})^2) \ne 0\} \\
&\cup \{(\alpha,\beta) \in \mathbb{Z}^2 : 0 \le \alpha \le m, 0 \le \beta \le n, (\alpha,\beta) \in \check{B}_j\} \\
&= \{(\alpha,\beta) \in \mathbb{Z}^2 : 0 \le \alpha \le m, 0 \le \beta \le n, \\
&\quad (1 - (\frac{z_\alpha - Z_j}{H})^2)(1 - (\frac{x_\beta - X_j}{H})^2) \ne 0\} \cup \hat{B}_j \ .
\end{aligned}
\tag{A.67}
$$

Since the cases where $(1 - \frac{z_\alpha - Z_j}{H})^2 (1 - \frac{x_\beta - X_j}{H})^2 = 0$ are exceptional we have by (A.67) that, in the average over j, $\#(B_j) \approx \#(\hat{B}_j)$ whence $C \approx \hat{C}$. We thus conclude that if $K = K_{C0,2D,P}$ then algorithms A1 and A2 have essentially the same cost. Secondly it is clear that the kernel $K = K_{C1,2D,P}$ has the same cost C as $K = K_{C0,2D,P}$ and the same cost $\hat{C}$ as $K = K_{C0,2D,P}$. Thirdly we consider $K = K_{C0,2D,R}$ and $K = K_{C1,2D,R}$ in which cases (see (A.48),(A.49)) the support of K is contained in $[-1,1] \times [-1,1]$ so that one can apply both algorithms. Furthermore by (A.48),(A.49), (A.54),(A.55), (A.57) we have

$$
\begin{aligned}
B_j &= \{(\alpha,\beta)\in\mathbb{Z}^2 : 0\le\alpha\le m, 0\le\beta\le n, K_{C0,2D,R}(\frac{y_{\alpha,\beta}-Y_j}{H})\neq 0\} \\
&= \{(\alpha,\beta)\in\mathbb{Z}^2 : 0\le\alpha\le m, 0\le\beta\le n, K_{C1,2D,R}(\frac{y_{\alpha,\beta}-Y_j}{H})\neq 0\} \\
&= \{(\alpha,\beta)\in\mathbb{Z}^2 : 0\le\alpha\le m, 0\le\beta\le n, \\
&\qquad (1-(\frac{z_\alpha-Z_j}{H})^2-(\frac{x_\beta-X_j}{H})^2)1_{[0,1]}((\frac{z_\alpha-Z_j}{H})^2+(\frac{x_\beta-X_j}{H})^2)\neq 0\} \\
&= \{(\alpha,\beta)\in\mathbb{Z}^2 : 0\le\alpha\le m, 0\le\beta\le n, (1-(\frac{z_\alpha-Z_j}{H})^2-(\frac{x_\beta-X_j}{H})^2)\neq 0\} \\
&\cup\{(\alpha,\beta)\in\mathbb{Z}^2 : 0\le\alpha\le m, 0\le\beta\le n, (\frac{z_\alpha-Z_j}{H})^2+(\frac{x_\beta-X_j}{H})^2\le 1\}\ . \qquad (A.68)
\end{aligned}
$$

It is clear by (A.68) that the kernel $K = K_{C1,2D,R}$ has the same cost C as $K = K_{C0,2D,R}$ (and we already mentioned that the cost $\hat{C}$ is the same for all kernels). Since the cases where $(1-(\frac{z_\alpha-Z_j}{H})^2-(\frac{x_\beta-X_j}{H})^2) = 0$ are exceptional we have by (A.68), in the average over j,

$$
\begin{aligned}
\#(B_j) \approx \#(\{(\alpha,\beta)\in\mathbb{Z}^2 : 0\le\alpha\le m, 0\le\beta\le n, \\
(\frac{z_\alpha-Z_j}{H})^2+(\frac{x_\beta-X_j}{H})^2\le 1\})\ . \qquad (A.69)
\end{aligned}
$$

Since the disc around Y_j with radius H has area πH^2 and the square around Y_j of side length $2H$ has area $4H^2$ we have, by (A.62),(A.69) that, in the average over j, $\#(B_j)\approx(\pi/4)\#(\check{B}_j)$. Under the assumption that $H \ll 1$, the average of $\#(\hat{B}_j)$ over j is approximately the average of $\#(\check{B}_j)$ over j whence, by (A.69), the costs satisfy $C=\sum_{j=1}^N \#(B_j)\approx(\pi/4)\sum_{j=1}^N \#(\check{B}_j)\approx(\pi/4)\sum_{j=1}^N \#(\hat{B}_j) = (\pi/4)\hat{C}$. We see that for the four kernels, $K_{C0,2D,P}, K_{C0,2D,R}, K_{C1,2D,P}, K_{C1,2D,R}$, the cost of algorithm A2 is not much larger than the cost of algorithm A1 which justifies the use of the more convenient algorithm A2.

A.3.3 Estimators of MISE and of related quantities

In this section we consider $MISE$ which is defined by

$$MISE(H) = MISE(H, Y) := E(\int_{\mathbb{R}^d} dy(\hat{f}(y, H, Y) - f(y))^2) . \quad (A.70)$$

MISE is an important figure of merit for the accuracy of $\hat{f}$ and so its minimization w.r.t. H is of great interest. In fact we want to use a $MISE$ driven bandwith selector, i.e., we define the optimal bandwith, H_{MISE}, by

$$H_{MISE} := argmin_{H>0}(MISE(H, Y)) . \quad (A.71)$$

Since we want to estimate and approximate H_{MISE} in Sections A.3.5,A.3.6, we first have to introduce quantities related with $MISE$. We define for real valued functions g, h

$$(g * h)(y) := \int_{\mathbb{R}^d} dy' g(y - y')h(y') . \quad (A.72)$$

Since $Y_1, ..., Y_N$ are independent identically distributed with probability density f, the expectation value of $\hat{f}$ reads as

$$E(\hat{f}(y, H, Y)) = E(\frac{1}{N}\sum_{j=1}^{N} K_H(y - Y_j)) = \frac{1}{N}\sum_{j=1}^{N} E(K_H(y - Y_j))$$
$$= \frac{1}{N}\sum_{j=1}^{N} \int_{\mathbb{R}^d} dy' f(y')K_H(y - y')$$
$$= \int_{\mathbb{R}^d} dy' f(y')K_H(y - y') = (K_H * f)(y) . \quad (A.73)$$

We define for $y \in \mathbb{R}^d$ and $H > 0$

$$RSE(y, H) = RSE(y, H, Y) := \hat{f}^2(y, H, Y) - 2\hat{f}(y, H, Y)f(y) , \quad (A.74)$$
$$SE(y, H) = SE(y, H, Y) := RSE(y, H, Y) + f^2(y)$$
$$= \hat{f}^2(y, H, Y) - 2\hat{f}(y, H, Y)f(y) + f^2(y) = (\hat{f}(y, H, Y) - f(y))^2 . \quad (A.75)$$

We define for real valued and square integrable functions g, h

$$(g,h)_2 := \int_{\mathbb{R}^d} dy g(y)h(y) \ . \tag{A.76}$$

We now define for $H > 0$

$$\begin{aligned}
&RISE(H) = RISE(H,Y) := \int_{\mathbb{R}^d} dy RSE(y,H,Y) \\
&= \int_{\mathbb{R}^d} dy(\hat{f}^2(y,H,Y) - 2\hat{f}(y,H,Y)f(y)) \\
&= (\hat{f}(\cdot,H,Y), \hat{f}(\cdot,H,Y))_2 - 2(\hat{f}(\cdot,H,Y), f)_2 \ , \qquad (A.77) \\
&ISE(H) = ISE(H,Y) := \int_{\mathbb{R}^d} dy SE(y,H,Y) = \int_{\mathbb{R}^d} dy(RSE(y,H,Y) + f^2(y)) \\
&\quad = RISE(H,Y) + \int_{\mathbb{R}^d} dy f^2(y) = \int_{\mathbb{R}^d} dy(\hat{f}(y,H,Y) - f(y))^2 \\
&\quad = (\hat{f}(\cdot,H,Y) - f, \hat{f}(\cdot,H,Y) - f)_2 \ . \qquad (A.78)
\end{aligned}$$

We now define for $y \in \mathbb{R}^d$ and $H > 0$

$$\begin{aligned}
&RMSE(y,H) = RMSE(y,H,Y) := E(RSE(y,H,Y)) \\
&\quad = E(\hat{f}^2(y,H,Y)) - 2f(y)E(\hat{f}(y,H,Y)) \\
&= E(\hat{f}^2(y,H,Y)) - 2f(y)(K_H * f)(y) \ , \qquad (A.79) \\
&MSE(y,H) = MSE(y,H,Y) := E(SE(y,H,Y)) = E((\hat{f}(y,H,Y) - f(y))^2) \\
&\quad = E(RSE(y,H,Y)) + f^2(y) = RMSE(y,H,Y) + f^2(y) \\
&\quad = E(\hat{f}^2(y,H,Y)) - 2f(y)(K_H * f)(y) + f^2(y) \ . \qquad (A.80)
\end{aligned}$$

Note that by (A.36)

$$\begin{aligned}
\hat{f}^2(y,H,Y) &= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} K_H(y-Y_i)K_H(y-Y_j) \\
&= \frac{1}{N^2}\sum_{i=1}^{N} K_H^2(y-Y_i) + \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j\neq i}^{N} K_H(y-Y_i)K_H(y-Y_j) \ , \qquad (A.81)
\end{aligned}$$

whence and since $Y_1, ..., Y_N$ are independent identically distributed with probability density f

$$
\begin{aligned}
&E(\hat{f}^2(y,H,Y)) = \frac{1}{N^2}\sum_{i=1}^{N} E(K_H^2(y-Y_i)) \\
&+\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j\neq i}^{N} E(K_H(y-Y_i)K_H(y-Y_j)) \\
&= \frac{1}{N^2}\sum_{i=1}^{N}\int_{\mathbb{R}^d} dy' f(y')K_H^2(y-y') \\
&\quad +\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j\neq i}^{N}\int_{\mathbb{R}^d} dy' f(y')\int_{\mathbb{R}^d} dy'' f(y'')K_H(y-y')K_H(y-y'') \\
&= \frac{1}{N}\int_{\mathbb{R}^d} dy' f(y')K_H^2(y-y') \\
&+\frac{N-1}{N}\int_{\mathbb{R}^d} dy' f(y')\int_{\mathbb{R}^d} dy'' f(y'')K_H(y-y')K_H(y-y'') \\
&= \frac{1}{N}(K_H^2 * f)(y) + \frac{N-1}{N}(K_H * f)^2(y)\ ,
\end{aligned}
\tag{A.82}
$$

so that by (A.79),(A.80)

$$
\begin{aligned}
RMSE(y,H,Y) &= E(\hat{f}^2(y,H,Y)) - 2f(y)(K_H * f)(y) \\
&= \frac{1}{N}(K_H^2 * f)(y) + \frac{N-1}{N}(K_H * f)^2(y) - 2f(y)(K_H * f)(y)\ ,
\end{aligned}
\tag{A.83}
$$

$$
\begin{aligned}
MSE(y,H,Y) &= RMSE(y,H,Y) + f^2(y) \\
&= \frac{1}{N}(K_H^2 * f)(y) + \frac{N-1}{N}(K_H * f)^2(y) - 2f(y)(K_H * f)(y) + f^2(y)\ .
\end{aligned}
\tag{A.84}
$$

We now define for $H > 0$

$$RMISE(H) = RMISE(H,Y) := E(RISE(H,Y)) = E(\int_{\mathbb{R}^d} dyRSE(y,H,Y))$$
$$= \int_{\mathbb{R}^d} dyRMSE(y,H,Y) = \int_{\mathbb{R}^d} dy(E(\hat{f}(y,H,Y)) - 2f(x)(K_H * f)(y))$$
$$= \int_{\mathbb{R}^d} dyE(\hat{f}^2(y,H,Y)) - 2(f, K_H * f)_2$$
$$= \int_{\mathbb{R}^d} dy(\frac{1}{N}(K_H^2 * f)(y) + \frac{N-1}{N}(K_H * f)^2(y)) - 2(f, K_H * f)_2$$
$$= \frac{1}{N}\int_{\mathbb{R}^d} dy(K_H^2 * f)(y) + \frac{N-1}{N}(K_H * f, K_H * f)_2 - 2(f, K_H * f)_2 \ , \qquad (A.85)$$

whence by (A.70),(A.78),(A.80), (A.84),(A.85)

$$MISE(H,Y) = E(\int_{\mathbb{R}^d} dy(\hat{f}(y,H,Y) - f(y))^2) = E(ISE(H,Y))$$
$$= E(\int_{\mathbb{R}^d} dySE(y,H,Y)) = \int_{\mathbb{R}^d} dyMSE(y,H,Y)$$
$$= \int_{\mathbb{R}^d} dy(RMSE(y,H,Y) + f^2(y)) = RMISE(H,Y) + \int_{\mathbb{R}^d} dyf^2(y)$$
$$= \frac{1}{N}\int_{\mathbb{R}^d} dy(K_H^2 * f)(y) + \frac{N-1}{N}(K_H * f, K_H * f)_2$$
$$-2(f, K_H * f)_2 + (f,f)_2 \ . \qquad (A.86)$$

Since $\int_{\mathbb{R}^d} dyf^2(y)$ is independent of H, we obtain from (A.71),(A.86) that

$$H_{MISE} = argmin_{H>0}(RMISE(H,Y)) \ . \qquad (A.87)$$

With (A.87) our aim of estimating H_{MISE} boils down to estimating $RMISE$.

To perform the asymptotic approximation of $MISE$ in Section A.3.4 it is convenient to define for $y \in \mathbb{R}^d$ and $H > 0$

$$BIAS(y,H) = BIAS(y,H,Y) := E(\hat{f}(y,H,Y)) - f(y) = (K_H * f)(y) - f(y) \ , \qquad (A.88)$$

$$VAR(y,H) = VAR(y,H,Y) := E(\hat{f}^2(y,H,Y)) - (E(\hat{f}(y,H,Y)))^2$$
$$= \frac{1}{N}(K_H^2 * f)(y) + \frac{N-1}{N}(K_H * f)^2(y) - (K_H * f)^2(y)$$
$$= \frac{1}{N}(K_H^2 * f)(y) - \frac{1}{N}(K_H * f)^2(y) \ , \qquad (A.89)$$

whence

$$(BIAS(y,H,Y))^2 + VAR(y,H,Y) = (E(\hat{f}(y,H,Y) - f(y))^2 + E(\hat{f}^2(y,H,Y))$$
$$-(E(\hat{f}(y,H,Y)))^2 = -2f(y)E(\hat{f}(y,H,Y)) + f^2(y) + E(\hat{f}^2(y,H,Y))$$
$$= E(\hat{f}^2(y,H,Y) - 2f(y)\hat{f}(y,H,Y) + f^2(y)) = E((\hat{f}(y,H,Y) - f(y))^2)$$
$$= MSE(y,H,Y)\ . \tag{A.90}$$

It follows from (A.86),(A.90) that for $H > 0$

$$MISE(H,Y) = \int_{\mathbb{R}^d} dyMSE(y,H,Y)$$
$$= \int_{\mathbb{R}^d} dy((BIAS(y,H,Y))^2 + VAR(y,H,Y))\ . \tag{A.91}$$

A.3.4 Asymptotic approximation of $MISE$

In this section we outline the asymptotic approximation of $MISE(H,Y)$ when

$$H \to 0\ , \qquad N \to \infty\ , \qquad (NH^d)^{-1} \to 0\ . \tag{A.92}$$

The resulting formula (A.99) is arguably the most important analytical fact about the kernel density estimators. Because of (A.91) $MISE(H,Y)$ has two terms and so the asymptotic approximation of $MISE(H,Y)$ is performed by doing Taylor expansion of $(BIAS(y,H,Y))^2$ w.r.t. H and by doing asymptotic expansion of $VAR(y,H,Y)$ w.r.t. H by Taylor expansion of $H^dVAR(y,H,Y)$ w.r.t. H. In this section we make the additional assumption on K that for $i,j = 1,...,d$

$$\int_{\mathbb{R}^d} dyy_iy_jK(y) = \delta_{i,j}\mu(K)\ , \tag{A.93}$$

where $\delta_{i,j}$ is the Kronecker symbol and where $\mu(K)$ is a constant depending only on K. Note however that the condition (A.93) is satisfied for all special kernels defined in Section A.3.1 as will be shown further below after (A.101). We first compute by

(A.34),(A.72)

$$(K_h * f)(y) = \frac{1}{H^d} \int_{\mathbb{R}^d} dy'' K(\frac{y-y''}{H}) f(y'')$$
$$= \int_{\mathbb{R}^d} dy' K(y') f(y - Hy') \ . \qquad (A.94)$$

Taylor expansion gives us

$$f(y - Hy') \approx f(y) - H \sum_{i=1}^{d} y'_i \frac{\partial f}{\partial y_i}(y) + \frac{H^2}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} y'_i y'_j \frac{\partial f}{\partial y_i \partial y_j}(y) \ . \qquad (A.95)$$

Inserting (A.95) into (A.94) yields, by (A.35),(A.93) and the evenness of K,

$$(K_H * f)(y) \approx f(y) + \frac{H^2}{2} \mu(K) \Delta f(y) \ , \qquad (A.96)$$

whence by (A.88)

$$(BIAS(y, H))^2 \approx \frac{H^4}{4} \mu^2(K) (\Delta f(y))^2 \ , \qquad (A.97)$$

where Δ is the Laplacian. We also compute by (A.34),(A.76),(A.89), (A.95),(A.96)

$$VAR(y, H, Y) = \frac{1}{NH^{2d}} \int_{\mathbb{R}^d} dy'' K^2(\frac{y-y''}{H}) f(y'') - \frac{1}{N} (K_H * f)^2(y)$$
$$= \frac{1}{NH^d} \int_{\mathbb{R}^d} dy' K^2(y') f(y - Hy') - \frac{1}{N} (f(y) + \frac{H^2}{2} \mu(K) \Delta f(y))^2$$
$$\approx \frac{1}{NH^d} \int_{\mathbb{R}^d} dy' K^2(y') \Big(f(y) - H \sum_{i=1}^{d} y'_i \frac{\partial f}{\partial y_i}(y) + \frac{H^2}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} y'_i y'_j \frac{\partial f}{\partial y_i \partial y_j}(y) \Big)$$
$$- \frac{1}{N} (f(y) + \frac{H^2}{2} \mu(K) \Delta f(y))^2 \approx \frac{1}{NH^d} f(y) \int_{\mathbb{R}^d} dy' K^2(y') - \frac{1}{N} (f(y)$$
$$+ \frac{H^2}{2} \mu(K) \Delta f(y))^2$$
$$\approx \frac{1}{NH^d} f(y) \int_{\mathbb{R}^d} dy' K^2(y') = \frac{1}{NH^d} f(y) (K, K)_2 \ . \qquad (A.98)$$

We conclude from (A.90),(A.97),(A.98)

$$MSE(y, H, Y) = (BIAS(y, H, Y))^2 + VAR(y, H, Y)$$
$$\approx \frac{H^4}{4} \mu^2(K) (\Delta f(y))^2 + \frac{1}{NH^d} f(y) (K, K)_2 \ ,$$

whence by (A.86)

$$\begin{aligned} MISE(H,Y) &= \int_{\mathbb{R}^d} dy MSE(y,H,Y) \\ &\approx \frac{H^4}{4}\mu^2(K)\int_{\mathbb{R}^d} dy(\Delta f(y))^2 + \frac{1}{NH^d}(K,K)_2 \\ &= \frac{H^4}{4}\mu^2(K)(\Delta f,\Delta f)_2 + \frac{1}{NH^d}(K,K)_2 =: AMISE(H,Y)\ . \end{aligned} \quad (A.99)$$

It follows from (A.99) that

$$\begin{aligned} H_{AMISE} &:= argmin_{H>0}(AMISE(H,Y)) \\ &= \left(\frac{d(K,K)_2}{N\mu^2(K)(\Delta f,\Delta f)_2}\right)^{1/(d+4)}\ , \end{aligned} \quad (A.100)$$

whence by (A.99)

$$AMISE(H_{AMISE},Y) = \frac{d+4}{4d}N^{-4/(d+4)}\left(\mu^{2d}(K)d^4((K,K)_2)^4((\Delta f,\Delta f)_2)^d\right)^{1/(d+4)}\ . \quad (A.101)$$

Equalities (A.99),(A.100),(A.101) are of practical and theoretical significance. In particular (A.101) quantifies the curse of dimensionality. Note that our derivation of (A.99) is schematic in some aspects and it can improved by rigorous asymptotic analysis. Nevertheless, (A.99) apparently is the result all textbooks agree on.

We now show that all the special kernels of Section A.3.1 satisfy the condition (A.93). In the univariate case K always satisfies (A.93) and we have

$$\mu(K) = \int_{\mathbb{R}} dy y^2 K(y)\ . \quad (A.102)$$

If in the univariate case K is of the radial form (see (A.42)) then by (A.42),(A.102)

$$\mu(K) = \int_{\mathbb{R}} dy y^2 \check{K}(y^2) = 2\int_0^\infty dy y^2 \check{K}(y^2) = \int_0^\infty dy' \sqrt{y'}\check{K}(y')\ . \quad (A.103)$$

It follows from (A.45),(A.46),(A.47), (A.103) that

$$\mu(K_{C0,1D}) = \int_0^\infty dy'\sqrt{y'}\check{K}_{C0,1D}(y') = \frac{3}{4}\int_0^1 dy'\sqrt{y'}(1-y') = \frac{1}{5}\ , \tag{A.104}$$

$$\mu(K_{C1,1D}) = \int_0^\infty dy'\sqrt{y'}\check{K}_{C1,1D}(y') = \frac{15}{16}\int_0^1 dy'\sqrt{y'}(1-y')^2 = \frac{1}{7}\ , \tag{A.105}$$

$$\mu(K_{Gauss,1D}) = \int_{\mathbb{R}} dy y^2 \check{K}_{Gauss,1D}(y^2)$$

$$= (2\pi)^{-1/2}\int_{\mathbb{R}} dy y^2 \exp(-y^2/2) = 1\ . \tag{A.106}$$

We now consider the bivariate case and we start with the product kernels. It follows from (A.38),(A.39),(A.40), (A.41)

$$\int_{\mathbb{R}^2} dy y_1 y_2 K_{C0,2D,P}(y) = \int_{\mathbb{R}^2} dy y_1 y_2 K_{C0,1D}(y_1)K_{C0,1D}(y_2) = 0\ ,$$
$$\int_{\mathbb{R}^2} dy y_i^2 K_{C0,2D,P}(y) = \int_{\mathbb{R}^2} dy y_i^2 K_{C0,1D}(y_1)K_{C0,1D}(y_2) = \int_{\mathbb{R}} dy_i y_i^2 K_{C0,1D}(y_i)\ ,$$
$$\int_{\mathbb{R}^2} dy y_1 y_2 K_{C1,2D,P}(y) = \int_{\mathbb{R}^2} dy y_1 y_2 K_{C1,1D}(y_1)K_{C1,1D}(y_2) = 0\ ,$$
$$\int_{\mathbb{R}^2} dy y_i^2 K_{C1,2D,P}(y) = \int_{\mathbb{R}^2} dy y_i^2 K_{C1,1D}(y_1)K_{C1,1D}(y_2) = \int_{\mathbb{R}} dy_i y_i^2 K_{C1,1D}(y_i)\ ,$$

whence (A.93) is fulfilled and we get from (A.38),(A.39),(A.93)

$$\mu(K_{C0,2D,P}) = \int_{\mathbb{R}^2} dy y_1^2 K_{C0,2D,P}(y) = \int_{\mathbb{R}} dy_1 y_1^2 K_{C0,1D}(y_1)$$

$$= \frac{3}{4}\int_{-1}^1 dy_1 y_1^2(1-y_1^2) = \frac{1}{5}\ , \tag{A.107}$$

$$\mu(K_{C1,2D,P}) = \int_{\mathbb{R}^2} dy y_1^2 K_{C1,2D,P}(y) = \int_{\mathbb{R}} dy_1 y_1^2 K_{C1,1D}(y_1)$$

$$= \frac{15}{16}\int_{-1}^1 dy_1 y_1^2(1-y_1^2)^2 = \frac{1}{7}\ . \tag{A.108}$$

If in the bivariate case K is of the radial form (see (A.42)) then by (A.42),(A.102) and by the substitution rule

$$\int_{\mathbb{R}^2} dy y_1 y_2 K(y) = \int_{\mathbb{R}^2} dy y_1 y_2 \check{K}(y_1^2+y_2^2) = -\int_{\mathbb{R}^2} dy y_1 y_2 \check{K}(y_1^2+y_2^2) = 0\ ,$$
$$\int_{\mathbb{R}^2} dy y_1^2 K(y) = \int_{\mathbb{R}^2} dy y_1^2 \check{K}(y_1^2+y_2^2) = \int_{\mathbb{R}^2} dy y_2^2 \check{K}(y_1^2+y_2^2)\ ,$$

whence (A.93) is fulfilled and we get from (A.42),(A.93)

$$\mu(K) = \int_{\mathbb{R}^2} dy y_1^2 K(y) = \int_{\mathbb{R}^2} dy y_1^2 \check{K}(y_1^2 + y_2^2) = \frac{1}{2} \int_{\mathbb{R}^2} dy y^T y \check{K}(y_1^2 + y_2^2)$$
$$= \pi \int_0^\infty dr r^3 \check{K}(r^2) = \frac{\pi}{2} \int_0^\infty dr' r' \check{K}(r') \ . \quad \text{(A.109)}$$

It follows from (A.48),(A.49),(A.50),(A.109) that

$$\mu(K_{C0,2D,R}) = \frac{\pi}{2} \int_0^\infty dr' r' \check{K}_{C0,2D}(r') = \int_0^1 dr' r'(1-r') = \frac{1}{6} \ , \quad \text{(A.110)}$$
$$\mu(K_{C1,2D,R}) = \frac{\pi}{2} \int_0^\infty dr' r' \check{K}_{C1,2D}(r') = \frac{3}{2} \int_0^1 dr' r'(1-r')^2 = \frac{1}{8} \ , \quad \text{(A.111)}$$
$$\mu(K_{Gauss,2D}) = \frac{\pi}{2} \int_0^\infty dr' r' \check{K}_{Gauss,2D}(r') = \frac{1}{4} \int_0^\infty dr' r' \exp(-r'/2) = 1 \ . \quad \text{(A.112)}$$

A.3.5 Least squares cross validation - general properties

Any technique which estimates H_{MISE} is called 'least squares cross validation'. Since $RMISE$ depends on f one has to estimate H_{MISE} in (A.87) and our estimator will be $\hat{H}_{MISE}$ in (A.178). In this section we estimate $RMISE$ by $LSCV$. Since the computational cost of $LSCV$ is of order N^2 we will, in Section A.3.6, by following Silverman approximate $LSCV$ by $\widehat{LSCV}$ and $\widehat{LSCV}$ by $LSCV_{Sil}$ since the computational cost of the latter is only of order N. We will thus define the estimator, $\hat{H}_{MISE}$, of H_{MISE} as the minumum bandwith w.r.t. $LSCV_{Sil}$. We first define for $y \in \mathbb{R}^d, H > 0$ and $i = 1, ..., N$

$$\hat{f}_{-i}(y, H) = \hat{f}_{-i}(y, H, Y) := \frac{1}{N-1} \sum_{j \neq i}^{N} K_H(y - Y_j) \ , \quad \text{(A.113)}$$

and

$$LSCV(H) = LSCV(H, Y) := \int_{\mathbb{R}^d} dy \hat{f}^2(y, H, Y) - \frac{2}{N} \sum_{i=1}^{N} \hat{f}_{-i}(Y_i, H, Y)$$
$$= \int_{\mathbb{R}^d} dy \hat{f}^2(y, H, Y) - \frac{2}{N(N-1)} \sum_{i=1}^{N} \sum_{j \neq i}^{N} K_H(Y_i - Y_j) \ . \quad \text{(A.114)}$$

At first sight $LSCV$ looks awkward because it employs the mysteriously looking $\hat{f}_{-i}$. However we will show below that LSCV has the important and useful property that it is an unbiased estimator of RMISE and even further below we will argue that LSCV is maybe the simplest possible unbiased estimator of RMISE! It follows from (A.113) that for $H > 0$ and $i = 1, ..., N$

$$
\begin{aligned}
E(\hat{f}_{-i}(Y_i, H, Y)) &= \frac{1}{N-1} \sum_{j \neq i}^{N} E(K_H(Y_i - Y_j)) \\
&= \frac{1}{N-1} \sum_{j \neq i}^{N} \int_{\mathbb{R}^d} dy f(y) \int_{\mathbb{R}^d} dy' f(y') K_H(y - y') \\
&= \int_{\mathbb{R}^d} dy f(y) \int_{\mathbb{R}^d} dy' f(y') K_H(y - y') \\
&= \int_{\mathbb{R}^d} dy f(y) (K_H * f)(y) = (f, K_H * f)_2 \, ,
\end{aligned}
\tag{A.115}
$$

so that by (A.114)

$$
\begin{aligned}
E(LSCV(H, Y)) &= \int_{\mathbb{R}^d} dy E(\hat{f}^2(y, H, Y)) - \frac{2}{N} \sum_{i=1}^{N} E(\hat{f}_{-i}(Y_i, H, Y)) \\
&= \int_{\mathbb{R}^d} dy E(\hat{f}^2(y, H, Y)) - 2(f, K_H * f)_2 \, ,
\end{aligned}
\tag{A.116}
$$

whence by (A.85)

$$
E(LSCV(H, Y)) = RMISE(H, Y) \, , \tag{A.117}
$$

i.e., LSCV is an unbiased estimator of RMISE. To get further insight into LSCV we define for $H > 0$

$$
K^{(2)}(y) := (K * K)(y) = \int_{\mathbb{R}^d} dy' K(y - y') K(y') \, , \tag{A.118}
$$

$$
\begin{aligned}
K_H^{(2)}(y) &:= (K_H * K_H)(y) = \int_{\mathbb{R}^d} dy' K_H(y - y') K_H(y') \\
&= \frac{1}{H^{2d}} \int_{\mathbb{R}^d} dy' K(\frac{y - y'}{H}) K(\frac{y'}{H}) \\
&= \frac{1}{H^d} \int_{\mathbb{R}^d} dy'' K(\frac{y}{H} - y'') K(y'') = \frac{1}{H^d} (K * K)(\frac{y}{H}) \\
&= \frac{1}{H^d} K^{(2)}(\frac{y}{H}) \, .
\end{aligned}
\tag{A.119}
$$

Note that since K is even, so are $K_H, K_H^{(2)}, K^{(2)}$. It follows from (A.81),(A.113),(A.119)

$$\int_{\mathbb{R}^d} dy \hat{f}^2(y,H,Y) = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{\mathbb{R}^d} dy K_H(y-Y_i)K_H(y-Y_j)$$
$$= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{\mathbb{R}^d} dy K_H(Y_j - Y_i + y)K_H(y)$$
$$= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{\mathbb{R}^d} dy K_H(-Y_j + Y_i - y)K_H(y)$$
$$= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}(K_H * K_H)(Y_i - Y_j) = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} K_H^{(2)}(Y_i - Y_j)\ , \tag{A.120}$$

$$\frac{1}{N}\sum_{i=1}^{N}\hat{f}_{-i}(Y_i,H) = \frac{1}{N(N-1)}\sum_{i=1}^{N}\sum_{j\neq i}^{N} K_H(Y_i - Y_j)$$
$$= \frac{1}{N(N-1)}\sum_{i=1}^{N}\sum_{j=1}^{N} K_H(Y_i - Y_j) - \frac{1}{N(N-1)}\sum_{i=1}^{N} K_H(0)$$
$$= \frac{1}{N(N-1)}\sum_{i=1}^{N}\sum_{j=1}^{N} K_H(Y_i - Y_j) - \frac{1}{N-1}K_H(0)\ , \tag{A.121}$$

so that by (A.114)

$$LSCV(H,Y) = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} K_H^{(2)}(Y_i - Y_j)$$
$$-\frac{2}{N(N-1)}\sum_{i=1}^{N}\sum_{j=1}^{N} K_H(Y_i - Y_j) + \frac{2}{N-1}K_H(0)\ . \tag{A.122}$$

Due to (A.122) the computational cost of $LSCV$ is of order N^2 which is forbiddingly large for real time applications with $N = 10^8$.

To better understand the awkward structure of $LSCV$ we first note that by

(A.73),(A.113)

$$E(\hat{f}_{-i}(y,H)) = \frac{1}{N-1}\sum_{j\neq i}^{N} E(K_H(y-Y_j)) = \frac{1}{N-1}\sum_{j\neq i}^{N}\int_{\mathbb{R}^d} dy' f(y) K_H(y-y')$$

$$= \int_{\mathbb{R}^d} dy' f(y') K_H(y-y') = (K_H * f)(y) = E(\hat{f}(y,H)) \ , \tag{A.123}$$

whence it seems plausible to replace the $\hat{f}_{-i}(Y_i, H, Y)$ in the definition (A.114) of $LSCV$ by $\hat{f}(Y_i, H, Y)$. Thus we modify $LSCV$ by defining for $H > 0$

$$\widehat{LSCV}(H,Y) := \int_{\mathbb{R}^d} dy \hat{f}^2(y,H,Y) - \frac{2}{N}\sum_{i=1}^{N}\hat{f}(Y_i,H,Y)$$

$$= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} K_H^{(2)}(Y_i - Y_j) - \frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} K_H(Y_i - Y_j)$$

$$= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} K_H^{*}(Y_i - Y_j) \ , \tag{A.124}$$

where for $y \in \mathbb{R}^d, H > 0$ we defined

$$K^*(y) := K^{(2)}(y) - 2K(y) = (K * K)(y) - 2K(y) \ , \tag{A.125}$$

$$K_H^*(y) := K_H^{(2)}(y) - 2K_H(y) = \frac{1}{H^d} K^{(2)}(\frac{y}{H}) - \frac{2}{H^d} K(\frac{y}{H})$$

$$= \frac{1}{H^d} K^*(\frac{y}{H}) \ . \tag{A.126}$$

Note that since $K_H, K_H^{(2)}$ are even, so are K^*, K_H^*. Eq. (A.124) is a straightforward modification of $LSCV$ whose definition is in fact simpler and looks more natural than the one of $LSCV$. However by (A.122),(A.124) we obtain

$$LSCV(H,Y) - \widehat{LSCV}(H,Y) = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} K_H^{(2)}(Y_i - Y_j)$$

$$-\frac{2}{N(N-1)}\sum_{i=1}^{N}\sum_{j=1}^{N} K_H(Y_i - Y_j) + \frac{2}{N-1} K_H(0)$$

$$-\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} K_H^{(2)}(Y_i - Y_j) + \frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} K_H(Y_i - Y_j)$$

$$= -\frac{2}{N^2(N-1)}\sum_{i=1}^{N}\sum_{j=1}^{N} K_H(Y_i - Y_j) + \frac{2}{N-1} K_H(0) \ . \tag{A.127}$$

While the estimator $LSCV$ of $RMISE$ is unbiased, the estimator $\widehat{LSCV}$ of $RMISE$ is biased due to (A.127). Moreover, asymptotic analysis indicates that the bias of $\widehat{LSCV}$ is not much smaller than $RMISE$ which indicates that $\widehat{LSCV}$ is (unlike $LSCV$) not a reliable estimator of $RMISE$. This indicates that it is not easy to define an unbiased estimator of $RMISE$ which has a simpler structure than $LSCV$ and it may even indicate that $LSCV$ is the 'simplest' unbiased estimator of $RMISE$. Thus we have somehow demystified the $\hat{f}_{-i}$ in (A.114).

A.3.6 Least squares cross validation - Silverman's algorithm

Following Silverman [Si] we approximate, in this section, $LSCV$ to reduce the computational cost of $LSCV$ from order N^2 to order N. We first approximate $LSCV$ by approximating the factor $1/(N-1)$ in (A.122) by $1/N$. Thus we obtain the definition

$$\begin{aligned}\widehat{LSCV}(H,Y) &:= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} K_H^{(2)}(Y_i-Y_j)\\ &-\frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} K_H(Y_i-Y_j)+\frac{2}{N}K_H(0)\\ &=\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} K_H^{*}(Y_i-Y_j)+\frac{2}{N}K_H(0)\\ &=\frac{1}{N^2H^d}\sum_{i=1}^{N}\sum_{j=1}^{N} K^{*}(\frac{Y_i-Y_j}{H})+\frac{2}{NH^d}K(0)\ . \end{aligned} \tag{A.128}$$

It follows from (A.122),(A.128) that

$$
\begin{aligned}
LSCV(H,Y) - \widehat{LSCV}(H,Y) &= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} K_H^{(2)}(Y_i - Y_j) \\
&- \frac{2}{N(N-1)}\sum_{i=1}^{N}\sum_{j=1}^{N} K_H(Y_i - Y_j) + \frac{2}{N-1}K_H(0) \\
&- \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} K_H^{(2)}(Y_i - Y_j) \\
&+ \frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} K_H(Y_i - Y_j) - \frac{2}{N}K_H(0) \\
= -\frac{2}{N^2(N-1)}\sum_{i=1}^{N}\sum_{j=1}^{N} K_H(Y_i - Y_j) &+ \frac{2}{N(N-1)}K_H(0)\ . \qquad \text{(A.129)}
\end{aligned}
$$

While the estimator $LSCV$ of $RMISE$ is unbiased, the estimator $\widehat{LSCV}$ of $RMISE$ is biased due to (A.129). Nevertheless using asymptotic analysis (small H, large N) one can argue by (A.129) that the bias of $\widehat{LSCV}$ is of order $1/N$ smaller than $RMISE$ so that the estimator $\widehat{LSCV}$ of $RMISE$ is as useful as the estimator $LSCV$.

We now continue following Silverman's approach by rewriting $\widehat{LSCV}$ as a quadrature (see (A.140)). We thus define the 'generator' $\widehat{LSCV}_{gen}$ of $\widehat{LSCV}$ for $y \in \mathbb{R}^d$, $H > 0$ by

$$
\widehat{LSCV}_{gen}(y,H,Y) := \frac{1}{N^2H^d}\sum_{i=1}^{N}\sum_{j=1}^{N} K^*(\frac{Y_i - Y_j}{H} - y)\ , \qquad \text{(A.130)}
$$

so that by (A.128)

$$
\widehat{LSCV}(H,Y) = \widehat{LSCV}_{gen}(0,H,Y) + \frac{2}{NH^d}K(0)\ . \qquad \text{(A.131)}
$$

Defining for $y \in \mathbb{R}^d$

$$u(y,Y) := (2\pi)^{-d/2}\frac{1}{N}\sum_{i=1}^{N} \exp(iy^T Y_i)\ , \tag{A.132}$$

$$v(y,Y) := (2\pi)^{d/2}|u(y,Y)|^2 = (2\pi)^{-d/2}\frac{1}{N^2}\sum_{i=1}^{N} \exp(iy^T Y_i)\sum_{j=1}^{N} \exp(-iy^T Y_j)$$

$$= (2\pi)^{-d/2}\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} \exp(iy^T(Y_i - Y_j))\ , \tag{A.133}$$

we obtain from (A.130), (A.51),(A.133)

$$\begin{aligned}
\widetilde{\widehat{LSCV}}_{gen}(y,H,Y) &= (2\pi)^{-d/2}\int_{\mathbb{R}^d} dy' \exp(iy^T y')\widehat{LSCV}_{gen}(y',H,Y) \\
&= (2\pi)^{-d/2}\frac{1}{N^2H^d}\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{\mathbb{R}^d} dy' \exp(iy^T y')K^*(\frac{Y_i - Y_j}{H} - y') \\
&= (2\pi)^{-d/2}\frac{1}{N^2H^d}\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{\mathbb{R}^d} dy'' \exp(iy^T(\frac{Y_i - Y_j}{H} - y''))K^*(y'') \\
&= (2\pi)^{-d/2}\frac{1}{N^2H^d}\sum_{i=1}^{N}\sum_{j=1}^{N} \exp(iy^T\frac{Y_i - Y_j}{H})\int_{\mathbb{R}^d} dy'' \exp(-iy^T y'')K^*(y'') \\
&= \frac{1}{H^d}v(\frac{y}{H},Y)\int_{\mathbb{R}^d} dy' \exp(-iy^T y')K^*(y') \\
&= \frac{1}{H^d}v(\frac{y}{H},Y)\int_{\mathbb{R}^d} dy' \exp(iy^T y')K^*(y') = \frac{1}{H^d}(2\pi)^{d/2}v(\frac{y}{H},Y)\tilde{K}^*(y) \\
&= (\frac{2\pi}{H})^d|u(\frac{y}{H},Y)|^2\tilde{K}^*(y)\ ,
\end{aligned} \tag{A.134}$$

where we also used the fact that K^* is even. Of course by (A.51)

$$\begin{aligned}
&(2\pi)^{-d/2}\int_{\mathbb{R}^d} dy\widetilde{\widehat{LSCV}}_{gen}(y,H,Y) \\
&= (2\pi)^{-d}\int_{\mathbb{R}^d} dy\int_{\mathbb{R}^d} dy' \exp(iy^T y')\widehat{LSCV}_{gen}(y',H,Y) \\
&= (2\pi)^{-d}\int_{\mathbb{R}^d} dy'\widehat{LSCV}_{gen}(y',H,Y)\int_{\mathbb{R}^d} dy \exp(iy^T y') \\
&= \int_{\mathbb{R}^d} dy'\widehat{LSCV}_{gen}(y',H,Y)\delta(y') \\
&= \widehat{LSCV}_{gen}(0,H,Y)\ ,
\end{aligned} \tag{A.135}$$

whence by (A.134)

$$\widehat{LSCV}_{gen}(0,H,Y) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} dy \widetilde{\widehat{LSCV}}_{gen}(y,H,Y)$$
$$= \frac{1}{H^d}(2\pi)^{d/2}\int_{\mathbb{R}^d} dy |u(\frac{y}{H},Y)|^2 \tilde{K}^*(y)$$
$$= (2\pi)^{d/2}\int_{\mathbb{R}^d} dy' |u(y',Y)|^2 \tilde{K}^*(Hy') \ , \tag{A.136}$$

so that by (A.131)

$$\widehat{LSCV}(H,Y) = (2\pi)^{d/2}\int_{\mathbb{R}^d} dy |u(y,Y)|^2 \tilde{K}^*(Hy) + \frac{2}{NH^d}K(0) \ . \tag{A.137}$$

We obtain from (A.118),(A.51)

$$\tilde{K}^{(2)}(y) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} dy' \exp(iy^T y') K^{(2)}(y')$$
$$= (2\pi)^{-d/2}\int_{\mathbb{R}^d} dy' \exp(iy^T y') \int_{\mathbb{R}^d} dy'' K(y'-y'')K(y'')$$
$$= (2\pi)^{-d/2}\int_{\mathbb{R}^d} dy'' K(y'') \int_{\mathbb{R}^d} dy' \exp(iy^T y') K(y'-y'')$$
$$= (2\pi)^{-d/2}\int_{\mathbb{R}^d} dy'' K(y'') \int_{\mathbb{R}^d} dy' \exp(iy^T (y'+y'')) K(y')$$
$$= (2\pi)^{-d/2}\int_{\mathbb{R}^d} dy'' \exp(iy^T y'') K(y'') \int_{\mathbb{R}^d} dy' \exp(iy^T y') K(y')$$
$$= \tilde{K}(y)\int_{\mathbb{R}^d} dy' \exp(iy^T y') K(y') = (2\pi)^{d/2}\tilde{K}^2(y) \ , \tag{A.138}$$

whence by (A.125),(A.51)

$$\tilde{K}^*(y) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} dy' \exp(iy^T y') K^*(y')$$
$$= (2\pi)^{-d/2}\int_{\mathbb{R}^d} dy' \exp(iy^T y')(K^{(2)}(y') - 2K(y')) = \tilde{K}^{(2)}(y) - 2\tilde{K}(y)$$
$$= (2\pi)^{d/2}\tilde{K}^2(y) - 2\tilde{K}(y) \ . \tag{A.139}$$

It follows from (A.137),(A.139) that

$$\widehat{LSCV}(H,Y) = (2\pi)^{d/2}\int_{\mathbb{R}^d} dy |u(y,Y)|^2 ((2\pi)^{d/2}\tilde{K}^2(Hy) - 2\tilde{K}(Hy))$$
$$+\frac{2}{NH^d}K(0) \ . \tag{A.140}$$

Note that (A.140) is exact and that it boils the computation of $\widehat{LSCV}$ down to an integration problem.

We thus move on with Silverman's approach by approximating $\widehat{LSCV}$ via linear binning to the equality (A.140). To keep the formalism concise and due to our aims we confine to the case $d = 2$ and we do linear binning where each point contributes to four grid points (this version of linear binning is called 'cloud-in-cell charge deposition' in Physics and it is employed by our density estimation 'Method 2' mentioned in Section 3.4). Using the definition (A.54) of the grid points $y_{\alpha,\beta}$ we partition $\mathbb{R}^2$ into the rectangles:

$$I_{\alpha,\beta} := [z_\alpha, z_{\alpha+1}) \times [x_\beta, z_{\beta+1}) = [\frac{\alpha}{m}, \frac{\alpha+1}{m}) \times [\frac{\beta}{n}, \frac{\beta+1}{n}) \ , \qquad (\alpha, \beta \in \mathbb{Z}) \ . \tag{A.141}$$

For convenience we assume that the integers m, n are even. We define for $j = 1, ..., N$

$$M_j := \underline{Int}(mZ_j) \ , \qquad N_j := \underline{Int}(nX_j) \ , \tag{A.142}$$

where $\underline{Int}$ denotes the greatest lower integer bound function on the reals and where the Z_j, X_j are given by (A.55). Thus (M_j, N_j) labels the rectangle surrounding Y_j, i.e., $I_{(M_j,N_j)}$ is the unique rectangle from the partition which contains Y_j. In particular the grid points $y_{M_j,N_j}, y_{M_j+1,N_j}, y_{M_j,N_j+1}, y_{M_j+1,N_j+1}$ are the left lower, right lower, upper left, upper right corner respectively of the rectangle $I_{(M_j,N_j)}$. Note that right, left, lower, upper are meant w.r.t. the convention where the z-axis is horizontal and the x-axis is vertical. The linear binning we consider here is the procedure where the 'unit charge' at Y_j is replaced ('deposited') by four fractional 'charges': the 'charge' $w_{LL,j}$ at y_{M_j,N_j}, the 'charge' $w_{LR,j}$ at y_{M_j+1,N_j}, the 'charge' $w_{UL,j}$ at y_{M_j,N_j+1}, and the 'charge' $w_{UR,j}$ at y_{M_j+1,N_j+1} where we define

$$\begin{aligned} w_{LL,j} &:= [1 - m(Z_j - z_{M_j})][1 - n(X_j - x_{N_j})] \ , \\ w_{LR,j} &:= m(Z_j - z_{M_j})[1 - n(X_j - x_{N_j})] \ , \\ w_{UL,j} &:= [1 - m(Z_j - z_{M_j})]n(X_j - x_{N_j}) \\ w_{UR,j} &:= m(Z_j - z_{M_j})n(X_j - x_{N_j}) \ . \end{aligned} \tag{A.143}$$

The interpretation of $w_{LL,j}, w_{LR,j}, w_{UL,j}, w_{UR,j}$ as charges will now be justified by proving (A.144),(A.150). It follows from (A.143) that $w_{LL,j} + w_{LR,j} = 1 - n(X_j - x_{N_j})$ and $w_{UL,j} + w_{UR,j} = n(X_j - x_{N_j})$, whence

$$w_{LL,j} + w_{LR,j} + w_{UL,j} + w_{UR,j} = 1 \ . \tag{A.144}$$

Moreover by (A.142) we have

$$0 \leq mZ_j - \underline{Int}(mZ_j) < 1 \ , \tag{A.145}$$

and by (A.54),(A.142)

$$m(Z_j - z_{M_j}) = mZ_j - M_j = mZ_j - \underline{Int}(mZ_j) \ , \tag{A.146}$$

whence

$$0 \leq m(Z_j - z_{M_j}) < 1 \ , \tag{A.147}$$

and analogously

$$0 \leq n(X_j - x_{N_j}) < 1 \ . \tag{A.148}$$

It follows from (A.147),(A.148) that

$$0 < 1 - m(Z_j - z_{M_j}) \leq 1 \ , \qquad 0 < 1 - n(X_j - x_{N_j}) \leq 1 \ . \tag{A.149}$$

We conclude from (A.143),(A.147),(A.148), (A.149)

$$0 < w_{LL,j} \leq 1 \;,\;\; 0 \leq w_{LR,j} < 1 \;,\;\; 0 \leq w_{UL,j} < 1 \;,\;\; 0 \leq w_{UR,j} < 1 \;. \quad \text{(A.150)}$$

Note that the $w_{LL,j}$ are independent identically distributed (the some holds for the $w_{LR,j}, w_{UL,j}, w_{UR,j}$ respectively). The above linear binning procedure can be interpreted in terms of probability measures as follows. The empiral measure, μ_1, determined by $Y_1, ..., Y_N$ is replaced in linear binning by the measure μ_2 where

$$\mu_1 := \frac{1}{N}\sum_{j=1}^{N} \varepsilon_{Y_j} \;, \quad \text{(A.151)}$$

$$\mu_2 := \frac{1}{N}\sum_{j=1}^{N} \left(w_{LL,j}\varepsilon_{y_{M_j,N_j}} + w_{LR,j}\varepsilon_{y_{M_j+1,N_j}} + w_{UL,j}\varepsilon_{y_{M_j,N_j+1}} + w_{UR,j}\varepsilon_{y_{M_j+1,N_j+1}}\right) , \quad \text{(A.152)}$$

where ε_y denotes the unit point measure at $y \in \mathbb{R}^2$. We define for $\alpha, \beta \in \mathbb{Z}$

$$\xi_{\alpha,\beta} := \frac{mn}{N}\sum_{j=1}^{N} \Big(w_{LL,j}\delta_{M_j-\alpha,N_j-\beta} + w_{LR,j}\delta_{M_j+1-\alpha,N_j-\beta} + w_{UL,j}\delta_{M_j-\alpha,N_j+1-\beta} + w_{UR,j}\delta_{M_j+1-\alpha,N_j+1-\beta}\Big) \;, \quad \text{(A.153)}$$

where $\delta_{\alpha,\beta}$ is the Kronecker symbol. Note that by (A.144),(A.153)

$$\begin{aligned}\sum_{\alpha\in\mathbb{Z}}\sum_{\beta\in\mathbb{Z}} \xi_{\alpha,\beta} &= \frac{mn}{N}\sum_{j=1}^{N}\sum_{\alpha\in\mathbb{Z}}\sum_{\beta\in\mathbb{Z}} \Big(w_{LL,j}\delta_{M_j-\alpha,N_j-\beta} + w_{LR,j}\delta_{M_j+1-\alpha,N_j-\beta} \\ &\quad + w_{UL,j}\delta_{M_j-\alpha,N_j+1-\beta} + w_{UR,j}\delta_{M_j+1-\alpha,N_j+1-\beta}\Big) \\ &= \frac{mn}{N}\sum_{j=1}^{N}(w_{LL,j} + w_{LR,j} + w_{UL,j} + w_{UR,j}) = \frac{mn}{N}\sum_{j=1}^{N} 1 = mn \;. \quad \text{(A.154)}\end{aligned}$$

We see by (A.153) (or by (A.152)) that $\xi_{\alpha,\beta}$ is proportional to the number of particles binned at $y_{\alpha,\beta}$. In fact (A.154) shows that the proportionality constant is N/mn, i.e., $N\xi_{\alpha,\beta}/mn$ is the number of particles binned at $y_{\alpha,\beta}$. Note that this number in general is not an integer since the binning procedure 'splits' each original particle into four particles. Defining

$$\check{I}_{\alpha,\beta} := [\frac{2\alpha-1}{2m}, \frac{2\alpha+1}{2m}) \times [\frac{2\beta-1}{2n}, \frac{2\beta+1}{2n}) \;, \qquad (\alpha, \beta \in \mathbb{Z}) \;. \quad \text{(A.155)}$$

we observe that the $\check{I}_{\alpha,\beta}$ form a partition of $\mathbb{R}^2$ into rectangles whose center points are $y_{\alpha,\beta}$ and that $1/mn$ is the volume of each $\check{I}_{\alpha,\beta}$. Note that the $\check{I}_{\alpha,\beta}$ are just translates of the $I_{\alpha,\beta}$. Since $N\xi_{\alpha,\beta}/mn$ is the number of particles binned at $y_{\alpha,\beta}$ it is also the number of particles binned in $\check{I}_{\alpha,\beta}$. Note that $y_{\alpha,\beta}$ is the only grid point in $\check{I}_{\alpha,\beta}$. On the other hand, since $1/mn$ is the volume of $\check{I}_{\alpha,\beta}$, the quantity $Nf(y_{\alpha,\beta})/mn$ approximates the number of particles in $\check{I}_{\alpha,\beta}$, i.e.,

$$\frac{Nf(y_{\alpha,\beta})}{mn} \approx \sum_{j=1}^{N} 1_{\check{I}_{\alpha,\beta}}(Y_j) \;. \quad \text{(A.156)}$$

We conclude that $Nf(y_{\alpha,\beta})/mn \approx N\xi_{\alpha,\beta}/mn$, i.e.,

$$f(y_{\alpha,\beta}) \approx \xi_{\alpha,\beta} \;. \quad \text{(A.157)}$$

This allows us to apply the midpoint rule w.r.t. the partition $\check{I}_{\alpha,\beta}$ which reads for a real valued function F on $\mathbb{R}^2$ as

$$\int_{\mathbb{R}^2} dy f(y) F(y) \approx \frac{1}{mn} \sum_{\alpha\in\mathbb{Z}} \sum_{\beta\in\mathbb{Z}} f(y_{\alpha,\beta}) F(y_{\alpha,\beta}) \;. \quad \text{(A.158)}$$

Applying (A.158) to (A.157) results in

$$\int_{\mathbb{R}^2} dy f(y) F(y) \approx \frac{1}{mn} \sum_{\alpha\in\mathbb{Z}} \sum_{\beta\in\mathbb{Z}} \xi_{\alpha,\beta} F(y_{\alpha,\beta}) \;. \quad \text{(A.159)}$$

On the other hand by the law of large numbers we have the Monte Carlo approximation

$$\int_{\mathbb{R}^2} dy f(y) F(y) \approx \frac{1}{N} \sum_{j=1}^{N} F(Y_j) \;, \quad \text{(A.160)}$$

whence by (A.159)

$$\frac{1}{N}\sum_{j=1}^{N} F(Y_j) \approx \frac{1}{mn}\sum_{\alpha\in\mathbb{Z}}\sum_{\beta\in\mathbb{Z}} \xi_{\alpha,\beta}F(y_{\alpha,\beta}) \ . \tag{A.161}$$

We will see below (see (A.163)) that we are interested in the case where $F(y) := \exp(iy^T t)$ where $t \in \mathbb{R}^2$. Applying then (A.161) we get

$$\frac{1}{N}\sum_{j=1}^{N} \exp(iy^T Y_j) \approx \frac{1}{mn}\sum_{\alpha\in\mathbb{Z}}\sum_{\beta\in\mathbb{Z}} \xi_{\alpha,\beta}\exp(iy^T y_{\alpha,\beta}) \ , \tag{A.162}$$

whence by (A.132)

$$u(y,Y) = \frac{1}{2\pi N}\sum_{j=1}^{N} \exp(iy^T Y_j) \approx \frac{1}{2\pi mn}\sum_{\alpha\in\mathbb{Z}}\sum_{\beta\in\mathbb{Z}} \xi_{\alpha,\beta}\exp(iy^T y_{\alpha,\beta}) \ , \tag{A.163}$$

so that by (A.140)

$$\begin{aligned}\widehat{LSCV}(H,Y) &= 2\pi\int_{\mathbb{R}^2} dy|u(y,Y)|^2(2\pi\tilde{K}^2(Hy) - 2\tilde{K}(Hy)) + \frac{2}{NH^2}K(0) \\ &\approx \frac{1}{2\pi m^2 n^2}\int_{\mathbb{R}^2} dy|\sum_{\alpha\in\mathbb{Z}}\sum_{\beta\in\mathbb{Z}} \xi_{\alpha,\beta}\exp(iy^T y_{\alpha,\beta})|^2(2\pi\tilde{K}^2(Hy) - 2\tilde{K}(Hy)) \\ &\quad + \frac{2}{NH^2}K(0) \ . \end{aligned} \tag{A.164}$$

Moving on with Silverman's procedure the integral in (A.164) will be approximated by the midpoint rule as follows. Defining for $a, b \in \mathbb{Z}$

$$s_{a,b} := 2\pi(a,b)^T \ , \quad \hat{I}_{a,b} := [\pi(2a-1), \pi(2a+1)) \times [\pi(2b-1), \pi(2b+1)) \ , \tag{A.165}$$

we observe that the $\hat{I}_{a,b}$ form a partition of $\mathbb{R}^2$ into squares whose center points are $s_{a,b}$ and that $4\pi^2$ is the volume of $\hat{I}_{a,b}$ whence the midpoint rule gives us for a real valued function F on $\mathbb{R}^2$

$$\int_{\mathbb{R}^2} dyF(y) \approx 4\pi^2\sum_{a\in\mathbb{Z}}\sum_{b\in\mathbb{Z}} F(s_{\alpha,\beta}) \ . \tag{A.166}$$

Thus approximating (A.164) by the midpoint rule results in

$$\widehat{LSCV}(H,Y) \approx \frac{2\pi}{m^2n^2}\sum_{a\in\mathbb{Z}}\sum_{b\in\mathbb{Z}}|\sum_{\alpha\in\mathbb{Z}}\sum_{\beta\in\mathbb{Z}}\xi_{\alpha,\beta}\exp(is_{a,b}^T y_{\alpha,\beta})|^2$$
$$\cdot(2\pi\tilde{K}^2(Hs_{a,b}) - 2\tilde{K}(Hs_{a,b})) + \frac{2}{NH^2}K(0)$$
$$= \frac{2\pi}{m^2n^2}\sum_{a\in\mathbb{Z}}\sum_{b\in\mathbb{Z}}|\sum_{\alpha\in\mathbb{Z}}\sum_{\beta\in\mathbb{Z}}\xi_{\alpha,\beta}\exp(2\pi i(\frac{a\alpha}{m}+\frac{b\beta}{n}))|^2$$
$$\cdot(2\pi\tilde{K}^2(Hs_{a,b}) - 2\tilde{K}(Hs_{a,b})) + \frac{2}{NH^2}K(0)\ . \qquad \text{(A.167)}$$

Defining the discrete Fourier transform of ξ for $a, b \in \mathbb{Z}$ by

$$\eta_{a,b} := \frac{1}{mn}\sum_{\alpha\in\mathbb{Z}}\sum_{\beta\in\mathbb{Z}}\xi_{\alpha,\beta}\exp(2\pi i(\frac{a\alpha}{m}+\frac{b\beta}{n}))\ , \qquad \text{(A.168)}$$

we obtain from (A.167)

$$\widehat{LSCV}(H,Y) \approx 2\pi\sum_{a\in\mathbb{Z}}\sum_{b\in\mathbb{Z}}|\eta_{a,b}|^2(2\pi\tilde{K}^2(Hs_{a,b}) - 2\tilde{K}(Hs_{a,b})) + \frac{2}{NH^2}K(0)\ . \qquad \text{(A.169)}$$

To make $\widehat{LSCV}$ amenable to the Fast Fourier Transform technique we now truncate the infinite sequence in (A.169) which is justified if the Y_j are concentrated in $[0,1]\times[0,1]$ (which is the case in the situation we are interested in). Then we have

$$1 - \frac{1}{N}\sum_{j=1}^{N} 1_{[0,1]\times[0,1]}(Y_j) \ll 1\ . \qquad \text{(A.170)}$$

Note that by (A.153),(A.170)

$$1 - \frac{1}{mn}\sum_{\alpha=0}^{m-1}\sum_{\beta=0}^{n-1}\xi_{\alpha,\beta} \ll 1\ . \qquad \text{(A.171)}$$

The lhs of (A.170),(A.171) are always nonnegative. Definining for $a, b \in \mathbb{Z}$

$$\hat{\eta}_{a,b} := \frac{1}{mn}\sum_{\alpha=0}^{m-1}\sum_{\beta=0}^{n-1}\xi_{\alpha,\beta}\exp(2\pi i(\frac{a\alpha}{m}+\frac{b\beta}{n}))\ , \qquad \text{(A.172)}$$

we observe by (A.168),(A.171) that

$$\eta_{a,b} \approx \hat{\eta}_{a,b} \ . \tag{A.173}$$

Thus replacing $\eta_{a,b}$ by $\hat{\eta}_{a,b}$ in (A.169) we obtain

$$\widehat{LSCV}(H,Y) \approx 2\pi \sum_{a\in\mathbb{Z}} \sum_{b\in\mathbb{Z}} |\hat{\eta}_{a,b}|^2 (2\pi \tilde{K}^2(Hs_{a,b}) - 2\tilde{K}(Hs_{a,b})) + \frac{2}{NH^2} K(0) \ . \tag{A.174}$$

Following Silverman, we restrict the a, b in (A.174) to the range $a = -m/2, ..., m/2$ and $b = -n/2, ..., n/2$ whence we obtain

$$\begin{aligned} &\widehat{LSCV}(H,Y) \approx LSCV_{Sil}(H,Y) \ , \\ &LSCV_{Sil}(H,Y) := 2\pi \sum_{a=-m/2}^{m/2} \sum_{b=-n/2}^{n/2} |\hat{\eta}_{a,b}|^2 (2\pi \tilde{K}^2(Hs_{a,b}) - 2\tilde{K}(Hs_{a,b})) \\ &\quad + \frac{2}{NH^2} K(0) \ . \end{aligned} \tag{A.175}$$

To apply the Fast Fourier Transform technique to the computation of $LSCV_{Sil}$, it can be convenient to have, in (A.175), the indices a, b starting at $a = 0, b = 0$ rather than at $a = -m/2, b = -n/2$. Thus we define for $a, b \in \mathbb{Z}$

$$\begin{aligned} \check{\eta}_{a,b} &:= \hat{\eta}_{a-m/2,b-n/2} = \frac{1}{mn} \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{n-1} \xi_{\alpha,\beta} \exp(2\pi i (\frac{(a-m/2)\alpha}{m} + \frac{(b-n/2)\beta}{n})) \\ &= \frac{1}{mn} \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{n-1} \xi_{\alpha,\beta} (-1)^{\alpha+\beta} \exp(2\pi i (\frac{a\alpha}{m} + \frac{b\beta}{n})) \ , \end{aligned} \tag{A.176}$$

whence by (A.175) we get the following alternative expression of $LSCV_{Sil}$:

$$\begin{aligned} LSCV_{Sil}(H,Y) &= 2\pi \sum_{a=0}^{m} \sum_{b=0}^{n} |\check{\eta}_{a,b}|^2 (2\pi \tilde{K}^2(Hs_{a-m/2,b-n/2}) - 2\tilde{K}(Hs_{a-m/2,b-n/2})) \\ &\quad + \frac{2}{NH^2} K(0) \ . \end{aligned} \tag{A.177}$$

Since the computational cost of each $\xi_{\alpha,\beta}$ is of order N, it follows from (A.176) that the computational cost of each $\check{\eta}_{a,b}$ is of order N, whence by (A.177) the computational cost of $LSCV_{Sil}$ is of order N. On the other hand since $LSCV_{Sil}$ approximates

$\widehat{LSCV}$ and $\widehat{LSCV}$ approximates the unbiased estimator $LSCV$ of $RMISE$ we define the estimator of H_{MISE} by

$$\hat{H}_{MISE} := argmin_{H>0}(LSCV_{Sil}(H, Y)) \; . \tag{A.178}$$

A.3.7 Practical considerations

I implemented the bivariate product Epanechnikov kernel $K_{C1,2D,P}$ into our code by using algorithm A2 of Section A.3.2. The accuracy obtained with this kernel, tested with the known initial spatial density ρ_g of Section 3.4, is competitive with that of the density estimation Methods 1 and 2 of Section 3.4. Moreover, in terms of computational cost, $K_{C1,2D,P}$ is competitive with density estimation Method 2 and outperforms density estimation Method 1. For more details on the performance of the kernel density density estimator in our code, see Section 3.4.3. We next aim to implement the cross validation formula (A.178). Another issue to be addressed is the fact that, in the situation of our code, the random variables $Y_1, ..., Y_N$ are not independent anymore when the code marches forward in s (although they are initially independent). However the dependence of the $Y_1, ..., Y_N$ may be weak and the $Y_1, ..., Y_N$ may still be identically distributed when the code marches forward in s. Note also that since in the previous sections we assumed that $Y_1, ..., Y_N$ are independent identically distributed, some results change when the $Y_1, ..., Y_N$ are dependent (in particular the asymptotic formulas for $MISE$ will change). Thus we plan to implement a routine in the code which quantifies the dependence of the $Y_1, ..., Y_N$ and tests if they are identically distributed.

A.4 Convergence study

I now discuss a technique, which is applied in Section 3.4.3 and which allows a convergence study of the error of various quantities computed by the code. We here concentrate on a convergence study w.r.t. the parameter $\mathcal{N}$, i.e., the particle number. Thus let Ψ be a normed space and let $\psi \in \Psi$ be an unknown element approximated by the elements $\psi(\mathcal{N}) \in \Psi$ where $\psi(\mathcal{N})$ denotes the approximant of ψ computed with $\mathcal{N}$ particles. Underlying the method is the assumption that, for $\mathcal{N} \to \infty$, the error $||\psi - \psi(\mathcal{N})||$ satisfies $||\psi - \psi(\mathcal{N})|| = \mathcal{O}(\mathcal{N}^{-d})$ where $d > 0$ is called the 'consistency order' of the approximant $\psi(\mathcal{N})$. Thus, by assumption, a $c > 0$ exists such that for large $\mathcal{N}$ we have

$$||\psi - \psi(\mathcal{N})|| \approx c\mathcal{N}^{-d} \ . \tag{A.179}$$

In fact the method we outline here allows to approximate d in terms of the $\psi(\mathcal{N})$ to arbitrary accuracy. Using the triangle inequality we have for arbitrary particle numbers $\mathcal{N}, \mathcal{N}'$

$$\begin{aligned} ||\psi - \psi(\mathcal{N})|| - ||\psi - \psi(\mathcal{N}')|| &\leq ||\psi(\mathcal{N}) - \psi(\mathcal{N}')|| \\ &\leq ||\psi - \psi(\mathcal{N})|| + ||\psi - \psi(\mathcal{N}')|| \ , \end{aligned}$$

whence, for particle numbers $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4$ which are constrained by

$$\mathcal{N}_2 \neq \mathcal{N}_4 \ , \tag{A.180}$$

we get

$$\begin{aligned} \frac{||\psi - \psi(\mathcal{N}_1)|| - ||\psi - \psi(\mathcal{N}_3)||}{||\psi - \psi(\mathcal{N}_2)|| + ||\psi - \psi(\mathcal{N}_4)||} &\leq \frac{||\psi(\mathcal{N}_1) - \psi(\mathcal{N}_3)||}{||\psi(\mathcal{N}_2) - \psi(\mathcal{N}_4)||} \\ &\leq \frac{||\psi - \psi(\mathcal{N}_1)|| + ||\psi - \psi(\mathcal{N}_3)||}{||\psi - \psi(\mathcal{N}_2)|| - ||\psi - \psi(\mathcal{N}_4)||} \ . \end{aligned} \tag{A.181}$$

If $\mathcal{N}_1, \mathcal{N}_3, \mathcal{N}_2, \mathcal{N}_4$ are sufficiently large then, by (A.179), we have, for $i = 1, 2, 3, 4$, $||\psi - \psi(\mathcal{N}_i)|| \approx c\mathcal{N}_i^{-d}$ whence

$$\frac{||\psi - \psi(\mathcal{N}_1)|| \mp ||\psi - \psi(\mathcal{N}_3)||}{||\psi - \psi(\mathcal{N}_2)|| \pm ||\psi - \psi(\mathcal{N}_4)||} \approx \frac{\mathcal{N}_1^{-d} \mp (\mathcal{N}_3)^{-d}}{(\mathcal{N}_2)^{-d} \pm (\mathcal{N}_4)^{-d}} = (\mathcal{N}_2/\mathcal{N}_1)^d \frac{1 \mp (\mathcal{N}_3/\mathcal{N}_1)^{-d}}{1 \pm (\mathcal{N}_4/\mathcal{N}_2)^{-d}} \ ,$$

so that by (A.181)

$$(\mathcal{N}_2/\mathcal{N}_1)^d \frac{1-(\mathcal{N}_3/\mathcal{N}_1)^{-d}}{1+(\mathcal{N}_4/\mathcal{N}_2)^{-d}} \lessapprox \frac{||\psi(\mathcal{N}_1)-\psi(\mathcal{N}_3)||}{||\psi(\mathcal{N}_2)-\psi(\mathcal{N}_4)||} \lessapprox (\mathcal{N}_2/\mathcal{N}_1)^d \frac{1+(\mathcal{N}_3/\mathcal{N}_1)^{-d}}{1-(\mathcal{N}_4/\mathcal{N}_2)^{-d}} \, . \tag{A.182}$$

Assuming in addition to (A.180) that

$$\mathcal{N}_3 > \mathcal{N}_1 \, , \qquad \mathcal{N}_4 > \mathcal{N}_2 \, , \tag{A.183}$$

we can take the logarithm in (A.182) and obtain

$$\begin{aligned} d\ln(\mathcal{N}_2/\mathcal{N}_1) + \ln(\frac{1-(\mathcal{N}_3/\mathcal{N}_1)^{-d}}{1+(\mathcal{N}_4/\mathcal{N}_2)^{-d}}) &\lessapprox \ln(\frac{||\psi(\mathcal{N}_1)-\psi(\mathcal{N}_3)||}{||\psi(\mathcal{N}_2)-\psi(\mathcal{N}_4)||}) \\ &\lessapprox d\ln(\mathcal{N}_2/\mathcal{N}_1) + \ln(\frac{1+(\mathcal{N}_3/\mathcal{N}_1)^{-d}}{1-(\mathcal{N}_4/\mathcal{N}_2)^{-d}}) \, . \end{aligned} \tag{A.184}$$

We will exploit (A.184) to approximate d whence, from now on, we assume, in addition to (A.180), (A.183), that $\mathcal{N}_1 \neq \mathcal{N}_2$. Without loss of generality we thus assume that

$$\mathcal{N}_2 > \mathcal{N}_1 \, , \tag{A.185}$$

whence, by (A.183),

$$\mathcal{N}_4 > \mathcal{N}_2 > \mathcal{N}_1 \, , \qquad \mathcal{N}_3 > \mathcal{N}_1 \, . \tag{A.186}$$

Dividing (A.184) by $\ln(\mathcal{N}_2/\mathcal{N}_1)$, we obtain

$$\begin{aligned} d + \frac{1}{\ln(\mathcal{N}_2/\mathcal{N}_1)} \ln(\frac{1-(\mathcal{N}_3/\mathcal{N}_1)^{-d}}{1+(\mathcal{N}_4/\mathcal{N}_2)^{-d}}) &\lessapprox \tilde{d} \\ &\lessapprox d + \frac{1}{\ln(\mathcal{N}_2/\mathcal{N}_1)} \ln(\frac{1+(\mathcal{N}_3/\mathcal{N}_1)^{-d}}{1-(\mathcal{N}_4/\mathcal{N}_2)^{-d}}) \, , \end{aligned} \tag{A.187}$$

where

$$\tilde{d} := \frac{1}{\ln(\mathcal{N}_2/\mathcal{N}_1)} \ln(\frac{||\psi(\mathcal{N}_1)-\psi(\mathcal{N}_3)||}{||\psi(\mathcal{N}_2)-\psi(\mathcal{N}_4)||}) \, . \tag{A.188}$$

Note that $\tilde{d}$ is the promised approximant of d. Choosing $\mathcal{N}_3/\mathcal{N}_1$ and $\mathcal{N}_4/\mathcal{N}_2$ sufficiently large, we can Taylor expand (A.187) w.r.t. $(\mathcal{N}_3/\mathcal{N}_1)^{-d}$ and $(\mathcal{N}_4/\mathcal{N}_2)^{-d}$ which results in

$$d - \frac{(\mathcal{N}_3/\mathcal{N}_1)^{-d} + (\mathcal{N}_4/\mathcal{N}_2)^{-d}}{\ln(\mathcal{N}_2/\mathcal{N}_1)} \lessapprox \tilde{d} \lessapprox d + \frac{(\mathcal{N}_3/\mathcal{N}_1)^{-d} + (\mathcal{N}_4/\mathcal{N}_2)^{-d}}{\ln(\mathcal{N}_2/\mathcal{N}_1)} ,$$

i.e.,

$$|d - \tilde{d}| \lessapprox \frac{(\mathcal{N}_3/\mathcal{N}_1)^{-d} + (\mathcal{N}_4/\mathcal{N}_2)^{-d}}{\ln(\mathcal{N}_2/\mathcal{N}_1)} . \tag{A.189}$$

To estimate the relative error, $|1 - \tilde{d}/d|$, made by $\tilde{d}$ we conclude from (A.189)

$$|1 - \frac{\tilde{d}}{d}| \lessapprox \frac{(\mathcal{N}_3/\mathcal{N}_1)^{-d} + (\mathcal{N}_4/\mathcal{N}_2)^{-d}}{d \ln(\mathcal{N}_2/\mathcal{N}_1)} . \tag{A.190}$$

Note that (A.186) contains all restrictions on $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4$ we made so far. Of course it follows from (A.186) that if $\mathcal{N}_1$ is sufficiently large such that (A.179) is a good approximation for $\mathcal{N} = \mathcal{N}_1$, then all $\mathcal{N}_i$ are sufficiently large such that (A.179) is a good approximation for $\mathcal{N} = \mathcal{N}_i$. If one imposes, for some $\varepsilon > 0$, the condition:

$$\frac{(\mathcal{N}_3/\mathcal{N}_1)^{-d} + (\mathcal{N}_4/\mathcal{N}_2)^{-d}}{d \ln(\mathcal{N}_2/\mathcal{N}_1)} \leq \varepsilon , \tag{A.191}$$

then, by (A.190), we get

$$|1 - \frac{\tilde{d}}{d}| \lessapprox \varepsilon . \tag{A.192}$$

Clearly (A.186) is equivalent to

$$\mathcal{N}_2 = k_1 \mathcal{N}_1 , \qquad \mathcal{N}_3 = k_2 \mathcal{N}_1 , \qquad \mathcal{N}_4 = k_3 \mathcal{N}_2 , \qquad k_1, k_2, k_3 > 1 , \tag{A.193}$$

which leaves $\mathcal{N}_1, k_1, k_2, k_3$ as the free parameters. Adding to (A.193) the constraints: $k_1 \leq k_2 = k_3$ we get the following convenient choice of $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4$:

$$\mathcal{N}_2 = k_1 \mathcal{N}_1 , \qquad \mathcal{N}_3 = k_2 \mathcal{N}_1 , \qquad \mathcal{N}_4 = k_1 k_2 \mathcal{N}_1 , \qquad k_2 \geq k_1 > 1 , \tag{A.194}$$

which leaves $\mathcal{N}_1, k_1, k_2$ as the only free parameters. Note that (A.194) entails (A.193) and also gives us the ordering

$$\mathcal{N}_4 > \mathcal{N}_3 \geq \mathcal{N}_2 > \mathcal{N}_1 . \tag{A.195}$$

Appendix B

Group actions

If X is a set, G a group with identity e_G and $L : G \times X \to X$ a function satisfying, for $g, h \in G, x \in X$,

$$L(e_G; x) = x \tag{B.1}$$

$$L(gh; x) = L(g; L(h; x)) \ , \tag{B.2}$$

then L is called a 'left G-action on X' and the pair (X, L) is called a 'left G-space'. Note that the group law of G is written multiplicatively in (B.2) and it is obvious how (B.2) would read if the group law of G is written additively (the latter convention is common if the group G is Abelian). It follows from (B.1),(B.2) that each $L(g; \cdot)$ is a bijection from X onto X. A left G-action L on X is called 'transitive' if for every pair of elements x, y of X a $g \in G$ exists such that $L(g; x) = y$. If G, G' are groups and $\Phi : G \to G'$ is a group homomorphism and if (X, L') is a left G'-space then (X, L) is a left G-space where I define, for $g \in G, x \in X$,

$$L(g; x) := L'(\Phi(g); x) \ . \tag{B.3}$$

In this work a topological group is defined in the common, broad sense as in [Hus]. If X is a topological space, G is a topological group, and (X, L) is a left G-space such that the L is continuous, then (X, L) is called a 'topological left G-space'. Of course in that case each $L(g; \cdot)$ is a homeomorphism from X onto X. In the important subcase when the topology of G is discrete (e.g., when $G = \mathbb{Z}$) the condition that L is continuous is equivalent to $L(g; \cdot)$ being continuous for all $g \in G$.

If $(X, L), (X', L')$ are left G-spaces and if $f : X \to X'$ is a function satisfying, for $g \in G, x \in X$,

$$f(L(g; x)) = L'(g; f(x)) \ , \tag{B.4}$$

then f is called a 'G-map from (X, L) to (X', L')'. G-maps are also called 'equivariant'. One calls $(X, L), (X', L')$ 'conjugate' if the G-map f is a bijection onto X'. In the special case $G = \mathbb{Z}$ the function f is a G-map iff (B.4) holds just for $g = 1, x \in X$.

If the G-map f is onto X' then the left G-space (X, L) is called an 'extension of the left G-space (X', L')'. In the special case where the extension (X, L) has the form $(X' \times Y, L)$ for some set Y and if f is the natural projection from from $X' \times Y$ onto X', then the left G-space (X, L) is called a 'skew product of the left G-space (X', L')'.

Remark:

(1) Let (X', L'), $(X' \times Y, L)$ be left G-spaces and let $(X' \times Y, L)$ be a skew product of (X', L'). This is a strong restriction on L, as follows.

By (B.2), we have, for $g \in G, x' \in X', y \in Y$,

$$L(g; x', y) = \begin{pmatrix} L'(g; x') \\ L''(g; x', y) \end{pmatrix} \ , \tag{B.5}$$

where the function $L'' : G \times X' \times Y \to Y$ satisfies, for $g, h \in G, x' \in X', y \in Y$,

$$L''(e_G; x', y) = y \ , \tag{B.6}$$

$$L''(gh; x', y) = L''(g; L'(h; x'), L''(h; x', y)) \ , \tag{B.7}$$

which is the announced restriction on L. □

If $(X, L), (X', L')$ are topological left G-spaces and if a continuous G-map f exists from (X, L) to (X', L') which is a homeomorphism onto X', then the topological left G-spaces $(X, L), (X', L')$ are called 'conjugate'. If $(X, L), (X', L')$ are topological left G-spaces and if a continuous G-map f exists from (X, L) to (X', L') such that f is onto X', then the topological left G-space (X, L) is called an 'extension of the topological left G-space (X', L')'. In the special case where the extension (X, L) has the form $(X' \times Y, L)$ for some topological space Y and if f is the natural projection from from $X' \times Y$ onto X', then the topological left G-space (X, L) is called a 'skew product of the topological left G-space (X', L')'. Note that $X' \times Y$ is equipped with the product topology.

If (X, L) is a topological left G-space and H is a topological group then a function $f \in \mathcal{C}(G \times X, H)$ is called a 'H-cocycle over the topological left G-space (X, L)' if, for $g, g' \in G, x \in X$,

$$f(gg', x) = f(g, L(g'; x))f(g', x) \; . \tag{B.8}$$

I define, for given X, G, H, the set $COC(X, G, H)$ as the collection of pairs (L, f) with the property that (X, L) is a topological left G-space and that f is a H-cocycle over (X, L). For literature on cocycles, see, e.g., [HK1, KR, Zi1]. Note also that two conventions for the definition of cocycles are used: my and the 'dual' one. In the latter convention (see e.g. [KR, Zi1]) $(f(g, x))^{-1}$, not $f(g, x)$, is a cocycle. However for convenience I stick to my convention which is the same as in [HK1].

Right G-actions are defined in direct analogy to left G-actions. In fact, if X is a set, G a group with identity e_G and $R : G \times X \to X$ a function satisfying, for $g, h \in G, x \in X$,

$$R(e_G; x) = x \; , \tag{B.9}$$

$$R(gh; x) = R(h; R(g; x)) \; , \tag{B.10}$$

then R is called a 'right G-action on X' and the pair (X, R) is called a 'right G-space'. Due to the close analogy of the concepts of right G-action and left G-action it is obvious how a topological right G-space, a G-map etc. are defined. Note that left G-spaces and right G-spaces are also called 'transformation groups'.

As is common, I will often skip the word 'left', i.e., I often call a left G-action a 'G-action', and a left G-space a G-space etc. This convention is especially useful if G is Abelian since in that case left and right G-actions are the same.

The following facts about right G-spaces are important for principal bundles (the latter are treated in Appendix E) so let (X, R) be a right G-space. Let the set X^* be defined by $X^* := \{(x, R(g; x)) : g \in G, x \in X\}$ and the function $\sigma_R : G \times X \to X^*$ be defined by $\sigma_R(g, x) := (x, R(g; x))$. Clearly σ_R is onto X^*. The right G-action R is called 'free' if, for all $x \in X$, the equality: $R(g; x) = x$ implies: $g = e_G$. It is easy to see that R is free iff σ_R is one-one. In fact, if $\sigma_R(g, x) = \sigma_R(g', x')$ then $(x, R(g; x)) = (x', R(g'; x'))$ whence, if R is free, $x = x', g = g'$ so that σ_R is one-one. Conversely, let $R(g; x) = x$. Thus $\sigma_R(g, x) = (x, R(g; x)) = (x, x) = (x, R(e_G; x)) = \sigma_R(e_G, x)$ whence, if σ_R is one-one, $g = e_G$ so that R is free. I thus have shown that R is free iff σ_R is one-one. Therefore, since σ_R is onto X^*, R is free iff σ_R is a bijection from $G \times X$ onto X^*. Of course if R is free the inverse σ_R^{-1} is well defined and one then defines the function $\tau_R : X^* \to G$ by $\tau_R := pr_1 \circ \sigma_R^{-1}$ where $pr_1(g, x) := g$. If R is free one calls τ_R the 'translation function' of R. Note that if R is free then for $g \in G, x \in X$ we have $R(\tau_R(x, R(g, x)); x) = R((pr_1 \circ \sigma_R^{-1})(x, R(g, x)); x) = R(pr_1(g, x); x) = R(g; x)$ whence for $x, x' \in X$ we have $R(\tau_R(x, x'); x) = x'$. Of course if R is free then τ_R is the only function $\tau : X^* \to G$ which satisfies, for $x, x' \in X$, $R(\tau(x, x'); x) = x'$. A topological right G-space (X, R) is called 'principal' if R is free and if τ_R is continuous.

If (X, R) is a right G-space and $x \in X$ then the set $\{R(g; x) : g \in G\}$ is called the 'orbit of x under R'. The set of orbits under R is denoted by X/R and the function

$p_R : X \to X/R$ is defined by

$$p_R(x) := \{R(g;x) : g \in G\} = \bigcup_{g\in G} \{R(g;x)\} . \tag{B.11}$$

Clearly p_R is onto X/R. Note that, for $x, y \in X$, we have that $p_R(x) = p_R(y)$ iff $y \in p_R(x)$. Thus, for $x \in X$,

$$\begin{aligned} &p_R^{-1}(p_R(\{x\})) = p_R^{-1}(\{p_R(x)\}) = \{y \in X : p_R(y) \in \{p_R(x)\}\} \\ &= \{y \in X : p_R(y) = p_R(x)\} = \{y \in X : y \in p_R(x)\} = p_R(x) . \end{aligned} \tag{B.12}$$

It follows from (B.11),(B.12) that for $A \subset X$

$$\begin{aligned} &p_R^{-1}(p_R(A)) = p_R^{-1}(p_R(\bigcup_{x\in A} \{x\})) = p_R^{-1}\Big(\bigcup_{x\in A} p_R(\{x\})\Big) = \bigcup_{x\in A} p_R^{-1}(p_R(\{x\})) \\ &= \bigcup_{x\in A} p_R(x) = \bigcup_{x\in A} \bigcup_{g\in G} \{R(g;x)\} = \bigcup_{g\in G} \bigcup_{x\in A} \{R(g;x)\} = \bigcup_{g\in G} R(g;A) . \end{aligned} \tag{B.13}$$

If X is a topological space and (X, R) is a topological right G-space then one equips X/R with the quotient topology w.r.t. p_R, i.e., a subset U of X/R is open iff $p_R^{-1}(U)$ is open in X. Thus the function p_R is identifying and one calls X/R an 'orbit space'. To show that p_R is open, let U be open in X whence, by (B.13),

$$p_R^{-1}(p_R(U)) = \bigcup_{g\in G} R(g;U) . \tag{B.14}$$

Since each $R(g;\cdot)$ is a homeomorphism from X onto X we have that $R(g;U)$ is open in X whence $\bigcup_{g\in G} R(g;U)$ is open in X. Thus, by (B.14), $p_R^{-1}(p_R(U))$ is open in X. Since the topology of X/R is the quotient topology w.r.t. p_R we have that $p_R(U)$ is open in X/R whence p_R is open.

There are many textbook treatments of group action. Two useful textbooks, dedicated to group actions, are [tDi2, Ka].

Appendix C

Topological concepts and facts

In this section I provide some concepts and facts from Topology, in particular some know-how about 'liftings' and 'factors' of 'bundles' and 'fiber structures' (see Definition C.1). This know-how is especially useful for continuous and 2π-periodic functions like $\Psi_{\omega,A}(n;\cdot)$ arising in the study of spin-orbit tori (ω, A). The concept of bundle is also of importance for me in Appendix E where I refine it to the concept of principal bundle. As in Appendix B, I present the material in such detail that it is essentially self contained.

'Hurewicz fibrations' (see Definition C.5) are fiber structures which satisfy a certain condition. In fact, for my purposes, a Hurewicz fibration has sufficient structure to obtain from a continuous function a lifting which is a continuous function as well. While liftings provide a tool to obtain continuous functions, factors provide another tool to obtain continuous functions (namely to turn 2π-periodic functions on $\mathbb{R}^k$ into functions on the 'k-torus' $\mathbb{T}^k$ defined below). For these matters I introduce with Definition C.2 four well-known fiber structures and demonstrate in Section C.1 that all four of them are Hurewicz fibrations. They will be used for liftings and one of them will be used for factors. Three of the four 'projections' (see Definition C.1) are covering maps (see Definition C.7). Note that fiber structures (and even Hurewicz fibrations) are rather simple concepts which do not involve any group actions. Thus in this section I neither employ the machinery of principal bundles nor do I need Category theory (see however Appendix E). The know-how I use about liftings and Hurewicz fibrations can be found in [Du, Sp] and the know-how about factors in [SZ]. See also [Bre, Di, Rot, tDi1]. My terminology is close to [Du, Hus].

C.1 Bundles, fiber structures and Hurewicz fibrations

In this section I choose my four fiber structures and show that they are Hurewicz fibrations. The search for liftings w.r.t. my fiber structures is the content of Sections C.2 and C.3. In Section C.3 this search will be facilitated by the use of 'factors' (see Definition C.1) w.r.t. one of the four fiber structures (the latter fiber structure is also used in Section D.2).

Definition C.1 *(Bundle, fiber structure, lifting, factor, cross section, locally trivial) Given topological spaces X, Y, I denote the set of continuous functions from X into Y by $\mathcal{C}(X,Y)$ and the set of homeomorphisms from X onto Y by $HOMEO(X,Y)$.*

A triple (E,p,B) is called a 'bundle' if E and B are topological spaces and if p is in $\mathcal{C}(E,B)$. A bundle (E,p,B) is called a 'fiber structure' if p is onto B. One calls E the 'total space', B the 'base space' and p the 'projection' of the bundle. For $b \in B$, $p^{-1}(b)$ is called the 'fibre of p over b' and its topology is defined as the relative topology from E.

If $\xi = (E,p,B)$ is a bundle, X is a topological space and $g \in \mathcal{C}(X,B)$, then $f \in \mathcal{C}(X,E)$ is called a 'lifting of f' w.r.t. the bundle ξ if $g = p \circ f$. If $g \in \mathcal{C}(E,X)$

then a $f \in \mathcal{C}(B, X)$ *is called a 'factor of g' w.r.t. the bundle* ξ *if* $g = f \circ p$. *If* $\sigma \in \mathcal{C}(B, E)$ *satisfies* $id_B = p \circ \sigma$, *where* id_B *is the identity map on* B, *then one calls* σ *a 'cross section of* ξ*'. The set of cross sections of* ξ *is denoted by* $\Gamma(\xi)$.

A fiber structure (E, p, B) *is called 'locally trivial' if for every* $b \in B$ *an open neighborhood* U *of* b, *a topological space* Y *and a homeomorphism* $\varphi : U \times Y \to p^{-1}(U)$ *onto* $p^{-1}(U)$ *exist such that, for all* $x \in U, y \in Y$, $p \circ \varphi(x, y) = x$ *where* $U \times Y$ *has the product topology,* U *has the relative topology from* B *and* $p^{-1}(U)$ *has the relative topology from* E. □

Remark:

(1) My notion of 'bundle' is from [Hus] and my notion of 'fiber structure' is from [Du] and all concrete examples of bundles in this work are fiber structures. Note that a bundle which has a cross section is a fiber structure. If $\xi = (E, p, B)$ is a fiber structure and X a topological space then, since p is onto B, every $g \in \mathcal{C}(E, X)$ has at most one factor w.r.t. ξ.

Clearly the concepts of bundle and fiber structure are trivial and the topologies of the fibres in a fiber structure are in general largely unrelated - in particular they are in general not homeomorphic. However a fiber structure has a lot of structure if it is locally trivial. In particular for locally trivial fiber structure (E, p, B), every $b \in B$ has an open neighborhood U such that the fibres $p^{-1}(u)$ with $u \in U$ are homeomorphic. We will see that the four fiber structures to be introduced in this section are locally trivial, a circumstance which makes it easy to show, again in this section, that all four of them are Hurewicz fibrations. □

Definition C.2 *A function on* $\mathbb{R}^k$ *is called '*2π*-periodic' if it is* 2π*-periodic in all* k *arguments. If* Y *is a topological space, I denote the set of* 2π*-periodic functions in* $\mathcal{C}(\mathbb{R}^k, Y)$ *by* $\mathcal{C}_{per}(\mathbb{R}^k, Y)$. *The set* $SO(3)$ *consists of those real* 3×3*–matrices* R *with*

$\det(R) = 1$ *for which* $R^T R = I_{3\times 3}$ *where* R^T *denotes the transpose of* R *and* $I_{3\times 3}$ *the* 3×3 *unit matrix. I define*

$$\mathcal{J} := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad SO_3(2) := \{\exp(2\pi x \mathcal{J}) : x \in \mathbb{R}\} \subset SO(3) , \quad \text{(C.1)}$$

and consider $SO_3(2)$ *as a topological subspace of* $SO(3)$*. Denoting the fractional part of a real number* x *by* $\lfloor x \rfloor$*, I obtain for* $x \in \mathbb{R}$

$$\exp(2\pi x \mathcal{J}) = \exp(2\pi \lfloor x \rfloor \mathcal{J}) = \begin{pmatrix} \cos(2\pi x) & -\sin(2\pi x) & 0 \\ \sin(2\pi x) & \cos(2\pi x) & 0 \\ 0 & 0 & 1 \end{pmatrix} . \quad \text{(C.2)}$$

Thus $SO_3(2)$ *is, under matrix multiplication, an Abelian subgroup of* $SO(3)$*. Clearly for every* $R \in SO_3(2)$ *a unique* $r \in [0, 1)$ *exists such that* $p_1(2\pi r) = \exp(2\pi r \mathcal{J})$ *and I abbreviate* $PH(R) := r$ *and call* $PH(R)$ *the 'phase of* R*'. The function* $p_1 : \mathbb{R} \to SO_3(2)$*, defined by* $p_1(y) := \exp(y\mathcal{J})$*, clearly belongs to* $\mathcal{C}_{per}(\mathbb{R}, SO_3(2))$ *and is onto* $SO_3(2)$ *whence* $(\mathbb{R}, p_1, SO_3(2))$ *is a fiber structure.*

I define the k-sphere $\mathbb{S}^k := \{x \in \mathbb{R}^{k+1} : |x| = 1\}$ *(k positive integer) and equip it with the relative topology from* $\mathbb{R}^{k+1}$*. I define the function* $p_2 : \mathbb{S}^3 \to SO(3)$ *by* $p_2(\bar{r})x := (2r_0^2 - 1)x + 2r(r^T x) + 2r_0(r \times x)$*, where* $\bar{r} =: (r_0, r) \in \mathbb{S}^3, r_0 \in \mathbb{R}, r \in \mathbb{R}^3$ *and* $x \in \mathbb{R}^3$*. Since the topology of* $SO(3)$ *is defined as the relative topology from* $\mathbb{R}^{3\times 3}$*,* $p_2 \in \mathcal{C}(\mathbb{S}^3, SO(3))$*. Note that the trace of* $p_2(\bar{r})$ *reads as* $Tr[p_2(\bar{r})] = 4r_0^2 - 1$*. On* $\mathbb{S}^3$ *one introduces a multiplication by* $(r_0, r)(s_0, s) = (r_0 s_0 - r^T s, r_0 s + s_0 r + r \times s)$ *where* $r_0, s_0 \in \mathbb{R}, r, s \in \mathbb{R}^3$*. One observes that* $\mathbb{S}^3$ *is a topological group whose unit element is* $(1, 0, 0, 0)^T$*. The inverse of* (r_0, r) *is* $(r_0, -r)$*. Moreover* p_2 *is a group homomorphism, i.e.* $p_2(\bar{r}\bar{s}) = p_2(\bar{r})p_2(\bar{s})$*. It is thus easy to show that* p_2 *is onto* $SO(3)$ *whence* $(\mathbb{S}^3, p_2, SO(3))$ *is a fiber structure.*

I define the function $p_3 : SO(3) \to \mathbb{S}^2$ *by* $p_3(R) := Re^3$*, where* e^3 *denotes the third unit vector, i.e.,* $e^3 = (0, 0, 1)^T$*. More generally,* e^i *denotes the i-th unit vector in any*

$\mathbb{R}^k$, *i.e.,* $(e^i)_i := 1$ *and, for* $i \neq j$, $(e^i)_j := 0$. *It is easy to see that* $p_3 \in \mathcal{C}(SO(3), \mathbb{S}^2)$ *and is onto* $\mathbb{S}^2$ *whence* $(SO(3), p_3, \mathbb{S}^2)$ *is a fiber structure.*

I define the complex unit circle $\mathbb{T} := \{x \in \mathbb{C} : |x| = 1\}$ *and the* k*-torus* $\mathbb{T}^k$, *i.e., the* k*-fold cartesian product of* $\mathbb{T}$ *(whenever I write* $\mathbb{T}^k$, *this implies that* k *is a positive integer). I consider* $\mathbb{T}$ *as a topological subspace of* $\mathbb{C}$ *and* $\mathbb{T}^k$ *as the topological product of its* k *factors. Defining* $p_{4,k} : \mathbb{R}^k \to \mathbb{T}^k$ *by* $p_{4,k}(\phi) := (\exp(i\phi_1), ..., \exp(i\phi_k))^T$ *it is easy to see that* $p_{4,k} \in \mathcal{C}_{per}(\mathbb{R}^k, \mathbb{T}^k)$ *and is onto* $\mathbb{T}^k$ *whence* $(\mathbb{R}^k, p_{4,k}, \mathbb{T}^k)$ *is a fiber structure.* □

Having defined my four fiber structures, the remaining task of this section is to show that all of them are Hurewicz fibrations. Since the notion of Hurewicz fibration is closely related to Homotopy Theory I first need

Definition C.3 *(Homotopic functions) Let* X, Y *be topological spaces and let* $f_i \in \mathcal{C}(X, Y)$ *be continuous functions where* $i = 0, 1$. *Then I write* $f_0 \simeq_Y f_1$ *if a* $h \in \mathcal{C}(X \times [0,1], Y)$ *exists such that* $h(\cdot, 0) = f_0$ *and* $h(\cdot, 1) = f_1$ *where* $X \times [0,1]$ *is equipped with the product topology and* $[0,1]$ *is equipped with the relative topology from* $\mathbb{R}$. *One then says that* f_0, f_1 *are 'homotopic w.r.t.* Y*'. It is easily shown (see, e.g., [Rot, Sp]) that* $\simeq_Y$ *is an equivalence relation on* $\mathcal{C}(X, Y)$ *and I denote by* $[X, Y]$ *the set of all equivalence classes.*

Note that for cartesian products like $X \times [0,1]$ *I choose the product topology if not mentioned otherwise. A* $g \in \mathcal{C}(X, Y)$ *is called 'nullhomotopic w.r.t.* Y*', if it is homotopic w.r.t.* Y *to a constant function in* $\mathcal{C}(X, Y)$. □

If two functions have different domain then they cannot be homotopic. It is also clear that, in the notation of Definition C.3, always functions exist in $\mathcal{C}(X, Y)$ which are nullhomotopic w.r.t. Y. Note that continuous functions with common domain are often not homotopic. Note that the suffix in $\simeq_Y$ is important. In fact, for every pair f_0, f_1 of continuous functions on a topological space X one can choose Y sufficiently large such that $f_0 \simeq_Y f_1$ [Du, Section XV.1]. Nevertheless one often does not mention Y when the context is clear.

Proposition C.4 *a) Let X and Y be topological spaces and let $g_i \in \mathcal{C}(\mathbb{R}^k, X)$ and $f_i \in \mathcal{C}(X, Y)$ where $i = 0, 1$. If $f_0 \simeq_Y f_1$ and $g_0 \simeq_X g_1$ then $f_1 \circ g_1 \simeq_Y f_0 \circ g_0$.*

b) If X is a topological space and if $g \in \mathcal{C}(\mathbb{R}^k, X)$ then g is nullhomotopic w.r.t. X.

c) Let X and Y be topological spaces and let Y be path-connected. Then all $g \in \mathcal{C}(X, Y)$ which are nullhomotopic w.r.t. Y, are homotopic w.r.t. Y. In other words, all $g \in \mathcal{C}(X, Y)$, which are nullhomotopic w.r.t. Y, belong to the same element of $[X, Y]$.

Proof of Proposition C.4a: Let X and Y be topological spaces and let $g_i \in \mathcal{C}(\mathbb{R}^k, X)$ and $f_i \in \mathcal{C}(X, Y)$ where $i = 0, 1$. Thus a $F \in \mathcal{C}(X \times [0, 1], Y)$ exists such that $F(\cdot, i) = f_i(\cdot)$ and a $G \in \mathcal{C}(\mathbb{R}^k \times [0, 1], X)$ exists such that $G(\cdot, i) = g_i(\cdot)$. The function $H : \mathbb{R}^k \times [0, 1] \to Y$, defined by $H(x, t) := F(G(x, t), t)$, is continuous and satisfies $H(x, i) = F(G(x, i), i) = F(g_i(x), i) = f_i(g_i(x))$. Thus $f_1 \circ g_1 \simeq_Y f_0 \circ g_0$. □

Proof of Proposition C.4b: See [Du, Section XV.1]. □

Proof of Proposition C.4c: See [SZ, Section 2.1]. □

It follows from Proposition C.4 that if X is a path-connected topological space, then all $g \in \mathcal{C}(\mathbb{R}^k, X)$ are homotopic w.r.t. X.

For a fiber structure (E, p, B) and a nonempty subset U of B the function $p|p^{-1}(U) : p^{-1}(U) \to U$ is onto U since p is onto B. Choosing for $p^{-1}(U)$ the relative topology from E and for U the relative topology from B, it is clear that $p|p^{-1}(U)$ is a continuous function whence $(p^{-1}(U), p|p^{-1}(U), U)$ is a fiber structure.

Definition C.5 *(Hurewicz fibration) Let X be a topological space. A fiber structure (E, p, B) is called a 'fibration for X' if it has the following property: if $G \in \mathcal{C}(X \times [0,1], B)$ and if $G(\cdot, 0)$ has a lifting f w.r.t. (E, p, B) then G has a lifting F w.r.t. (E, p, B) such that $f(\cdot) = F(\cdot, 0)$.*

A fiber structure (E, p, B) is called a 'Hurewicz fibration' if it is a fibration for arbitrary topological spaces X.

A fiber structure (E, p, B) is called a 'local Hurewicz fibration' if every $b \in B$ has a neighborhood U such that the fiber structure $(p^{-1}(U), p|p^{-1}(U), U)$ is a Hurewicz fibration. Recall that $p^{-1}(U)$ has the relative topology from E and that U has the relative topology from B. □

Note that the concept of local Hurewicz fibration will play a role in the proof of Lemma C.6.

One sees by Definition C.5 that liftings w.r.t. Hurewicz fibrations can be found by the following method. If (E, p, B) is a Hurewicz fibration and if one looks for a lifting of a continuous function $g : X \to B$ w.r.t. (E, p, B) then one just tries to find a continuous function $g' : X \to B$ with $g \simeq_B g'$ which is so simple that a lifting of g' w.r.t. (E, p, B) can be easily found. As a matter of fact, in Sections C.2, C.3 I will often apply this method.

To show that my four fiber structures are Hurewicz fibrations, the following lemma is crucial.

Lemma C.6 *(Homotopy Lifting Theorem) Let (E, p, B) be a fiber structure which is locally trivial and let B be a compact Hausdorff space. Then (E, p, B) is a Hurewicz fibration.*

Proof of Lemma C.6: Since B is a compact Hausdorff space, the claim follows by applying [Du, Corollary XX.3.6] if (E, p, B) is a local Hurewicz fibration.

Thus I only have to show that (E, p, B) is a local Hurewicz fibration so let $b \in B$. By Definition C.1 an open neighborhood U of b, a topological space Y and a homeomorphism $\varphi : U \times Y \to p^{-1}(U)$ onto $p^{-1}(U)$ exist such that, for all $b \in U, y \in Y$, $p \circ \varphi(b, y) = b$. I only have to show that the fiber structure $(p^{-1}(U), p|p^{-1}(U), U)$ is a Hurewicz fibration. Thus let $G \in \mathcal{C}(X \times [0,1], U)$ and let $g(\cdot) := G(\cdot, 0)$ have a lifting f w.r.t. $(p^{-1}(U), p|p^{-1}(U), U)$. I define the function $F : X \times [0,1] \to p^{-1}(U)$ by $F(x,t) := \varphi\Big(G(x,t), pr_2(\varphi^{-1}(f(x)))\Big)$ where pr_2 is the projection on the second factor, i.e., $pr_2(b, y) = y$. Since φ is a homeomorphism onto $p^{-1}(U)$, F is a continous function. Clearly $p(F(x,t)) = G(x,t)$ whence F is a lifting of G w.r.t. $(p^{-1}(U), p|p^{-1}(U), U)$. Furthermore, for every $e \in p^{-1}(U)$, we have $e = \varphi(\varphi^{-1}(e)) = \varphi\Big(pr_1(\varphi^{-1}(e)), pr_2(\varphi^{-1}(e))\Big) = \varphi\Big(p(e), pr_2(\varphi^{-1}(e))\Big)$ where pr_1 is the projection on the second factor, i.e., $pr_1(b, y) = b$. Hence $F(x, 0) = \varphi\Big(G(x,0), pr_2(\varphi^{-1}(F(x,0)))\Big)$. Since also $F(x,0) = \varphi\Big(G(x,0), pr_2(\varphi^{-1}(f(x)))\Big)$ and since φ is a bijection I conclude that $F(\cdot, 0) = f(\cdot)$. Since b and X were chosen arbitrarily I thus have shown that (E, p, B) is a local Hurewicz fibration. □

Since the base spaces $SO_3(2), SO(3), \mathbb{S}^2$ and $\mathbb{T}^k$ of my four fiber structures are compact Hausdorff spaces, one sees by Lemma C.6 that my aim of proving that these fiber structures are Hurewicz fibrations reduces to showing that they are locally trivial.

I first introduce

Definition C.7 *(Covering map) Let X, Y be topological spaces and $p \in \mathcal{C}(X, Y)$ be onto Y. Then p is called a 'covering map w.r.t. X and Y' if every point of Y has an open neighbourhood U such that $p^{-1}(U)$ is a disjoint union $\bigcup_{\lambda \in \Lambda} U_\lambda$ of open sets $U_\lambda \subset X$ with $p(U_\lambda) = U$ and such that every $p|U_\lambda : U_\lambda \to U$ is a homeomorphism onto U.* □

To relate the fiber structures $(\mathbb{R}, p_{4,1}, \mathbb{T})$ and $(\mathbb{R}, p_1, SO_3(2))$ I define the function $q : \mathbb{T} \to SO_3(2)$ by $q(\exp(ix)) := \exp(x\mathcal{J})$ where $x \in \mathbb{R}$.

Proposition C.8 *a) $p_{4,k}$ is a covering map w.r.t. $\mathbb{R}^k$ and $\mathbb{T}^k$.*

b) The function q is a homeomorphism from $\mathbb{T}$ onto $SO_3(2)$ and satisfies $q \circ p_{4,1} = p_1$. Furthermore p_1 is a covering map w.r.t. $\mathbb{R}$ and $SO_3(2)$.

c) p_2 is a covering map w.r.t. $\mathbb{S}^3$ and $SO(3)$.

d) Let $p : E \to B$ be a covering map w.r.t. topological spaces E, B. Then (E, p, B) is a locally trivial fiber structure.

e) The fiber structure $(SO(3), p_3, \mathbb{S}^2)$ is locally trivial.

Proof of Proposition C.8a: See [SZ, Section 6.1]. □

Proof of Proposition C.8b: The function $q' : SO_3(2) \to \mathbb{T}$, defined by $q'(\exp(x\mathcal{J})) := \exp(ix)$ where $x \in \mathbb{R}$, is inverse to q. Clearly q and q' are continuous so that q is a homeomorphism from $\mathbb{T}$ onto $SO_3(2)$. Furthermore $q \circ p_{4,1}(x) = q(\exp(ix)) = \exp(x\mathcal{J}) = p_1(x)$ whence $q \circ p_{4,1} = p_1$.

To show that p_1 is a covering map, let y be in $SO_3(2)$ and let $y' := q'(y) \in \mathbb{T}$. Since, by Proposition C.8a, $p_{4,1}$ is a covering map w.r.t. $\mathbb{R}$ and $\mathbb{T}$, there is an open neighbourhood U' of y' such that $p_{4,1}^{-1}(U')$ is a disjoint union $\bigcup_{\lambda \in \Lambda} U_\lambda$ of open sets $U_\lambda \subset \mathbb{R}$ with $p_{4,1}(U_\lambda) = U'$ and such that every $p_{4,1}|U_\lambda : U_\lambda \to U'$ is a homeomorphism onto U'. Since q is a homeomorphism we have that $U := q(U')$ is an open neighbourhood of y. Furthermore $p_1^{-1}(U) = (q \circ p_{4,1})^{-1}(U) = p_{4,1}^{-1}(q^{-1}(U)) = p_{4,1}^{-1}(q'(U)) = p_{4,1}^{-1}(U') = \bigcup_{\lambda \in \Lambda} U_\lambda$. Also $p_1(U_\lambda) = q \circ p_{4,1}(U_\lambda) = q(U') = U$ and $p_1|U_\lambda = q \circ p_{4,1}|U_\lambda$ is a homeomorphism onto $q(U') = U$.

Since y is an arbitrary element in $SO_3(2)$, I thus have shown that p_1 is a covering map w.r.t. $\mathbb{R}$ and $SO_3(2)$. □

Proof of Proposition C.8c: See [SZ, Section 6.1]. □

Proof of Proposition C.8d: Let $p : E \to B$ be a covering map w.r.t. topological spaces E, B. Clearly (E, p, B) is a fiber structure.

To show that this fiber structure is locally trivial, let $b \in B$. Thus b has an open neighbourhood U such that $p^{-1}(U)$ is a disjoint union $\bigcup_{\lambda \in \Lambda} U_\lambda$ of open sets $U_\lambda \subset E$ with $p(U_\lambda) = U$ and such that every $p|U_\lambda : U_\lambda \to U$ is a homeomorphism onto U. I pick for Λ the discrete topology. Hence the function $p' : p^{-1}(U) \to U \times \Lambda$ defined, for $e \in U_\lambda$ by $p'(e) := (p(e), \lambda)$, is a homeomorphism onto $U \times \Lambda$. The inverse of p' is a homeomorphism $\varphi : U \times \Lambda \to p^{-1}(U)$ and, for $e \in p^{-1}(U)$, we have $p(\varphi(p(e), \lambda)) = p(\varphi(p'(e))) = p(e)$.

Since $b \in B$ is an arbitrary point I conclude that the fiber structure (E, p, B) is locally trivial. □

Proof of Proposition C.8e: See for example [Bre, Section II.13],[Sw, Section 4]. □

I conclude from Lemma C.6 and Proposition C.8:

Corollary C.9 *The fiber structures* $(\mathbb{R}, p_1, SO_3(2))$, $(\mathbb{S}^3, p_2, SO(3))$,$(SO(3), p_3, \mathbb{S}^2)$ *and*
$(\mathbb{R}^k, p_{4,k}, \mathbb{T}^k)$ *are Hurewicz fibrations.* □

I will use Corollary C.9 to obtain liftings w.r.t. the four fiber structures. I will use the fiber structure $(\mathbb{R}^k, p_{4,k}, \mathbb{T}^k)$ to obtain factors (see Section C.3) and to show that certain subsets of $\mathbb{R}^k$ are dense (see Section D.2).

C.2 Basic liftings

Crucial for this work are liftings of functions g on the domain $\mathbb{R}^k$ w.r.t. my four fiber structures $(\mathbb{R}, p_1, SO_3(2))$, $(\mathbb{S}^3, p_2, SO(3))$,$(SO(3), p_3, \mathbb{S}^2)$, $(\mathbb{R}^k, p_{4,k}, \mathbb{T}^k)$ and in this section I will provide basic properties of those liftings. As a byproduct I will obtain the concepts of phase function, $SO_3(2)$-index, $SO(3)$-index, and $\mathbb{S}^3$-index. The following lemma is essential for this section.

Lemma C.10 *Let (E, p, B) be a Hurewicz fibration and X be a topological space. Then the following hold. If $g \in \mathcal{C}(X, B)$ is nullhomotopic w.r.t. B then it has a lifting f w.r.t. (E, p, B). Each of these f is nullhomotopic w.r.t. E. If $g \in \mathcal{C}(\mathbb{R}^k, B)$ then it has a lifting w.r.t. (E, p, B).*

Proof of Lemma C.10: Let $g \in \mathcal{C}(X, B)$ be nullhomotopic w.r.t. B. Then a $G \in \mathcal{C}(X \times [0, 1], B)$ exists such that $g(\cdot) = G(\cdot, 1)$ and such that $g'(\cdot) := G(\cdot, 0)$ is a constant function. Because p is onto B, a constant function $f' : X \to E$ exists such that $g' = p \circ f'$. Since (E, p, B) is a Hurewicz fibration it follows that a $F \in \mathcal{C}(X \times [0, 1], E)$ exists such that $G = p \circ F$ and $f'(\cdot) = F(\cdot, 0)$. Clearly $f := F(\cdot, 1)$ is a lifting of g w.r.t. (E, p, B) and f is nullhomotopic w.r.t. E.

To prove the second claim, let $X = \mathbb{R}^k$. Then, by Proposition C.4b, g is nullhomotopic w.r.t. B whence, by the first claim, g has a lifting w.r.t. (E, p, B). □

Theorem C.11 *a) Let $g \in \mathcal{C}(\mathbb{R}^k, SO_3(2))$. Then g has a lifting f w.r.t. $(\mathbb{R}, p_1, SO_3(2))$, i.e., a $f \in \mathcal{C}(\mathbb{R}^k, \mathbb{R})$ exists such that $g = p_1 \circ f = \exp(\mathcal{J} f)$. Any lifting $\tilde{f}$ of g w.r.t. $(\mathbb{R}, p_1, SO_3(2))$ has the form $\tilde{f}(\phi) = f(\phi) + 2\pi N$ where N is an integer. Furthermore, for every integer N, $\tilde{f}$ is a lifting of g w.r.t. $(\mathbb{R}, p_1, SO_3(2))$.*

b) Let $g \in \mathcal{C}_{per}(\mathbb{R}^k, SO_3(2))$. Then every lifting f of g w.r.t. $(\mathbb{R}, p_1, SO_3(2))$ has the form $f(\phi) = f_{per}(\phi) + N^T\phi$ where $N \in \mathbb{Z}^k$ and where $f_{per} \in \mathcal{C}_{per}(\mathbb{R}^k, \mathbb{R})$. Furthermore

N is uniquely determined by g.

c) Let $g \in \mathcal{C}(\mathbb{R}^m, \mathbb{T}^k)$. Then g has a lifting f w.r.t. $(\mathbb{R}^k, p_{4,k}, \mathbb{T}^k)$, i.e., a $f \in \mathcal{C}(\mathbb{R}^m, \mathbb{R}^k)$ exists such that $g = p_{4,k} \circ f = (\exp(if_1), ..., \exp(if_k))$. Any lifting $\tilde{f}$ of g w.r.t. $(\mathbb{R}^k, p_{4,k}, \mathbb{T}^k)$ has the form $\tilde{f}(\phi) = f(\phi) + 2\pi N$ where $N \in \mathbb{Z}^k$. Furthermore, for every $N \in \mathbb{Z}^k$, $\tilde{f}$ is a lifting of g w.r.t. $(\mathbb{R}^k, p_{4,k}, \mathbb{T}^k)$.

d) Let $g \in \mathcal{C}_{per}(\mathbb{R}^m, \mathbb{T}^k)$. Then every lifting f of g w.r.t. $(\mathbb{R}^k, p_{4,k}, \mathbb{T}^k)$ has the form $f(\phi) = f_{per}(\phi) + N\phi$ where $N \in \mathbb{Z}^{k\times m}$ and where $f_{per} \in \mathcal{C}_{per}(\mathbb{R}^m, \mathbb{R}^k)$. Furthermore N is uniquely determined by g.

e) Let $g \in \mathcal{C}(\mathbb{R}^k, \mathbb{S}^2)$. Then g has a lifting f w.r.t. $(SO(3), p_3, \mathbb{S}^2)$, i.e., a function $f \in \mathcal{C}(\mathbb{R}^k, SO(3))$ exists such that $g = p_3 \circ f = fe^3$.

Proof of Theorem C.11a: Let $g \in \mathcal{C}(\mathbb{R}^k, SO_3(2))$. Since, by Corollary C.9, $(\mathbb{R}, p_1, SO_3(2))$ is a Hurewicz fibration we know from Lemma C.10 that g has a lifting f w.r.t. $(\mathbb{R}, p_1, SO_3(2))$, i.e., a $f \in \mathcal{C}(\mathbb{R}^k, \mathbb{R})$ exists such that $g = p_1 \circ f = \exp(\mathcal{J} f)$. If $\tilde{f}$ is any lifting of g w.r.t. $(\mathbb{R}, p_1, SO_3(2))$, then

$$I_{3\times 3} = \exp(\mathcal{J}(f - \tilde{f})) = \begin{pmatrix} \cos(f-\tilde{f}) & -\sin(f-\tilde{f}) & 0 \\ \sin(f-\tilde{f}) & \cos(f-\tilde{f}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the remaining claim follows from the continuity of $f, \tilde{f}$. □

Proof of Theorem C.11b: Let $g \in \mathcal{C}_{per}(\mathbb{R}^k, SO_3(2))$. By Theorem C.11a a lifting f of g w.r.t. $(\mathbb{R}, p_1, SO_3(2))$ exists. Since g is 2π-periodic, we have for $i = 1, ..., k$, $I_{3\times 3} = g(\phi + 2\pi e^i) g^T(\phi) = \exp(\mathcal{J} f(\phi + 2\pi e^i) - \mathcal{J} f(\phi))$. Since f is continuous I conclude that for $i = 1, ..., k$ an integer N_i exists such that $f(\phi + 2\pi e^i) - f(\phi) = 2\pi N_i$. Therefore the function $f_{per} : \mathbb{R}^k \to \mathbb{R}$, defined by $f_{per}(\phi) := f(\phi) - N^T\phi$, is in $\mathcal{C}_{per}(\mathbb{R}^k, \mathbb{R})$, where $N := (N_1, ..., N_k)$. That N is uniquely determined by g follows by applying once again Theorem C.11a. □

Proof of Theorem C.11c: Let $g \in \mathcal{C}(\mathbb{R}^m, \mathbb{T}^k)$. Since, by Corollary C.9, $(\mathbb{R}^k, p_{4,k}, \mathbb{T}^k)$ is a Hurewicz fibration we know from Lemma C.10 that g has a lifting f w.r.t. $(\mathbb{R}^k, p_{4,k}, \mathbb{T}^k)$, i.e., a $f \in \mathcal{C}(\mathbb{R}^m, \mathbb{R}^k)$ exists such that $g = p_{4,k} \circ f = (\exp(if_1), ..., \exp(if_k))$. If $\tilde{f}$ is any lifting of g w.r.t. $(\mathbb{R}^k, p_{4,k}, \mathbb{T}^k)$, then

$$(1, ..., 1) = (\exp(if_1 - i\tilde{f}_1), ..., \exp(if_k - i\tilde{f}_k)) ,$$

and the remaining claim follows from the continuity of $f, \tilde{f}$. □

Proof of Theorem C.11d: Let $g \in \mathcal{C}_{per}(\mathbb{R}^m, \mathbb{T}^k)$. By Theorem C.11c a lifting f of g w.r.t. $(\mathbb{R}^k, p_{4,k}, \mathbb{T}^k)$ exists. Since g is 2π-periodic, we have for $i = 1, ..., m$,

$$\begin{aligned}(1, ..., 1) &= (g_1(\phi + 2\pi e^i)\overline{g_1(\phi)}, ..., g_k(\phi + 2\pi e^i)\overline{g_k(\phi)}) \\ &= (\exp(if_1(\phi + 2\pi e^i)) \exp(-if_1(\phi)), ..., \exp(if_k(\phi + 2\pi e^i)) \exp(-if_k(\phi))) .\end{aligned}$$

Since f is continuous I conclude that for $i = 1, ..., m, j = 1, ..., k$ an integer $N_{j,i}$ exists such that $f_j(\phi + 2\pi e^i) - f_j(\phi) = 2\pi N_{j,i}$. Therefore the function $f_{per} : \mathbb{R}^k \to \mathbb{R}$, defined by $f_{per}(\phi) := f(\phi) - N\phi$, is in $\mathcal{C}_{per}(\mathbb{R}^k, \mathbb{R})$, where N is the $k \times m$-matrix with elements $N_{j,i}$. That N is uniquely determined by g follows by applying once again Theorem C.11c. □

Proof of Theorem C.11e: Let $g \in \mathcal{C}(\mathbb{R}^k, \mathbb{S}^2)$. Since, by Corollary C.9, $(SO(3), p_3, \mathbb{S}^2)$ is a Hurewicz fibration we know from Lemma C.10 that g has a lifting f w.r.t. $(SO(3), p_3, \mathbb{S}^2)$, i.e., a function $f \in \mathcal{C}(\mathbb{R}^k, SO(3))$ exists such that $g = p_3 \circ f = fe^3$. □

Definition C.12 *($SO_3(2)$-index, $\mathbb{S}^3$-index, phase function) Let $g \in \mathcal{C}_{per}(\mathbb{R}^k, SO_3(2))$. Then the constant $N \in \mathbb{Z}^k$ in Theorem C.11b will be called the '$SO_3(2)$-index of g' and I define the function $Ind_{2,k} : \mathcal{C}_{per}(\mathbb{R}^k, SO_3(2)) \to \mathbb{Z}^k$ by $Ind_{2,k}(g) := N$. It follows from Theorem C.11a,b that for every $g \in \mathcal{C}_{per}(\mathbb{R}^k, SO_3(2))$ there exists a unique $h \in \mathcal{C}_{per}(\mathbb{R}^k, \mathbb{R})$ such that $g(\phi) = \exp(\mathcal{J}[N^T\phi + 2\pi h(\phi)])$ and $h(0) \in [0, 1)$ where $N = Ind_{2,k}(g)$. I call h the 'phase function' of g and abbreviate $PHF(g) := h$.*

Let $f \in \mathcal{C}(\mathbb{R}^k, \mathbb{S}^3)$ and let, for $i = 1, ..., k$, a s_i exist in $\{1, -1\}$ such that, for all ϕ, $f(\phi + 2\pi e^i) = s_i f(\phi)$, i.e., $f(\phi + 2\pi e^i) = (s_i, 0, 0, 0)^T f(\phi)$. Then $s := (s_1, ..., s_k)^T \in \{1, -1\}^k$ is called the '$\mathbb{S}^3$-index of f' and I denote the collection of those functions by $\mathcal{C}^{\pm}_{per}(\mathbb{R}^k, \mathbb{S}^3)$. I define the function $Ind_{1,k} : \mathcal{C}^{\pm}_{per}(\mathbb{R}^k, \mathbb{S}^3) \to \{1, -1\}^k$ by $Ind_{1,k}(f) := s$ where s is the $\mathbb{S}^3$-index of f. The $\mathbb{S}^3$-index s is uniquely determined by f since $\mathbb{S}^3$ is a group and $(1, 0, 0, 0)^T$ is its identity whence $f(\phi + 2\pi e^i) f^{-1}(\phi) = (s_i, 0, 0, 0)^T$. Clearly $\mathcal{C}_{per}(\mathbb{R}^k, \mathbb{S}^3)$ consists of those functions in $\mathcal{C}^{\pm}_{per}(\mathbb{R}^k, \mathbb{S}^3)$ whose $\mathbb{S}^3$-index is the identity. I consider $\{1, -1\}$ as a multiplicative group with identity 1 and $\{1, -1\}^k$ as the k-fold direct product of $\{1, -1\}$. Note also that $(1, ..., 1)^T$ is the identity of the group $\{1, -1\}^k$ and that each f in $\mathcal{C}^{\pm}_{per}(\mathbb{R}^k, \mathbb{S}^3)$ is 4π-periodic in its k arguments. $\square$

Theorem C.13 *a) Let $g \in \mathcal{C}(\mathbb{R}^k, SO(3))$. Then g has a lifting $\tilde{g}$ w.r.t. $(\mathbb{S}^3, p_2, SO(3))$, i.e., a $\tilde{g} \in \mathcal{C}(\mathbb{R}^k, \mathbb{S}^3)$ exists such that $g = p_2 \circ \tilde{g}$. Any lifting $\tilde{f}$ of g w.r.t. $(\mathbb{S}^3, p_2, SO(3))$ has the form $\tilde{f} = (\kappa, 0, 0, 0)^T \tilde{g} = \kappa \tilde{g}$ where $\kappa \in \{1, -1\}$, i.e., g has exactly the two liftings $\pm\tilde{g}$.*

b) If $\tilde{g} \in \mathcal{C}^{\pm}_{per}(\mathbb{R}^k, \mathbb{S}^3)$ then $p_2 \circ \tilde{g} \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$. Let $g \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$. Then both liftings $\pm\tilde{f}$ of g w.r.t. $(\mathbb{S}^3, p_2, SO(3))$ have an $\mathbb{S}^3$-index, i.e., are elements of $\mathcal{C}^{\pm}_{per}(\mathbb{R}^k, \mathbb{S}^3)$. Furthermore, both liftings $\pm\tilde{f}$ have the same $\mathbb{S}^3$-index. If $h \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ is a constant function then both liftings of h w.r.t. $(\mathbb{S}^3, p_2, SO(3))$ are constant functions and their $\mathbb{S}^3$-index is the identity.

c) The set $\mathcal{C}_{per}(\mathbb{R}^k, \mathbb{S}^3)$ consists of those functions in $\mathcal{C}^{\pm}_{per}(\mathbb{R}^k, \mathbb{S}^3)$ whose $\mathbb{S}^3$-index is the identity. If $\tilde{g}, \tilde{g}' \in \mathcal{C}^{\pm}_{per}(\mathbb{R}^k, \mathbb{S}^3)$ have $\mathbb{S}^3$-indices s, s' respectively then their product (under pointwise multiplication) $\tilde{g}\tilde{g}'$ is in $\mathcal{C}^{\pm}_{per}(\mathbb{R}^k, \mathbb{S}^3)$ and has $\mathbb{S}^3$-index ss'. The set $\mathcal{C}^{\pm}_{per}(\mathbb{R}^k, \mathbb{S}^3)$ is a group under pointwise multiplication of $\mathbb{S}^3$ valued functions. The function $Ind_{1,k}$ is a group homomorphism of the multiplicative group $\mathcal{C}^{\pm}_{per}(\mathbb{R}^k, \mathbb{S}^3)$ into the multiplicative group $\{1, -1\}^k$.

d) Let $G \in \mathcal{C}(\mathbb{R}^k \times [0, 1], SO(3))$ such that, for all $t \in [0, 1]$, $G(\cdot, t) \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$.

Then there exists a lifting $\tilde{G}$ w.r.t. $(\mathbb{S}^3, p_2, SO(3))$, i.e., a $\tilde{G} \in \mathcal{C}(\mathbb{R}^k \times [0,1], \mathbb{S}^3)$ exists such that $G = p_2 \circ \tilde{G}$. Any lifting $\tilde{H}$ w.r.t. $(\mathbb{S}^3, p_2, SO(3))$ has the form $\tilde{H} = (\kappa, 0, 0, 0)^T \tilde{G} = \kappa \tilde{G}$ where $\kappa \in \{1, -1\}$, i.e., G has exactly the two liftings $\pm\tilde{G}$. Moreover, for $t \in [0,1]$, $\tilde{G}(\cdot, t)$ and $-\tilde{G}(\cdot, t)$ are in $\mathcal{C}^{\pm}_{per}(\mathbb{R}^k, \mathbb{S}^3)$ and $Ind_{1,k}(\tilde{G}(\cdot, 0)) = Ind_{1,k}(\tilde{G}(\cdot, t)) = Ind_{1,k}(-\tilde{G}(\cdot, t)) = Ind_{1,k}(-\tilde{G}(\cdot, 0))$.

e) The set $\mathcal{C}_{per}(\mathbb{R}^k, SO_3(2))$ is a group under pointwise multiplication of $SO_3(2)$-valued functions. The function $Ind_{2,k}$ is a group homomorphism from the multiplicative group $\mathcal{C}_{per}(\mathbb{R}^k, SO_3(2))$ onto the additive group $\mathbb{Z}^k$.

Proof of Theorem C.13a: Let $g \in \mathcal{C}(\mathbb{R}^k, SO(3))$. Since, by Corollary C.9, $(\mathbb{S}^3, p_2, SO(3))$ is a Hurewicz fibration we know from Lemma C.10 that g has a lifting $\tilde{g}$ w.r.t. $(\mathbb{S}^3, p_2, SO(3))$, i.e., a $\tilde{g} \in \mathcal{C}(\mathbb{R}^k, \mathbb{S}^3)$ exists such that $g = p_2 \circ \tilde{g}$. If $\tilde{f}$ is any lifting of g w.r.t. $(\mathbb{S}^3, p_2, SO(3))$ then $p_2 \circ \tilde{g} = p_2 \circ \tilde{f}$. Recalling from Definition C.2 that p_2 is a homomorphism from the group $\mathbb{S}^3$ into the group $SO(3)$ I conclude that $p_2(\tilde{f}(\phi)\tilde{g}^{-1}(\phi)) = p_2(\tilde{f}(\phi))p_2(\tilde{g}^{-1}(\phi)) = p_2(\tilde{f}(\phi))(p_2(\tilde{g}(\phi)))^{-1} = g(\phi))(g(\phi))^{-1} = I_{3\times 3}$. By Definition C.2 we have $p_2^{-1}(I_{3\times 3}) = \{(1,0,0,0)^T, (-1,0,0,0)^T\}$ whence $\tilde{f}(\phi)\tilde{g}^{-1}(\phi) \in \{(1,0,0,0)^T, (-1,0,0,0)^T\}$. The continuity of $\tilde{f}(\phi)\tilde{g}^{-1}(\phi)$ in ϕ gives me that $\tilde{f}(\phi)\tilde{g}^{-1}(\phi)$ is independent of ϕ whence either $\tilde{f}\tilde{g}^{-1} = (1,0,0,0)^T$ or $\tilde{f}\tilde{g}^{-1} = (-1,0,0,0)^T$. Thus g has exactly the two liftings $\pm\tilde{g}$ w.r.t. $(\mathbb{S}^3, p_2, SO(3))$. □

Proof of Theorem C.13b: Let $\tilde{g} \in \mathcal{C}^{\pm}_{per}(\mathbb{R}^k, \mathbb{S}^3)$ and let me abbreviate $s := Ind_{1,k}(\tilde{g})$. Thus $\tilde{g}(\phi + 2\pi e^i) = (s_i, 0, 0, 0)^T \tilde{g}(\phi)$. Since p_2 is a group homomorphism and since $p_2(\pm 1, 0, 0, 0) = I_{3\times 3}$ I obtain $p_2(\tilde{g}(\phi + 2\pi e^i)) = p_2((s_i, 0, 0, 0)^T \tilde{g}(\phi))$
$= p_2(s_i, 0, 0, 0)p_2(\tilde{g}(\phi)) = p_2(\tilde{g}(\phi))$ whence $p_2 \circ \tilde{g} \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$.

Let $g \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$. By Theorem C.13a a lifting $\tilde{f}$ of g w.r.t. $(\mathbb{S}^3, p_2, SO(3))$ exists. Since $g = p_2 \circ \tilde{f}$ is 2π-periodic we have for $i = 1, ..., k$ that $I_{3\times 3} = g(\phi + 2\pi e^i)g^{-1}(\phi) = p_2(\tilde{f}(\phi + 2\pi e^i))(p_2(\tilde{f}(\phi)))^{-1} = p_2(\tilde{f}(\phi + 2\pi e^i))p_2(\tilde{f}^{-1}(\phi)) = p_2(\tilde{f}(\phi + 2\pi e^i)\tilde{f}^{-1}(\phi))$ where I also used the fact that p_2 is a group homomorphism. By the definition of p_2 it follows that $\tilde{f}(\phi + 2\pi e^i)\tilde{f}^{-1}(\phi) \in \{(1,0,0,0)^T, (-1,0,0,0)^T\}$. By the continuity of $\tilde{f}(\phi + 2\pi e^i)\tilde{f}^{-1}(\phi)$ in ϕ I conclude that $\tilde{f}(\phi + 2\pi e^i)\tilde{f}^{-1}(\phi)$ is independent of ϕ whence $\tilde{f}(\phi + 2\pi e^i) = (\kappa_i, 0, 0, 0)^T \tilde{f}(\phi)$ with $\kappa_i \in \{1, -1\}$ so that $\tilde{f}$ has the $\mathbb{S}^3$-index $(\kappa_1, ..., \kappa_k)^T$. Clearly

$$
\begin{aligned}
-\tilde{f}(\phi + 2\pi e^i) &= (-1,0,...,0)^T \tilde{f}(\phi + 2\pi e^i) = (-1,0,...,0)^T (\kappa_i, 0,0,0)^T \tilde{f}(\phi) \\
&= (\kappa_i, 0,0,0)^T (-1,0,...,0)^T \tilde{f}(\phi) = (\kappa_i, 0,0,0)^T (-\tilde{f}(\phi)) \; ,
\end{aligned}
$$

whence $-\tilde{f}$ has the same $\mathbb{S}^3$-index as $\tilde{f}$.

Let $h \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ be a constant function having a constant value, say x, and let $\pm\tilde{h}$ be the liftings of h w.r.t. $(\mathbb{S}^3, p_2, SO(3))$. Since p_2 is onto $SO(3)$, there exists $\tilde{x} \in \mathbb{S}^3$ such that $p_2(\tilde{x}) = x$. Because p_2 is a group homomorphism and $p_2^{-1}(I_{3\times 3}) = \{(1,0,0,0)^T, (-1,0,0,0)^T\}$, the range of $\tilde{h}$ is a subset of $\{\tilde{x}, -\tilde{x}\}$ whence, by the continuity of $\tilde{h}$, $\tilde{h}$ is constant and its $\mathbb{S}^3$-index is the identity. I conclude that both liftings of h w.r.t. $(\mathbb{S}^3, p_2, SO(3))$ are constant functions and their $\mathbb{S}^3$-index is the identity. □

Proof of Theorem C.13c: Since the $\mathbb{S}^3$-index of a function $\tilde{g} \in \mathcal{C}^{\pm}_{per}(\mathbb{R}^k, \mathbb{S}^3)$ is the identity iff $\tilde{g}$ is 2π-periodic one observes that the set $\mathcal{C}_{per}(\mathbb{R}^k, \mathbb{S}^3)$ consists of those functions in $\mathcal{C}^{\pm}_{per}(\mathbb{R}^k, \mathbb{S}^3)$ whose $\mathbb{S}^3$-index is the identity. Let $\tilde{g}, \tilde{g}' \in \mathcal{C}^{\pm}_{per}(\mathbb{R}^k, \mathbb{S}^3)$ and let me abbreviate $s := Ind_{1,k}(\tilde{g}), s' := Ind_{1,k}(\tilde{g}')$. Thus, for $\phi \in \mathbb{R}^k, i = 1, ..., k$, I compute

$$
\begin{aligned}
&\tilde{g}(\phi + 2\pi e^i)\tilde{g}'(\phi + 2\pi e^i) = (s_i, 0, ..., 0)^T \tilde{g}(\phi)(s'_i, 0, ..., 0)^T \tilde{g}'(\phi) \\
&= (s_i, 0, ..., 0)^T (s'_i, 0, ..., 0)^T \tilde{g}(\phi)\tilde{g}'(\phi) = (s_i s'_i, 0, ..., 0)^T \tilde{g}(\phi)\tilde{g}'(\phi) \; , \qquad \text{(C.3)}
\end{aligned}
$$

where in the second equality I used the fact that $(\pm 1, 0, ..., 0)^T$ belong to the center of the group $\mathbb{S}^3$. Since $\tilde{g}\tilde{g}' \in \mathcal{C}(\mathbb{R}^k, \mathbb{S}^3)$ I conclude from (C.3) that $\tilde{g}\tilde{g}' \in \mathcal{C}^{\pm}_{per}(\mathbb{R}^k, \mathbb{S}^3)$ and $Ind_{1,k}(\tilde{g}\tilde{g}') = ss'$. Using again the fact that $(\pm 1, 0, ..., 0)^T$ belong to the center of the group $\mathbb{S}^3$, one obtains

$$\tilde{g}^{-1}(\phi + 2\pi e^i) = ((s_i, 0, ..., 0)^T \tilde{g}(\phi))^{-1} = (\tilde{g}(\phi))^{-1}(s_i, 0, ..., 0)^T$$
$$= (s_i, 0, ..., 0)^T (\tilde{g}(\phi))^{-1} ,$$

whence $\tilde{g}^{-1} \in \mathcal{C}^{\pm}_{per}(\mathbb{R}^k, \mathbb{S}^3)$. Here I also used the fact that $\tilde{g}^{-1} \in \mathcal{C}(\mathbb{R}^k, \mathbb{S}^3)$ which follows from the facts that $\tilde{g} \in \mathcal{C}(\mathbb{R}^k, \mathbb{S}^3)$ and that $\mathbb{S}^3$ is a topological group. Since $\mathcal{C}^{\pm}_{per}(\mathbb{R}^k, \mathbb{S}^3)$ is a subgroup of the multiplicative group $\mathcal{C}(\mathbb{R}^k, \mathbb{S}^3)$ and since, for $\tilde{g}, \tilde{g}' \in \mathcal{C}^{\pm}_{per}(\mathbb{R}^k, \mathbb{S}^3)$ we have $\tilde{g}\tilde{g}', \tilde{g}^{-1} \in \mathcal{C}^{\pm}_{per}(\mathbb{R}^k, \mathbb{S}^3)$ I conclude that the set $\mathcal{C}^{\pm}_{per}(\mathbb{R}^k, \mathbb{S}^3)$ is a subgroup of $\mathcal{C}_{per}(\mathbb{R}^k, \mathbb{S}^3)$. In particular, since $Ind_{1,k}(\tilde{g}\tilde{g}') = ss'$, $Ind_{1,k}$ is a group homomorphism of the multiplicative group $\mathcal{C}^{\pm}_{per}(\mathbb{R}^k, \mathbb{S}^3)$ into the multiplicative group $\{1, -1\}^k$. □

Proof of Theorem C.13d: Let $G \in \mathcal{C}(\mathbb{R}^k \times [0,1], SO(3))$ such that, for all $t \in [0,1]$, $G(\cdot, t) \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$. By Theorem C.13a, $G(\cdot, 0)$ has a lifting of g w.r.t. $(\mathbb{S}^3, p_2, SO(3))$ and by Corollary C.9, $(\mathbb{S}^3, p_2, SO(3))$ is a Hurewicz fibration. Thus, by Definition C.5, a $\tilde{G} \in \mathcal{C}(\mathbb{R}^k \times [0,1], \mathbb{S}^3)$ exists such that $G = p_2 \circ \tilde{G}$. It thus follows by Theorem C.13b that, for all $t \in [0,1]$, we have that $\tilde{G}(\cdot, t) \in \mathcal{C}^{\pm}_{per}(\mathbb{R}^k, \mathbb{S}^3)$ whence $\tilde{G}(\cdot, t)$ has a $\mathbb{S}^3$-index, say $s(t)$. By the group multiplication in $\mathbb{S}^3$ and due to Definition C.12, we have, for $i = 1, ..., k$, $\tilde{G}(\phi + 2\pi e^i, t)(\tilde{G}(\phi, t))^{-1} = (s_i(t), 0, 0, 0)^T$. By the continuity of $\tilde{G}$ one concludes that $s_i(t)$ is continuous in t whence constant.

Let $\tilde{H}$ be an arbitrary lifting of G w.r.t. $(\mathbb{S}^3, p_2, SO(3))$. By Theorem C.13a, for $t \in [0,1]$, a $\kappa(t) \in \{1, -1\}$ exists such that $\tilde{H}(\cdot, t) = (\kappa(t), 0, 0, 0)^T \tilde{G}(\cdot, t)$ whence $\tilde{H}(\cdot, t)\tilde{G}^{-1}(\cdot, t) = (\kappa(t), 0, 0, 0)^T$. Since $\tilde{G}$ and $\tilde{H}$ are continuous functions and $\mathbb{S}^3$ is a topological group, it follows that κ is constant. It follows by Theorem C.13b that $Ind_{1,k}(\tilde{G}(\cdot, 0)) = Ind_{1,k}(\tilde{G}(\cdot, t)) = Ind_{1,k}(-\tilde{G}(\cdot, t)) = Ind_{1,k}(-\tilde{G}(\cdot, 0))$. □

Proof of Theorem C.13e: Since $SO_3(2)$ is a topological group w.r.t. matrix multiplication, $\mathcal{C}(\mathbb{R}^k, SO_3(2))$ is a group under pointwise multiplication of $SO_3(2)$-valued functions. Let $g, g' \in \mathcal{C}_{per}(\mathbb{R}^k, SO_3(2))$. Since $SO_3(2)$ is a topological group w.r.t.

matrix multiplication, it follows that gg' and g^{-1} are in $\mathcal{C}_{per}(\mathbb{R}^k, SO_3(2))$ whence $\mathcal{C}_{per}(\mathbb{R}^k, SO_3(2))$ is a subgroup of $\mathcal{C}(\mathbb{R}^k, SO_3(2))$. By Definition C.12 we have

$$g(\phi) = \exp(\mathcal{J}[N^T\phi + 2\pi h(\phi)]) , \qquad g'(\phi) = \exp(\mathcal{J}[N'^T\phi + 2\pi h'(\phi)]) ,$$

where $N := Ind_{2,k}(g), N' := Ind_{2,k}(g')$ and $h := PHF(g), h' := PHF(g')$. Clearly

$$g(\phi)g'(\phi) = \exp(\mathcal{J}[(N + N')^T\phi + 2\pi h(\phi) + 2\pi h'(\phi)]) ,$$

whence $Ind_{2,k}(gg') = N + N' = Ind_{2,k}(g) + Ind_{2,k}(g')$ so that $Ind_{2,k}$ is a group homomorphism. Of course $Ind_{2,k}$ is onto $\mathbb{Z}^k$ which completes the proof that $Ind_{2,k}$ is a group homomorphism from $\mathcal{C}_{per}(\mathbb{R}^k, SO_3(2))$ onto $\mathbb{Z}^k$. $\square$

Dealing with liftings of functions $g \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ w.r.t. $(\mathbb{S}^3, p_2, SO(3))$ is, in the context of polarized beams in storage rings, called the 'quaternion formalism'. We see by Theorem C.13 that every continuous function $g \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ has two counterparts $\pm\tilde{g} \in \mathcal{C}^{\pm}_{per}(\mathbb{R}^k, \mathbb{S}^3)$ in the quaternion formalism. Beyond its importance for the study of $[\mathbb{T}^k, SO(3)]$ (see Section C.3), the quaternion formalism also has advantages in terms of numerical efficiency (this aspect is not covered in this work - see however the references mentioned in the context of the code SPRINT in Section 8.5).

Definition C.14 *($SO(3)$-index) Let $g \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$. Then the common $\mathbb{S}^3$-index of both liftings $\pm\tilde{f}$ of g in Theorem C.13b will be called the '$SO(3)$-index of g' and I define the function $Ind_{3,k} : \mathcal{C}_{per}(\mathbb{R}^k, SO(3)) \to \{1, -1\}^k$ by $Ind_{3,k}(g) := Ind_{1,k}(\tilde{f})$. Note that, by Theorem C.13b, the $SO(3)$-index of a constant function in $\mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ is the identity. Furthermore I define the function $Ind_{4,k} : \mathcal{C}(\mathbb{T}^k, SO(3)) \to \{1, -1\}^k$ by $Ind_{4,k}(F) := Ind_{3,k}(F \circ p_{4,k})$ and I call $Ind_{4,k}(F)$ the '$SO(3)$-index of F'. For $s \in \{1, -1\}^k$ I define the function $\tilde{g}^{(s)}_k \in \mathcal{C}(\mathbb{R}^k, \mathbb{S}^3)$ by*

$$\tilde{g}^{(s)}_k(\phi) := \left(\cos(\frac{1}{4}\sum_{i=1}^{k}(1 - s_i)\phi_i), 0, 0, \sin(\frac{1}{4}\sum_{i=1}^{k}(1 - s_i)\phi_i)\right)^T , \qquad \text{(C.4)}$$

and the function $g_k^{(s)} \in \mathcal{C}_{per}(\mathbb{R}^k, SO_3(2))$ *by* $g_k^{(s)}(\phi) := \exp(\frac{1}{2}\mathcal{J}\sum_{i=1}^k(1-s_i)\phi_i)$. *Clearly* $\tilde{g}_k^{(s)}$ *has the* $\mathbb{S}^3$*-index* s *whence* $\tilde{g}_k^{(s)} \in \mathcal{C}_{per}^{\pm}(\mathbb{R}^k, \mathbb{S}^3)$. *Thus every* $s \in \{1,-1\}^k$ *is the* $\mathbb{S}^3$*-index* s *of some function in* $\mathcal{C}_{per}^{\pm}(\mathbb{R}^k, \mathbb{S}^3)$ *whence, by recalling Theorem C.13c, the group homomorphism* $Ind_{1,k}$ *is onto* $\{1,-1\}^k$. *Note also that* $\frac{1}{2}(1-s_1, ..., 1-s_k)^T$ *is the* $SO_3(2)$*-index of* $g_k^{(s)}$. □

Theorem C.15 *a) If* $g, g' \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ *with* $SO(3)$*-indices* s, s' *respectively then their product* gg' *is in* $\mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ *and has* $SO(3)$*-index* ss'. *The set* $\mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ *is a group under pointwise multiplication of* $SO(3)$ *valued functions. The function* $Ind_{3,k}$ *is a group homomorphism of the multiplicative group* $\mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ *onto the multiplicative group* $\{1,-1\}^k$.

b) Let $g \in \mathcal{C}(\mathbb{R}^k, SO_3(2))$ *and let* $f \in \mathcal{C}(\mathbb{R}^k, \mathbb{R})$ *be a lifting of* g *w.r.t.* $(\mathbb{R}, p_1, SO_3(2))$. *Then the function* $\tilde{g} \in \mathcal{C}(\mathbb{R}^k, \mathbb{S}^3)$, *defined by*

$$\tilde{g}(\phi) := \left(\cos(\frac{f(\phi)}{2}), 0, 0, \sin(\frac{f(\phi)}{2})\right)^T, \tag{C.5}$$

is a lifting of g *w.r.t.* $(\mathbb{S}^3, p_2, SO(3))$. *If* $g \in \mathcal{C}_{per}(\mathbb{R}^k, SO_3(2))$ *then*
$Ind_{3,k}(g) = ((-1)^{N_1}, ..., (-1)^{N_k})^T$ *where* $N := Ind_{2,k}(g)$.

c) Let $s \in \{1,-1\}^k$. *Then* $g_k^{(s)} = p_2 \circ \tilde{g}_k^{(s)}$, *i.e.,* $\tilde{g}_k^{(s)}$ *is a lifting of* $g_k^{(s)}$ *w.r.t.* $(\mathbb{S}^3, p_2, SO(3))$. *Moreover* $Ind_{3,k}(g_k^{(s)}) = s$.

Proof of Theorem C.15a: Let $g, g' \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ with $SO(3)$-indices s, s' respectively and let $\tilde{g}, \tilde{g}'$ be liftings of g, g' w.r.t. $(\mathbb{S}^3, p_2, SO(3))$. Clearly, by Definition C.14, $\tilde{g}, \tilde{g}'$ have $\mathbb{S}^3$-indices s, s' respectively. Since $SO(3)$ is a topological group, $gg' \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ whence gg' has an $SO(3)$-index. Because p_2 is a homomorphism we have $gg' = p_2(\tilde{g})p_2(\tilde{g}') = p_2(\tilde{g}\tilde{g}')$. Since, by Theorem C.13c, $\tilde{g}\tilde{g}'$ has $\mathbb{S}^3$-index ss', one finds that gg' has $SO(3)$-index ss'. Of course $g^{-1} = g^T$ whence $g^{-1} \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$. Since $\mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ is a subset of the multiplicative group $\mathcal{C}(\mathbb{R}^k, SO(3))$ and since $gg', g^{-1} \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ one concludes that

$\mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ is a subgroup of $\mathcal{C}(\mathbb{R}^k, SO(3))$. In particular, since $Ind_{3,k}(gg') = ss' = Ind_{3,k}(g)Ind_{3,k}(g')$, $Ind_{3,k}$ is a group homomorphism of the multiplicative group $\mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ into the multiplicative group $\{1,-1\}^k$. Since $Ind_{1,k}(\tilde{g}_k^{(s)}) = s$ and since, by Theorem C.13b, $p_2 \circ \tilde{g}_k^{(s)} \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ we have that $Ind_{3,k}(p_2 \circ \tilde{g}_k^{(s)}) = s$ whence $Ind_{3,k}$ is onto $\{1,-1\}^k$. □

Proof of Theorem C.15b: Let $g \in \mathcal{C}(\mathbb{R}^k, SO_3(2))$ and let f be a lifting of g w.r.t. $(\mathbb{R}, p_1, SO_3(2))$. Then the function $\tilde{g} \in \mathcal{C}(\mathbb{R}^k, \mathbb{S}^3)$, defined by (C.5), satisfies, for $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$, by using Definition C.2,

$$
\begin{aligned}
&p_2(\tilde{g}(\phi))x = p_2\Big(\cos(\frac{f(\phi)}{2}), 0, 0, \sin(\frac{f(\phi)}{2})\Big)x \\
&= \Big(2\cos^2(\frac{f(\phi)}{2}) - 1\Big)x + 2\sin^2(\frac{f(\phi)}{2})x_3 e^3 + 2\cos(\frac{f(\phi)}{2})\sin(\frac{f(\phi)}{2})(e^3 \times x) \\
&= \cos(f(\phi))x + (1 - \cos(f(\phi))x_3 e^3 + \sin(f(\phi))(e^3 \times x) \ ,
\end{aligned}
$$

whence $p_2(\tilde{g}(\phi))) = \exp(\mathcal{J} f(\phi)) = (p_1 \circ f)(\phi) = g(\phi)$, i.e., $\tilde{g}$ is a lifting of g w.r.t. $(\mathbb{S}^3, p_2, SO(3))$. Let g in addition be in $\mathcal{C}_{per}(\mathbb{R}^k, SO_3(2))$. By Theorem C.11b and Definition C.12 there exists a $f_{per} \in \mathcal{C}_{per}(\mathbb{R}^k, \mathbb{R})$ such that

$$
f(\phi) = N^T\phi + f_{per}(\phi) \ , \tag{C.6}
$$

where $N := Ind_{2,k}(g)$. It follows from (C.5) and (C.6) that for $j = 1, ..., k$

$$\tilde{g}(\phi + 2\pi e^j) = \Big(\cos(\frac{1}{2}f(\phi + 2\pi e^j)), 0, 0, \sin(\frac{1}{2}f(\phi + 2\pi e^j))\Big)^T$$
$$= \begin{pmatrix} \cos(\frac{1}{2}[N^T(\phi + 2\pi e^j) + f_{per}(\phi + 2\pi e^j)]) \\ 0 \\ 0 \\ \sin(\frac{1}{2}[N^T(\phi + 2\pi e^j) + f_{per}(\phi + 2\pi e^j)]) \end{pmatrix}$$
$$= \Big(\cos(\pi N_j + \frac{1}{2}[N^T\phi + f_{per}(\phi)]), 0, 0, \sin(\pi N_j + \frac{1}{2}[N^T\phi + f_{per}(\phi)])\Big)^T$$
$$= (-1)^{N_j}\Big(\cos(\frac{1}{2}[N^T\phi + f_{per}(\phi)]), 0, 0, \sin(\frac{1}{2}[N^T\phi + f_{per}(\phi)])\Big)^T$$
$$= (-1)^{N_j}\Big(\cos(\frac{f(\phi)}{2}), 0, 0, \sin(\frac{f(\phi)}{2})\Big)^T = (-1)^{N_j}\tilde{g}(\phi) \ .$$

Thus $((-1)^{N_1}, ..., (-1)^{N_k})^T$ is the $\mathbb{S}^3$-index of $\tilde{g}$
whence $((-1)^{N_1}, ..., (-1)^{N_k})^T = Ind_{1,k}(\tilde{g}) = Ind_{3,k}(p_2 \circ \tilde{g}) = Ind_{3,k}(g)$. □

Proof of Theorem C.15c: Let $s \in \{1, -1\}^k$. We first observe, by Definition C.14, that the function $f \in \mathcal{C}(\mathbb{R}^k, \mathbb{R})$, defined by $f(\phi) := \frac{1}{2}\sum_{i=1}^k(1 - s_i)\phi_i$, is a lifting of $g_k^{(s)}$ w.r.t. $(\mathbb{R}, p_1, SO_3(2))$. Thus, by Theorem C.15b, the function $\tilde{g} \in \mathcal{C}(\mathbb{R}^k, \mathbb{S}^3)$, defined by

$$\tilde{g}(\phi) := \Big(\cos(\frac{1}{4}\sum_{i=1}^k(1 - s_i)\phi_i), 0, 0, \sin(\frac{1}{4}\sum_{i=1}^k(1 - s_i)\phi_i)\Big)^T \ , \qquad \text{(C.7)}$$

is a lifting of $g_k^{(s)}$ w.r.t. $(\mathbb{S}^3, p_2, SO(3))$. However, $\tilde{g}$ in (C.7) is equal to $\tilde{g}_k^{(s)}$ whence $\tilde{g}_k^{(s)}$ is a lifting of $g_k^{(s)}$ w.r.t. $(\mathbb{S}^3, p_2, SO(3))$. Since, by Definition C.14, $Ind_{1,k}(\tilde{g}_k^{(s)}) = s$, I conclude by Definition C.14 that $Ind_{3,k}(g_k^{(s)}) = s$. □

Since $Ind_{2,k}(g_k^{(s)}) = \frac{1}{2}(1 - s_1, ..., 1 - s_k)^T$, the claim of Theorem C.15c, that $Ind_{3,k}(g_k^{(s)}) = s$, confirms the last claim of Theorem C.15b.

C.3 Liftings of 2π-periodic functions on $\mathbb{R}^k$ and basic properties of $[\mathbb{T}^k, SO(3)]$

With Section C.2 I have obtained a string of theorems about liftings w.r.t. the four fiber structures in Corollary C.9, giving important clues about $\mathcal{C}_{per}(\mathbb{R}^k, X)$ for various topological spaces X. The final touch on $\mathcal{C}_{per}(\mathbb{R}^k, X)$ will be provided in the present section where I make systematic use of factors of functions $g \in \mathcal{C}_{per}(\mathbb{R}^k, X)$ w.r.t. $(\mathbb{R}^k, p_{4,k}, \mathbb{T}^k)$. Most importantly, the factors will allow me to define equivalence classes on $\mathcal{C}_{per}(\mathbb{R}^k, X)$ in terms of the homotopy classes in $\mathcal{C}(\mathbb{T}^k, X)$. This, in turn, will give insight into the relevance of the $SO_3(2)$-index, $SO(3)$-index, and $\mathbb{S}^3$-index for Homotopy Theory and, in particular, will allow me to determine the homotopy classes in $\mathcal{C}(\mathbb{T}^k, SO(3))$ for $k = 1, 2, 3$.

Lemma C.16 *Let X' be a set and $g' : \mathbb{R}^k \to X'$ be a 2π-periodic function. Then there exists one and only one function $f' : \mathbb{T}^k \to X'$ such that $g' = f' \circ p_{4,k}$.*

Let X be a topological space and $g \in \mathcal{C}_{per}(\mathbb{R}^k, X)$. Then there exists one and only one function $f \in \mathcal{C}(\mathbb{T}^k, X)$ such that $g = f \circ p_{4,k}$, i.e., g has the unique factor f w.r.t. $(\mathbb{R}^k, p_{4,k}, \mathbb{T}^k)$.

Proof of Lemma C.16: Let X' be a set and $g' : \mathbb{R}^k \to X'$ be a 2π-periodic function. Since $p_{4,k}$ is onto $\mathbb{T}^k$, f' is unique (if it exists). To prove existence I define the function $f' : \mathbb{T}^k \to X'$ by $f'(\exp(i2\pi x_1), ..., \exp(i2\pi x_k)) := g'(2\pi x)$ where $x = (x_1, ..., x_k)^T \in [0, 1)^k$. Clearly, for arbitrary $x \in \mathbb{R}^k$, we have

$$
\begin{aligned}
& g'(2\pi x) = g'(2\pi\lfloor x_1\rfloor, ..., 2\pi\lfloor x_k\rfloor) = f'(\exp(i2\pi\lfloor x_1\rfloor), ..., \exp(i2\pi\lfloor x_k\rfloor)) \\
& = f'(\exp(i2\pi x_1), ..., \exp(i2\pi x_k)) = f'(p'_{4,k}(2\pi x)) \ , \qquad \text{(C.8)}
\end{aligned}
$$

where in the first and third equalities I used the 2π-periodicity of g'. It follows from (C.8) that $g' = f' \circ p_{4,k}$.

Let X be a topological space and $g \in \mathcal{C}_{per}(\mathbb{R}^k, X)$. By the first claim there exists one and only one function $f : \mathbb{T}^k \to X$ such that $g = f \circ p_{4,k}$. To prove the continuity of f I first note by Proposition C.8a that $p_{4,k}$ is a covering map w.r.t. $\mathbb{R}^k$ and $\mathbb{T}^k$. It follows (see [SZ, 6.1.3]) that $p_{4,k}$ is identifying whence (see [SZ, 1.2.9]) f is continuous. Using Definition C.1, this implies that f is the unique factor of g w.r.t. $(\mathbb{R}^k, p_{4,k}, \mathbb{T}^k)$. □

Lemma C.16 leads to the following definition.

Definition C.17 *Let X be a topological space. Then, by using Lemma C.16, I define, for every positive integer k, the function $FAC_k(\cdot; X) : \mathcal{C}_{per}(\mathbb{R}^k, X) \to \mathcal{C}(\mathbb{T}^k, X)$ by $FAC_k(g; X) := f$ where f is the unique factor of $g \in \mathcal{C}_{per}(\mathbb{R}^k, X)$ w.r.t. $(\mathbb{R}^k, p_{4,k}, \mathbb{T}^k)$. Let $g_i \in \mathcal{C}_{per}(\mathbb{R}^k, X)$ where $i = 0, 1$. Then g_0 and g_1 are called '2π-homotopic w.r.t. X', written $g_0 \simeq_X^{2\pi} g_1$, if $FAC_k(g_0; X) \simeq_X FAC_k(g_1; X)$. Moreover, a $g \in \mathcal{C}_{per}(\mathbb{R}^k, X)$ is called '2π-nullhomotopic w.r.t. X' if $FAC_k(g; X)$ is nullhomotopic w.r.t. X.* □

Proposition C.18 *a) Let X be a topological space and $G \in \mathcal{C}(\mathbb{R}^k \times [0,1], X)$ such that each $G(\cdot, t)$ is in $\mathcal{C}_{per}(\mathbb{R}^k, X)$. Then the function $F : \mathbb{T}^k \times [0,1] \to X$, defined by $F(\cdot, t) := FAC_k(G(\cdot, t), X)$, is in $\mathcal{C}(\mathbb{T}^k \times [0,1], X)$.*

b) Let X be a topological space and let $g_i \in \mathcal{C}_{per}(\mathbb{R}^k, X)$ where $i = 0, 1$. Then $g_0 \simeq_X^{2\pi} g_1$ iff a $G \in \mathcal{C}(\mathbb{R}^k \times [0,1], X)$ exists such that $G(\cdot, i) = g_i$ and $G(\cdot, t) \in \mathcal{C}_{per}(\mathbb{R}^k, X)$. Moreover $\simeq_X^{2\pi}$ is an equivalence relation on $\mathcal{C}_{per}(\mathbb{R}^k, X)$. Furthermore a $h_0 \in \mathcal{C}_{per}(\mathbb{R}^k, X)$ is 2π-nullhomotopic w.r.t. X iff a constant function $h_1 \in \mathcal{C}_{per}(\mathbb{R}^k, X)$ exists such that $h_0 \simeq_X^{2\pi} h_1$.

c) Let X be a path-connected topological space. Then all functions in $\mathcal{C}_{per}(\mathbb{R}^k, X)$, which are 2π-nullhomotopic w.r.t. X, are 2π-homotopic w.r.t. X.

d) Let X and Y be topological spaces and let $g_i \in \mathcal{C}_{per}(\mathbb{R}^k, X)$ and $f_i \in \mathcal{C}(X, Y)$ where

$i = 0, 1$. *Then the* $f_i \circ g_i$ *are in* $\mathcal{C}_{per}(\mathbb{R}^k, Y)$ *and, if* $f_0 \simeq_Y f_1$ *and* $g_0 \simeq_X^{2\pi} g_1$*, then* $f_1 \circ g_1 \simeq_Y^{2\pi} f_0 \circ g_0$.

e) If $g_0, g_1 \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ *with* $g_0 \simeq_{SO(3)}^{2\pi} g_1$ *then* $Ind_{3,k}(g_0) = Ind_{3,k}(g_1)$. *If* $g \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ *is* 2π*-nullhomotopic w.r.t.* $SO(3)$ *then* $Ind_{3,k}(g)$ *is the identity.*

f) Let X *be a topological space. Let* g *be in* $\mathcal{C}_{per}(\mathbb{R}^k, X)$ *and* $\phi_0 \in \mathbb{R}^k$. *Then* $g(\cdot) \simeq_X^{2\pi} g(\cdot + \phi_0)$ *and* $Ind_{3,k}(g(\cdot)) = Ind_{3,k}(g(\cdot + \phi_0))$.

Proof of Proposition C.18a: Let X be a topological space and $G \in \mathcal{C}(\mathbb{R}^k \times [0,1], X)$ such that each $G(\cdot, t)$ is in $\mathcal{C}_{per}(\mathbb{R}^k, X)$. I define the function $F : \mathbb{T}^k \times [0,1] \to X$ by $F(\cdot, t) := FAC_k(G(\cdot, t), X)$. Of course, $G(\phi, t) = F(p_{4,k}(\phi), t)$ whence $G = F \circ h$ where the function $h : \mathbb{R}^k \times [0,1] \to \mathbb{T}^k \times [0,1]$ is defined by $h(\phi, t) := (p_{4,k}(\phi), t)$. We know from the proof of Lemma C.16 that $p_{4,k}$ is identifying. Since $[0,1]$ is compact and Hausdorff, I conclude that the function h is identifying (see [Du, Section XII.4]). Because $G = F \circ h$ and h is identifying I thus conclude that F is continuous (see [SZ, 1.2.9]).

Proof of Proposition C.18b: Let X be a topological space and let $g_i \in \mathcal{C}_{per}(\mathbb{R}^k, X)$ where $i = 0, 1$. I abbreviate $f_i := FAC_k(g_i; X) \in \mathcal{C}_{per}(\mathbb{T}^k, X)$.

I first assume that $g_0 \simeq_X^{2\pi} g_1$. Then, by Definition C.17, $f_0 \simeq_X f_1$ whence a function $F \in \mathcal{C}(\mathbb{T}^k \times [0,1], X)$ exists such that $F(\cdot, i) = f_i(\cdot)$. The function $G : \mathbb{R}^k \times [0,1] \to X$, defined by $G(\phi, t) := F(p_{4,k}(\phi), t)$, is continuous and $G(\phi, t)$ is 2π-periodic in ϕ whence $G(\cdot, t) \in \mathcal{C}_{per}(\mathbb{R}^k, X)$. Moreover $G(\phi, i) = F(p_{4,k}(\phi), i) = f_i(p_{4,k}(\phi)) = g_i(\phi)$.

To prove the other direction I assume that a function $G \in \mathcal{C}(\mathbb{R}^k \times [0,1], X)$ exists such that $G(\cdot, i) = g_i$ and such that each $G(\cdot, t)$ is in $\mathcal{C}_{per}(\mathbb{R}^k, X)$. I define the function $F : \mathbb{T}^k \times [0,1] \to X$ by $F(\cdot, t) := FAC_k(G(\cdot, t), X)$. Clearly $F(\cdot, i) = f_i$ and, by Proposition C.18a, $F \in \mathcal{C}(\mathbb{T}^k \times [0,1], X)$. Therefore $f_0 \simeq_X f_1$ whence $g_0 \simeq_X^{2\pi} g_1$.

The second claim follows from the facts that $\simeq_X$ is an equivalence relation on $\mathcal{C}(\mathbb{T}^k, X)$ and that $FAC_k(\cdot, X)$ is a function from $\mathcal{C}_{per}(\mathbb{R}^k, X)$ into $\mathcal{C}(\mathbb{T}^k, X)$.

To prove the third claim I first consider a $h_0 \in \mathcal{C}_{per}(\mathbb{R}^k, X)$ which is 2π-nullhomotopic w.r.t. X. Then, by Definition C.17, $FAC_k(h_0, X)$ is nullhomotopic w.r.t. X whence a function $K \in \mathcal{C}(\mathbb{T}^k \times [0,1], X)$ exists such that $K(\cdot, 0) = FAC_k(h_0, X)$ and such $K(\cdot, 1)$ is a constant function. Clearly the function $h_1 \in \mathcal{C}_{per}(\mathbb{R}^k, X)$, defined by $h_1 := K(\cdot, 1) \circ p_{4,k}$, is constant and satisfies $K(\cdot, 1) = FAC_k(h_1, X)$. Thus $FAC_k(h_1, X) = K(\cdot, 1) \simeq_X K(\cdot, 0) = FAC_k(h_0, X)$ whence $h_0 \simeq_X^{2\pi} h_1$, i.e., h_0 is 2π-homotopic w.r.t. X to the constant function h_1.

To prove the other direction I consider $h_0, h_1 \in \mathcal{C}_{per}(\mathbb{R}^k, X)$ such that h_1 is constant and $h_0 \simeq_X^{2\pi} h_1$. Thus $FAC_k(h_0, X) \simeq_X FAC_k(h_1, X)$ and $FAC_k(h_1, X)$ is constant. It follows that $FAC_k(h_0, X)$ is nullhomotopic w.r.t. X whence h_0 is 2π-nullhomotopic w.r.t. X. □

Proof of Proposition C.18c: Let X be a path-connected topological space and let $g_0, g_1 \in \mathcal{C}_{per}(\mathbb{R}^k, X)$ be 2π-nullhomotopic w.r.t. X. Thus $FAC_k(g_0, X), FAC_k(g_1, X)$ are nullhomotopic w.r.t. X. Since X is path-connected I conclude from Proposition C.4c that $FAC_k(g_0, X) \simeq_X FAC_k(g_1, X)$. It follows from Definition C.17 that $g_0 \simeq_X^{2\pi} g_1$. □

Proof of Proposition C.18d: Let X and Y be topological spaces and let $g_i \in \mathcal{C}_{per}(\mathbb{R}^k, X)$ and $f_i \in \mathcal{C}(X, Y)$ where $i = 0, 1$. Clearly the $f_i \circ g_i$ are in $\mathcal{C}_{per}(\mathbb{R}^k, Y)$. Let also $f_0 \simeq_Y f_1$ and $g_0 \simeq_X^{2\pi} g_1$. Thus a $F \in \mathcal{C}(X \times [0,1], Y)$ exists such that $F(\cdot, i) = f_i(\cdot)$. Furthermore, by Proposition C.18b, a $G \in \mathcal{C}(\mathbb{R}^k \times [0,1], X)$ exists such that $G(\cdot, i) = g_i(\cdot)$ and such that each $G(\cdot, t)$ is in $\mathcal{C}_{per}(\mathbb{R}^k, X)$. The function $H : \mathbb{R}^k \times [0,1] \to Y$, defined by $H(x,t) := F(G(x,t), t)$, is continuous and satisfies $H(x, i) = F(G(x,i), i) = F(g_i(x), i) = f_i(g_i(x))$. Furthermore each $H(\cdot, t)$ is in $\mathcal{C}_{per}(\mathbb{R}^k, Y)$. Using again Proposition C.18b, we thus have shown that

$f_1 \circ g_1 \simeq_Y^{2\pi} f_0 \circ g_0$. □

Proof of Proposition C.18e: Let $g_0, g_1 \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ with $g_0 \simeq_{SO(3)}^{2\pi} g_1$. By Proposition C.18b a $G \in \mathcal{C}(\mathbb{R}^k \times [0,1], SO(3))$ exists such that $g_i(\cdot) = G(\cdot, i)$ and $G(\cdot, t) \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ where $i = 0, 1$. It follows from Theorem C.13d that a $\tilde{G} \in \mathcal{C}(\mathbb{R}^k \times [0,1], \mathbb{S}^3)$ exists such that $G = p_2 \circ \tilde{G}$ and such that, for all $t \in [0,1]$, $\tilde{G}(\cdot, t) \in \mathcal{C}_{per}^{\pm}(\mathbb{R}^k, \mathbb{S}^3)$ and $Ind_{1,k}(\tilde{G}(\cdot, 0)) = Ind_{1,k}(\tilde{G}(\cdot, t))$. Defining $\tilde{g}_0, \tilde{g}_1 \in \mathcal{C}_{per}^{\pm}(\mathbb{R}^k, \mathbb{S}^3)$ by $\tilde{g}_i(\cdot) := \tilde{G}(\cdot, i)$ we get $Ind_{1,k}(\tilde{g}_0) = Ind_{1,k}(\tilde{G}(\cdot, 0)) = Ind_{1,k}(\tilde{G}(\cdot, 1)) = Ind_{1,k}(\tilde{g}_1)$ and $g_i = p_2 \circ \tilde{g}_i$ whence $Ind_{3,k}(g_0) = Ind_{3,k}(p_2 \circ \tilde{g}_0) = Ind_{1,k}(\tilde{g}_0) = Ind_{1,k}(\tilde{g}_1) = Ind_{3,k}(p_2 \circ \tilde{g}_1) = Ind_{3,k}(g_1)$.

Let $g \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ be 2π-nullhomotopic w.r.t. $SO(3)$. Thus, by Proposition C.18b, a constant function $h \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ exists such that $g \simeq_{SO(3)}^{2\pi} h$. Since, by Definition C.14, $Ind_{3,k}(h) = (1, ..., 1)^T$ one concludes from the first claim that $Ind_{3,k}(g) = (1, ..., 1)^T$. □

Proof of Proposition C.18f: Let X be a topological space. Let g be in $\mathcal{C}_{per}(\mathbb{R}^k, X)$ and $\phi_0 \in \mathbb{R}^k$. I define the function $G \in \mathcal{C}(\mathbb{R}^k \times [0,1], X)$ by $G(\phi, t) := g(\phi + t\phi_0)$. Clearly $G(\cdot, 0) = g(\cdot)$, $G(\cdot, 1) = g(\cdot + \phi_0)$ and each $G(\cdot, t)$ is in $\mathcal{C}_{per}(\mathbb{R}^k, X)$. Thus, by Proposition C.18b, $g(\cdot) \simeq_X^{2\pi} g(\cdot + \phi_0)$ whence, by Proposition C.18e, $Ind_{3,k}(g(\cdot)) = Ind_{3,k}(g(\cdot + \phi_0))$. □

Definition C.19 *Let X be a topological space. Using the fact from Proposition C.18b that $\simeq_X^{2\pi}$ is an equivalence relation on $\mathcal{C}_{per}(\mathbb{R}^k, X)$ I denote the set of equivalence classes w.r.t. $\simeq_X^{2\pi}$ by $[\mathbb{R}^k, X]_{2\pi}$.*

Let $\tilde{g}_0, \tilde{g}_1 \in \mathcal{C}_{per}^{\pm}(\mathbb{R}^k, \mathbb{S}^3)$. Then, by Theorem C.13b, $p_2 \circ \tilde{g}_0, p_2 \circ \tilde{g}_1 \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ and I write $\tilde{g}_0 \simeq_{\mathbb{S}^3}^{2\pi,\pm} \tilde{g}_1$ if $p_2 \circ \tilde{g}_0 \simeq_{SO(3)}^{2\pi} p_2 \circ \tilde{g}_1$. Clearly $\simeq_{\mathbb{S}^3}^{2\pi,\pm}$ is an equivalence relation on $\mathcal{C}_{per}^{\pm}(\mathbb{R}^k, \mathbb{S}^3)$. I denote by $[\mathbb{R}^k, \mathbb{S}^3]_{2\pi}^{\pm}$ the set of equivalence classes w.r.t. $\simeq_{\mathbb{S}^3}^{2\pi,\pm}$. □

Recalling Definitions C.17,C.19 and using the fact that $FAC_k(\cdot, X)$ is onto $\mathcal{C}(\mathbb{T}^k, X)$, it follows that $[\mathbb{R}^k, X]_{2\pi}$ and $[\mathbb{T}^k, X]$ have the same cardinality. Moreover, by Theorem C.13b, each $g \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ has a lifting $\tilde{f} \in \mathcal{C}^{\pm}_{per}(\mathbb{R}^k, \mathbb{S}^3)$ w.r.t. $(\mathbb{S}^3, p_2, SO(3))$ whence $[\mathbb{R}^k, \mathbb{S}^3]^{\pm}_{2\pi}$ and $[\mathbb{R}^k, SO(3)]_{2\pi}$ have the same cardinality so that $[\mathbb{R}^k, \mathbb{S}^3]^{\pm}_{2\pi}$, $[\mathbb{R}^k, SO(3)]_{2\pi}$, and $[\mathbb{T}^k, SO(3)]$ have the same cardinality.

Proposition C.20 *Let G be a topological group and X be a topological space. Then the following hold.*

a) Let $g, g_0, g_1 \in \mathcal{C}(X, G)$. Then $g_0 \simeq_G g_1$ iff $g_0 g \simeq_G g_1 g$ and $g_0 \simeq_G g_1$ iff $g g_0 \simeq_G g g_1$.

b) Let $f, f_0, f_1 \in \mathcal{C}_{per}(\mathbb{R}^k, G)$. Then $f_0 \simeq^{2\pi}_G f_1$ iff $f_0 f \simeq^{2\pi}_G f_1 f$ and $f_0 \simeq^{2\pi}_G f_1$ iff $f f_0 \simeq^{2\pi}_G f f_1$.

Proof of Proposition C.20a: Let G be a topological group, X be a topological space and $g, g_0, g_1 \in \mathcal{C}(X, G)$.

If $g_0 \simeq_G g_1$ then a $F \in \mathcal{C}(X \times [0,1], G)$ exists with $F(\cdot, i) = g_i(\cdot)$ so that, since G is a topological group, $gF, Fg \in \mathcal{C}(X \times [0,1], G)$ with $g(\cdot)F(\cdot, i) = g(\cdot)g_i(\cdot)$ and $F(\cdot, i)g(\cdot) = g_i(\cdot)g(\cdot)$ whence $g_0 g \simeq_G g_1 g$ and $g g_0 \simeq_G g g_1$.

To prove the other direction let $g_0 g \simeq_G g_1 g$. Thus $H \in \mathcal{C}(X \times [0,1], G)$ exists with $H(\cdot, i) = g_i(\cdot)g(\cdot)$. Since G is a topological group, $Hg^{-1} \in \mathcal{C}(X \times [0,1], G)$ with $H(\cdot, i)g^{-1} = g_i(\cdot)$ whence $g_0 \simeq_G g_1$. Analogously, $g g_0 \simeq_G g g_1$ implies $g_0 \simeq_G g_1$. □

Proof of Proposition C.20b: Let $f, f_0, f_1 \in \mathcal{C}_{per}(\mathbb{R}^k, G)$. I abbreviate $g' := FAC_k(f, G) \in \mathcal{C}(\mathbb{T}^k, G)$ and $g'_i := FAC_k(f_i, G) \in \mathcal{C}(\mathbb{T}^k, G)$ where $i = 0, 1$. Clearly $FAC_k(f_i f, G) \circ p_{4,k} = f_i f = (FAC_k(f_i, G) \circ p_{4,k})(FAC_k(f, G) \circ p_{4,k}) = (g'_i \circ p_{4,k})(g' \circ p_{4,k}) = (g'_i g') \circ p_{4,k}$ whence $g'_i g' = FAC_k(f_i f, G)$ and, analogously, $g' g'_i = FAC_k(f f_i, G)$.

I first assume that $f_0 \simeq_G^{2\pi} f_1$. Thus, by Definition C.17, $g'_0 \simeq_G g'_1$ whence, by Proposition C.20a, $g'_0 g' \simeq_G g'_1 g'$ and $g' g'_0 \simeq_G g' g'_1$. Thus $FAC_k(f_0 f, G) = g'_0 g' \simeq_G g'_1 g' = FAC_k(f_1 f, G)$ and $FAC_k(f f_0, G) = g' g'_0 \simeq_G g' g'_1 = FAC_k(f f_1, G)$ whence, by Definition C.17, $f_0 f \simeq_G^{2\pi} f_1 f$ and $f f_0 \simeq_G^{2\pi} f f_1$.

To prove the other direction let $f_0 f \simeq_G^{2\pi} f_1 f$. Thus, by Definition C.17, $g'_0 g' = FAC_k(f_0 f, G) \simeq_G FAC_k(f_1 f, G) = g'_1 g'$ whence, by Proposition C.20a, $g'_0 \simeq_G g'_1$ so that, by Definition C.17, $f_0 \simeq_G^{2\pi} f_1$. Analogously, $f f_0 \simeq_G^{2\pi} f f_1$ implies $f_0 \simeq_G^{2\pi} f_1$.
□

The following definition provides important tools I need for studying $[\mathbb{R}^3, SO(3)]_{2\pi}$ and $[\mathbb{T}^3, SO(3)]$.

Definition C.21 *(deg, Deg, DEG)*

As is well known [tDi1, Section II.9], since the topological space $\mathbb{T}^3$ carries the structure of a compact, orientable, connected three-dimensional C^∞ manifold without boundary, two functions in $\mathcal{C}(\mathbb{T}^3, \mathbb{S}^3)$ are homotopic w.r.t. $\mathbb{S}^3$ iff they have the same degree. The 'degree' $deg(F)$ of a function $F \in \mathcal{C}(\mathbb{T}^3, \mathbb{S}^3)$ is an integer, defined in an analytic fashion, as follows [tDi1, Section II.9]. For any C^∞ function $\hat{F}$ in $\mathcal{C}(\mathbb{T}^3, \mathbb{S}^3)$ one picks a regular value y of $\hat{F}$ and defines the 'degree' of $\hat{F}$ by

$$deg(\hat{F}) := \begin{cases} \sum_{x \in \hat{F}^{-1}(y)} sig(T_x \hat{F}) & \text{if } \hat{F}^{-1}(y) \neq \emptyset \\ 0 & \text{if } \hat{F}^{-1}(y) = \emptyset \end{cases},$$

where $T_x \hat{F}$ is the derivative of $\hat{F}$ at x and where $sig(T_x \hat{F}) = 1$ if $T_x \hat{F}$ is orientation preserving and $= -1$ otherwise. Note that y being a regular value of $\hat{F}$ means that either $\hat{F}^{-1}(y) = \emptyset$ or that, for every $x \in \hat{F}^{-1}(y)$, the linear function $T_x \hat{F}$ is nonsingular. One can show that the integer $deg(\hat{F})$ is independent of the choice of y and is the same for any C^∞ function in $\mathcal{C}(\mathbb{T}^3, \mathbb{S}^3)$ which is homotopic to $\hat{F}$ w.r.t. $\mathbb{S}^3$. Thus, for every $F \in \mathcal{C}(\mathbb{T}^3, \mathbb{S}^3)$, one defines $deg(F) := deg(\hat{F})$, where $\hat{F}$ is any C^∞ function *in* $\mathcal{C}(\mathbb{T}^3, \mathbb{S}^3)$ *which is homotopic to* F *w.r.t.* $\mathbb{S}^3$ *(note that there is always such a* $\hat{F}$*). Furthermore, by the aforementioned properties of* $\mathbb{T}^3$*, there exists, for every integer* n*, a function* $F \in \mathcal{C}(\mathbb{T}^3, \mathbb{S}^3)$ *whose degree is* n *whence the degree induces a bijection from* $[\mathbb{T}^3, \mathbb{S}^3]$ *onto* $\mathbb{Z}$*. Moreover a function* $F \in \mathcal{C}(\mathbb{T}^3, \mathbb{S}^3)$ *is nullhomotopic w.r.t.* $\mathbb{S}^3$ *iff* $deg(F) = 0$*.*

If $\tilde{g} \in \mathcal{C}_{per}(\mathbb{R}^3, \mathbb{S}^3)$*, I define* $Deg(\tilde{g}) := deg(FAC_3(\tilde{g}, \mathbb{S}^3)) \in \mathbb{Z}$*. Since* $\mathbb{S}^3$ *is a path-connected and since, by Definition C.2, for* $x \in \mathbb{S}^3$*,* $-x = x(-1,0,0,0)^T$*, a function* $f \in \mathcal{C}([0,1], \mathbb{S}^3)$ *exists such that* $f(0) = (1,0,0,0)^T$ *and* $f(1) = (-1,0,0,0)^T$*. Thus, since* $\mathbb{S}^3$ *is a topological group, for* $F \in \mathcal{C}(\mathbb{T}^3, \mathbb{S}^3)$*, I define* $G \in \mathcal{C}(\mathbb{T}^3 \times [0,1], \mathbb{S}^3)$ *by* $G(z,t) := F(z)f(t)$*. Clearly* $G(z,0) = F(z), G(z,1) = -F(z)$ *whence, for* $F \in \mathcal{C}(\mathbb{T}^3, \mathbb{S}^3)$*, we have* $F \simeq_{\mathbb{S}^3} (-F)$ *so that* $deg(F) = deg(-F)$*. It follows that if* $\tilde{g} \in \mathcal{C}_{per}(\mathbb{R}^3, \mathbb{S}^3)$*, then* $Deg(-\tilde{g}) = deg(FAC_3(-\tilde{g}, \mathbb{S}^3)) = deg(-FAC_3(\tilde{g}, \mathbb{S}^3)) = deg(FAC_3(\tilde{g}, \mathbb{S}^3)) = Deg(\tilde{g})$*. The equality* $Deg(-\tilde{g}) = Deg(\tilde{g})$ *will be needed for the definition of* DEG *in the following paragraph.*

Let $g \in \mathcal{C}_{per}(\mathbb{R}^3, SO(3))$ *and let* $\pm\tilde{g}$ *be the liftings of* g *w.r.t.* $(\mathbb{S}^3, p_2, SO(3))$*. Abbreviating* $s := Ind_{3,3}(g)$*, we get, by Definition C.14,* $Ind_{1,3}(\pm\tilde{g}) = s$ *whence, by Theorem C.13c and Definition C.14,* $Ind_{1,3}(\pm(\tilde{g}\tilde{g}_3^{(s)})) = Ind_{1,3}((\pm\tilde{g})\tilde{g}_3^{(s)})$ $= Ind_{1,3}(\pm\tilde{g})Ind_{1,3}(\tilde{g}_3^{(s)}) = ss = (1,...,1)^T$ *so that, by Definition C.12,* $\pm(\tilde{g}\tilde{g}_3^{(s)}) \in \mathcal{C}_{per}(\mathbb{R}^3, \mathbb{S}^3)$*. I define* $DEG(g) := Deg(\tilde{g}\tilde{g}_3^{(s)})$*. Note that this definition is meaningful since, by the previous paragraph,* $DEG(g) = Deg(\tilde{g}\tilde{g}_3^{(s)}) = Deg(-\tilde{g}\tilde{g}_3^{(s)})$*, i.e., the definition of* $DEG(g)$ *is independent of the choice of the lifting* $\tilde{g}$*.*

Note finally that while the degree 'deg' is an ubiquitious definition, the definition of 'Deg' and 'DEG' is introduced here just for the purposes of the present work. □

Remarkably, parts c) and f) of the following theorem reveal, for $k = 1, 2, 3$, the structure of $[\mathbb{R}^k, SO(3)]_{2\pi}$ and $[\mathbb{T}^k, SO(3)]$ solely in terms of $Ind_{3,k}$ and DEG.

Theorem C.22 *a) For $k = 1, 2$, all functions in $\mathcal{C}(\mathbb{T}^k, \mathbb{S}^3)$ are nullhomotopic w.r.t. $\mathbb{S}^3$ and $[\mathbb{T}^k, \mathbb{S}^3]$ is a singleton.*

b) Let $g \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ where $k = 1, 2$. Then $Ind_{3,k}(g)$ is the identity iff g is 2π-nullhomotopic w.r.t. $SO(3)$.

c) Let $g_0, g_1 \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ and $k = 1, 2$. Then $g_0 \simeq^{2\pi}_{SO(3)} g_1$ iff $Ind_{3,k}(g_0) = Ind_{3,k}(g_1)$. Let $F_0, F_1 \in \mathcal{C}(\mathbb{T}^k, SO(3))$ and $k = 1, 2$. Then $F_0 \simeq_{SO(3)} F_1$ iff $Ind_{3,k}(F_0 \circ p_{4,k}) = Ind_{3,k}(F_1 \circ p_{4,k})$.

d) Let the $SO(3)$-index of $g_0, g_1 \in \mathcal{C}_{per}(\mathbb{R}^3, SO(3))$ be the identity. Then $g_0 \simeq^{2\pi}_{SO(3)} g_1$ iff $DEG(g_0) = DEG(g_1)$.

e) Let $g \in \mathcal{C}_{per}(\mathbb{R}^3, SO(3))$ and let me abbreviate $s := Ind_{3,3}(g)$. Then $DEG(g) = DEG(gg_3^{(s)})$.

f) Let $g_0, g_1 \in \mathcal{C}_{per}(\mathbb{R}^3, SO(3))$. Then $g_0 \simeq^{2\pi}_{SO(3)} g_1$ iff $Ind_{3,3}(g_0) = Ind_{3,3}(g_1)$ and $DEG(g_0) = DEG(g_1)$. Let $F_0, F_1 \in \mathcal{C}(\mathbb{T}^3, SO(3))$. Then $F_0 \simeq_{SO(3)} F_1$ iff $Ind_{3,3}(F_0 \circ p_{4,3}) = Ind_{3,3}(F_1 \circ p_{4,3})$ and $DEG(F_0 \circ p_{4,3}) = DEG(F_1 \circ p_{4,3})$.

g) Let $g_0, g_1 \in \mathcal{C}_{per}(\mathbb{R}^k, SO_3(2))$. Then $g_0 \simeq^{2\pi}_{SO(3)} g_1$ iff $Ind_{3,k}(g_0) = Ind_{3,k}(g_1)$. Moreover a $g \in \mathcal{C}_{per}(\mathbb{R}^k, SO_3(2))$ is 2π-nullhomotopic w.r.t. $SO(3)$ iff $Ind_{3,k}(g)$ is the identity. Furthermore a $g \in \mathcal{C}_{per}(\mathbb{R}^k, SO_3(2))$ is 2π-nullhomotopic w.r.t. $SO(3)$ iff the components of $Ind_{2,k}(g)$ are even integers.

Proof of Theorem C.22a: The topological space $\mathbb{T}^k$ carries the structure of a k-dimensional C^∞ manifold without boundary. It thus follows, for $k = 1, 2$, that all functions in $\mathcal{C}(\mathbb{T}^k, \mathbb{S}^3)$ are nullhomotopic w.r.t. $\mathbb{S}^3$ [Bre, Section II.11]. Since $\mathbb{S}^3$ is path-connected, this implies by Proposition C.4c that, for $k = 1, 2$, $[\mathbb{T}^k, \mathbb{S}^3]$ is a singleton. □

Proof of Theorem C.22b: Let $g_0 \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ where $k = 1, 2$. Let $\tilde{g}_0$ be a lifting of g_0 w.r.t. $(\mathbb{S}^3, p_2, SO(3))$.

I first assume that the $SO(3)$-index of g_0 is the identity. Thus, by Definition C.14, $Ind_{1,k}(\tilde{g}_0)$ is the identity whence, by Definition C.12, $\tilde{g}_0 \in \mathcal{C}_{per}(\mathbb{R}^k, \mathbb{S}^3)$ and I define $F_0 \in \mathcal{C}(\mathbb{T}^k, \mathbb{S}^3)$ by $F_0 := FAC_k(\tilde{g}_0, \mathbb{S}^3)$. By Theorem C.22a, F_0 is nullhomotopic w.r.t. $\mathbb{S}^3$ whence a constant function $F_1 \in \mathcal{C}(\mathbb{T}^k, \mathbb{S}^3)$ exists such that $F_0 \simeq_{\mathbb{S}^3} F_1$. It follows that $p_2 \circ F_1$ is a constant function in $\mathcal{C}(\mathbb{T}^k, SO(3))$ and that, by Proposition C.4a, $p_2 \circ F_0 \simeq_{SO(3)} p_2 \circ F_1$. Applying Definition C.17, one concludes that $p_2 \circ F_0 \circ p_{4,k} \simeq^{2\pi}_{SO(3)} p_2 \circ F_1 \circ p_{4,k}$. Note that $p_2 \circ F_0 \circ p_{4,k} = p_2 \circ \tilde{g}_0 = g_0$. Defining $g_1 \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ by $g_1 := p_2 \circ F_1 \circ p_{4,k}$, one observes that g_1 is constant and that $g_0 \simeq^{2\pi}_{SO(3)} g_1$. Since g_1 is constant one concludes from Proposition C.18b that g_0 is 2π-nullhomotopic w.r.t. $SO(3)$.

To prove the other direction, let $g \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ be 2π-nullhomotopic w.r.t. $SO(3)$. Thus, by Proposition C.18b, a constant function $f \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ exists such that $f \simeq^{2\pi}_{SO(3)} g$. Therefore Proposition C.18e gives me $Ind_{3,k}(f) = Ind_{3,k}(g)$. Since f is a constant function in $\mathcal{C}_{per}(\mathbb{R}^k, SO(3))$, it follows from Definition C.14 that $Ind_{3,k}(f)$ is the identity whence $Ind_{3,k}(g)$ is the identity. □

Proof of Theorem C.22c: Let $g_0, g_1 \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ where $k = 1, 2$. If $g_0 \simeq^{2\pi}_{SO(3)} g_1$ then, by Proposition C.18e, $Ind_{3,k}(g_0) = Ind_{3,k}(g_1)$. To prove the converse implication, let $Ind_{3,k}(g_0) = Ind_{3,k}(g_1) =: s$. Clearly, by Theorem C.15a,c, we have $Ind_{3,k}(g_i g_k^{(s)}) = (1, ..., 1)^T$ where $i = 0, 1$. It follows from Theorem C.22b that $g_0 g_k^{(s)}, g_1 g_k^{(s)}$ are 2π-nullhomotopic w.r.t. $SO(3)$. This implies, by Proposition C.18c, that $g_0 g_k^{(s)} \simeq^{2\pi}_{SO(3)} g_1 g_k^{(s)}$. Applying now Proposition C.20b one concludes that $g_0 \simeq^{2\pi}_{SO(3)} g_1$.

To prove the second claim let $F_0, F_1 \in \mathcal{C}(\mathbb{T}^k, SO(3))$ and $k = 1, 2$. Defining $g_i' := F_i \circ p_{4,k} \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ one observes that $F_i = FAC_k(g_i', SO(3))$ where $i = 0, 1$.

I first assume that $F_0 \simeq_{SO(3)} F_1$. Definition C.17 gives me $g'_0 \simeq^{2\pi}_{SO(3)} g'_1$ so that, by Proposition C.18e, $Ind_{3,k}(F_0 \circ p_{4,k}) = Ind_{3,k}(g'_0) = Ind_{3,k}(g'_1) = Ind_{3,k}(F_1 \circ p_{4,k})$.

To prove the other direction, let $Ind_{3,k}(F_0 \circ p_{4,k}) = Ind_{3,k}(F_1 \circ p_{4,k})$ whence $Ind_{3,k}(g'_0) = Ind_{3,k}(g'_1)$. Thus by the first claim $g'_0 \simeq^{2\pi}_{SO(3)} g'_1$. Applying Definition C.17 one concludes that $F_0 \simeq_{SO(3)} F_1$. □

Proof of Theorem C.22d: Let the $SO(3)$-index of $g_0, g_1 \in \mathcal{C}_{per}(\mathbb{R}^3, SO(3))$ be the identity.

I first assume that $g_0 \simeq^{2\pi}_{SO(3)} g_1$. Thus, by Proposition C.18b, a $G \in \mathcal{C}(\mathbb{R}^3 \times [0,1], SO(3))$ exists such that $G(\cdot, i) = g_i$ and $G(\cdot, t) \in \mathcal{C}_{per}(\mathbb{R}^3, SO(3))$ where $i = 0, 1$ and $t \in [0,1]$. By Theorem C.13d a lifting $\tilde{G} \in \mathcal{C}(\mathbb{R}^3 \times [0,1], \mathbb{S}^3)$ of G exists w.r.t. $(\mathbb{S}^3, p_2, SO(3))$ such that $\tilde{G}(\cdot, t) \in \mathcal{C}^{\pm}_{per}(\mathbb{R}^3, \mathbb{S}^3)$ and $Ind_{1,3}(\tilde{G}(\cdot, 0)) = Ind_{1,3}(\tilde{G}(\cdot, t))$. I define $\tilde{g}'_i \in \mathcal{C}^{\pm}_{per}(\mathbb{R}^3, \mathbb{S}^3)$ by $\tilde{g}'_i(\cdot) := \tilde{G}(\cdot, i)$ where $i = 0, 1$. Since $p_2 \circ \tilde{g}'_i(\cdot) = p_2 \circ \tilde{G}(\cdot, i) = G(\cdot, i) = g_i(\cdot)$ one obtains from Definition C.14 that $(1,1,1)^T = Ind_{3,3}(g_i) = Ind_{1,3}(\tilde{g}'_i) = Ind_{1,3}(\tilde{G}(\cdot, i)) = Ind_{1,3}(\tilde{G}(\cdot, t))$ whence, by Definition C.12, $\tilde{g}'_i, \tilde{G}(\cdot, t) \in \mathcal{C}_{per}(\mathbb{R}^3, \mathbb{S}^3)$ where $i = 0, 1$ and $t \in [0,1]$. I can thus define $F'_i \in \mathcal{C}(\mathbb{T}^3, \mathbb{S}^3)$ by $F'_i := FAC_3(\tilde{g}'_i, \mathbb{S}^3)$ where $i = 0, 1$. Since, for $i = 0, 1$ and $t \in [0,1]$, $\tilde{G}(\cdot, t) \in \mathcal{C}_{per}(\mathbb{R}^3, \mathbb{S}^3)$ and $\tilde{g}'_i(\cdot) = \tilde{G}(\cdot, i)$ we have, by Proposition C.18b, that $\tilde{g}'_0 \simeq^{2\pi}_{\mathbb{S}^3} \tilde{g}'_1$ whence, by Definition C.17, $F'_0 \simeq_{\mathbb{S}^3} F'_1$. However, by Definition C.21, $F'_0 \simeq_{\mathbb{S}^3} F'_1$ implies $deg(F'_0) = deg(F'_1)$. Of course, for $i = 0, 1$, we have, by Definition C.21, $deg(F'_i) = Deg(\tilde{g}'_i)$ whence $Deg(\tilde{g}'_0) = Deg(\tilde{g}'_1)$. Furthermore, for $i = 0, 1$, we have $s := Ind_{3,3}(g_i) = (1,1,1)^T$ whence, by Definition C.14, $\tilde{g}^{(s)}_3$ is the constant function in $\mathcal{C}_{per}(\mathbb{R}^3, \mathbb{S}^3)$ with value $(1,0,0,0)^T$, i.e., $\tilde{g}^{(s)}_3$ is identity of the group $\mathcal{C}^{\pm}_{per}(\mathbb{R}^3, \mathbb{S}^3)$. Thus, for $i = 0, 1$, we have, by Definition C.21, $DEG(g_i) = Deg(\tilde{g}'_i \tilde{g}^{(s)}_3) = Deg(\tilde{g}'_i (1,0,0,0)^T) = Deg(\tilde{g}'_i)$ whence $DEG(g_0) = DEG(g_1)$.

To prove the other direction let $DEG(g_0) = DEG(g_1)$. Let $\tilde{g}_i$ be a lifting of g_i w.r.t. $(\mathbb{S}^3, p_2, SO(3))$ where $i = 0, 1$. Clearly, by Definition C.14, the $\mathbb{S}^3$-index of $\tilde{g}_i$ is the identity whence, by Definition C.12, $\tilde{g}_i$ is in $\mathcal{C}_{per}(\mathbb{R}^k, \mathbb{S}^3)$ where $i = 0, 1$. I define $F_i \in \mathcal{C}(\mathbb{T}^3, \mathbb{S}^3)$ by $F_i := FAC_3(\tilde{g}_i, \mathbb{S}^3)$ where $i = 0, 1$. Recalling that, for $i = 0, 1$, $s = Ind_{3,3}(g_i) = (1, 1, 1)^T$ and that $\tilde{g}_3^{(s)}$ is the constant function in $\mathcal{C}_{per}(\mathbb{R}^3, \mathbb{S}^3)$ with value $(1, 0, 0, 0)^T$ we get, by Definition C.21, that $DEG(g_i) = Deg(\tilde{g}_i \tilde{g}_3^{(s)}) = Deg(\tilde{g}_i (1, 0, 0, 0)^T) = Deg(\tilde{g}_i)$. Thus $Deg(\tilde{g}_0) = Deg(\tilde{g}_1)$ whence, by Definition C.21, $deg(F_0) = deg(F_1)$. Applying again Definition C.21, we get $F_0 \simeq_{\mathbb{S}^3} F_1$ whence, by Definition C.17, $\tilde{g}_0 \simeq_{\mathbb{S}^3}^{2\pi} \tilde{g}_1$. It follows from Proposition C.18d that $g_0 = p_2 \circ \tilde{g}_0 \simeq_{SO(3)}^{2\pi} p_2 \circ \tilde{g}_1 = g_1$. □

Proof of Theorem C.22e: Let $g \in \mathcal{C}_{per}(\mathbb{R}^3, SO(3))$ and let me abbreviate $s := Ind_{3,3}(g)$. Let $\pm\tilde{g}$ be the liftings of g w.r.t. $(\mathbb{S}^3, p_2, SO(3))$. Definition C.21 gives $DEG(g) = Deg(\tilde{g}\tilde{g}_3^{(s)})$. To compute $DEG(gg_3^{(s)})$ we recall that p_2 is a group homomorphism whence, by Theorem C.15c, $p_2 \circ (\tilde{g}\tilde{g}_3^{(s)}) = (p_2 \circ \tilde{g})(p_2 \circ \tilde{g}_3^{(s)}) = gg_3^{(s)}$ so that $\tilde{g}' := \tilde{g}\tilde{g}_3^{(s)} \in \mathcal{C}_{per}^{\pm}(\mathbb{R}^3, \mathbb{S}^3)$ is a lifting of $gg_3^{(s)}$ w.r.t. $(\mathbb{S}^3, p_2, SO(3))$. Moreover, by Definition C.14, $Ind_{1,3}(\tilde{g}) = Ind_{1,3}(\tilde{g}_3^{(s)}) = s$ whence, by Theorem C.13c, $s' := Ind_{1,3}(\tilde{g}') = Ind_{1,3}(\tilde{g}\tilde{g}_3^{(s)}) = Ind_{1,3}(\tilde{g})Ind_{1,3}(\tilde{g}_3^{(s)}) = ss = (1, 1, 1)^T$ so that, by Definition C.21, $DEG(gg_3^{(s)}) = Deg(\tilde{g}'\tilde{g}_3^{(s')}) = Deg(\tilde{g}\tilde{g}_3^{(s)}\tilde{g}_3^{(s')})$. Recalling the proof of Theorem C.22d, $\tilde{g}_3^{(s')}$ is the identity of the group $\mathcal{C}_{per}^{\pm}(\mathbb{R}^3, \mathbb{S}^3)$ whence $\tilde{g}\tilde{g}_3^{(s)}\tilde{g}_3^{(s')} = \tilde{g}\tilde{g}_3^{(s)}$ so that $DEG(gg_3^{(s)}) = Deg(\tilde{g}\tilde{g}_3^{(s)}\tilde{g}_3^{(s')}) = Deg(\tilde{g}\tilde{g}_3^{(s)}) = DEG(g)$. □

Proof of Theorem C.22f: Let $g_0, g_1 \in \mathcal{C}_{per}(\mathbb{R}^3, SO(3))$. I first assume that $g_0 \simeq_{SO(3)}^{2\pi} g_1$. Then, by Proposition C.18e, $Ind_{3,3}(g_0) = Ind_{3,3}(g_1) =: s$. To prove that $DEG(g_0) = DEG(g_1)$, we recall that $SO(3)$ is a topological group whence, by Proposition C.20b,

$g_0 g_3^{(s)} \simeq_{SO(3)}^{2\pi} g_1 g_3^{(s)}$. Since $Ind_{3,3}(g_i) = s$ and, by Theorem C.15c, $Ind_{3,3}(g_3^{(s)}) = s$ one obtains from Theorem C.15a that $Ind_{3,3}(g_i g_3^{(s)}) = Ind_{3,3}(g_i)Ind_{3,3}(g_3^{(s)}) = ss = (1, 1, 1)^T$ where $i = 0, 1$. Thus and since $g_0 g_3^{(s)} \simeq_{SO(3)}^{2\pi} g_1 g_3^{(s)}$ Theorem C.22d gives me $DEG(g_0 g_3^{(s)}) = DEG(g_1 g_3^{(s)})$. Since $Ind_{3,3}(g_0) = Ind_{3,3}(g_1) = s$, Theorem C.22e gives me $DEG(g_i g_3^{(s)}) = DEG(g_i)$ whence $DEG(g_0) = DEG(g_1)$.

To prove the other direction, let $Ind_{3,3}(g_0) = Ind_{3,3}(g_1) =: s'$ and $DEG(g_0) = DEG(g_1)$. Theorem C.22e gives me $DEG(g_0 g_3^{(s')}) = DEG(g_0) = DEG(g_1) = DEG(g_1 g_3^{(s')})$ and Theorem C.15a gives me $Ind_{3,3}(g_i g_3^{(s')}) = Ind_{3,3}(g_i) Ind_{3,3}(g_3^{(s')}) = s's' = (1,1,1)^T$ where $i = 0,1$. This implies by Theorem C.22d that $g_0 g_3^{(s')} \simeq^{2\pi}_{SO(3)} g_1 g_3^{(s')}$. Applying Proposition C.20b we get $g_0 \simeq^{2\pi}_{SO(3)} g_1$ which completes the proof of the first claim.

To prove the second claim let $F_0, F_1 \in \mathcal{C}(\mathbb{T}^3, SO(3))$. I abbreviate $g'_i := F_i \circ p_{4,3} \in \mathcal{C}_{per}(\mathbb{R}^3, SO(3))$ whence $F_i = FAC_3(g'_i, SO(3))$ where $i = 0,1$. By Definition C.17 we have $F_0 \simeq_{SO(3)} F_1$ iff $g'_0 \simeq^{2\pi}_{SO(3)} g'_1$. Thus, by the first claim, $F_0 \simeq_{SO(3)} F_1$ iff $Ind_{3,3}(g'_0) = Ind_{3,3}(g'_1)$ and $DEG(g'_0) = DEG(g'_1)$. By the definition of g'_0, g'_1 one thus concludes that $F_0 \simeq_{SO(3)} F_1$ iff $Ind_{3,3}(F_0 \circ p_{4,3}) = Ind_{3,3}(F_1 \circ p_{4,3})$ and $DEG(F_0 \circ p_{4,3}) = DEG(F_1 \circ p_{4,3})$. □

Proof of Theorem C.22g: Let $g, g' \in \mathcal{C}_{per}(\mathbb{R}^k, SO_3(2))$. By Definition C.12 we have, for $\phi \in \mathbb{R}^k$,

$$g(\phi) = \exp(\mathcal{J}[N^T\phi + 2\pi f(\phi)]) \ , \quad g'(\phi) = \exp(\mathcal{J}[N'^T\phi + 2\pi f'(\phi)]) \ , \tag{C.9}$$

where $N = (N_1, ..., N_k)^T := Ind_{2,k}(g), N' = (N'_1, ..., N'_k)^T := Ind_{2,k}(g')$ and $f := PHF(g), f' := PHF(g')$.

I first assume that $g \simeq^{2\pi}_{SO(3)} g'$. Then, by Proposition C.18e, $Ind_{3,k}(g) = Ind_{3,k}(g')$. To prove the other direction, let $Ind_{3,k}(g) = Ind_{3,k}(g')$. I define the functions $G, G' \in \mathcal{C}(\mathbb{R}^k \times [0,1], SO(3))$ by

$$G(\phi, t) := \exp(\mathcal{J}[N^T\phi + t2\pi f(\phi)]) \ , \quad G'(\phi, t) := \exp(\mathcal{J}[N'^T\phi + t2\pi f'(\phi)]) \ . \tag{C.10}$$

By (C.9),(C.10) we have $G(\cdot,1) = g(\cdot), G'(\cdot,1) = g'(\cdot)$. Also $G(\cdot,t), G'(\cdot,t) \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ whence, by defining $h, h' \in \mathcal{C}_{per}(\mathbb{R}^k, SO_3(2))$ for $\phi \in \mathbb{R}^k$,

$$h(\phi) := G(\phi,0) = \exp(\mathcal{J}N^T\phi)\ , \qquad h'(\phi) := G'(\phi,0) = \exp(\mathcal{J}N'^T\phi)\ , \quad \text{(C.11)}$$

we get from Proposition C.18b that $h(\cdot) = G(\cdot,0) \simeq^{2\pi}_{SO(3)} G(\cdot,1) = g(\cdot)$ and $h'(\cdot) = G'(\cdot,0) \simeq^{2\pi}_{SO(3)} G'(\cdot,1) = g'(\cdot)$. Since the aim is to show that $g \simeq^{2\pi}_{SO(3)} g'$, we are done if I show that $h \simeq^{2\pi}_{SO(3)} h'$, i.e., by Proposition C.18b, I just have to find a $H \in \mathcal{C}(\mathbb{R}^k \times [0,1], SO(3))$ such that $H(\cdot,0) = h(\cdot)$,$H(\cdot,1) = h'(\cdot)$ and $H(\cdot,t) \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$. Since $h \simeq^{2\pi}_{SO(3)} g$ and $h' \simeq^{2\pi}_{SO(3)} g'$ we have, by Proposition C.18e, that $Ind_{3,k}(h) = Ind_{3,k}(g) = Ind_{3,k}(g') = Ind_{3,k}(h')$. Clearly, by (C.11) and Definition C.12, we have $Ind_{2,k}(h) = N, Ind_{2,k}(h') = N'$ whence, by Theorem C.15b,

$$((-1)^{N_1}, ..., (-1)^{N_k})^T = Ind_{3,k}(h) = Ind_{3,k}(h') = ((-1)^{N'_1}, ..., (-1)^{N'_k})^T\ . \quad \text{(C.12)}$$

I now define, for $j = 1, ..., k, \phi \in \mathbb{R}$, the functions $h_j, h'_j \in \mathcal{C}_{per}(\mathbb{R}, SO_3(2))$ by

$$h_j(\phi) := \exp(\mathcal{J}N_j\phi)\ , \qquad h'_j(\phi) := \exp(\mathcal{J}N'_j\phi)\ , \quad \text{(C.13)}$$

whence (C.11) gives me, for $\phi \in \mathbb{R}^k$,

$$\begin{aligned} h(\phi) &= \exp(\mathcal{J}N_1\phi_1)\cdots\exp(\mathcal{J}N_k\phi_k) = h_1(\phi_1)\cdots h_k(\phi_k)\ , \\ h'(\phi) &= \exp(\mathcal{J}N'_1\phi_1)\cdots\exp(\mathcal{J}N'_k\phi_k) = h'_1(\phi_1)\cdots h'_k(\phi_k)\ . \end{aligned} \quad \text{(C.14)}$$

By (C.13) we have $Ind_{2,1}(h_j) = N_j, Ind_{2,1}(h'_j) = N'_j$ whence, by (C.12) and Theorem C.15b,

$$Ind_{3,1}(h_j) = (-1)^{N_j} = (-1)^{N'_j} = Ind_{3,1}(h'_j)\ , \quad \text{(C.15)}$$

where $j = 1, ..., k$. Applying Theorem C.22c one observes by (C.15) that $h_j \simeq^{2\pi}_{SO(3)} h'_j$ whence, by Proposition C.18b, a $H_j \in \mathcal{C}(\mathbb{R} \times [0,1], SO(3))$ exists such that $H_j(\cdot,0) =$

$h_j(\cdot)$, $H_j(\cdot,1) = h'_j(\cdot)$ and $H_j(\cdot,t) \in \mathcal{C}_{per}(\mathbb{R}, SO(3))$ where $j = 1,...,k$. I define $H \in \mathcal{C}(\mathbb{R}^k \times [0,1], SO(3))$ by

$$H(\phi,t) := H_1(\phi_1,t)\cdots H_k(\phi_k,t) \ . \tag{C.16}$$

Clearly $H(\cdot,t) \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$. It follows from (C.14),(C.16) that, for $\phi \in \mathbb{R}^k$,

$$H(\phi,0) = H_1(\phi_1,0)\cdots H_k(\phi_k,0) = h_1(\phi_1)\cdots h_k(\phi_k) = h(\phi) \ ,$$
$$H(\phi,1) = H_1(\phi_1,1)\cdots H_k(\phi_k,1) = h'_1(\phi_1)\cdots h'_k(\phi_k) = h'(\phi) \ ,$$

whence, by Proposition C.18b, $h \simeq^{2\pi}_{SO(3)} h'$ so that $g \simeq^{2\pi}_{SO(3)} g'$. This concludes the proof of the first claim, i.e., the claim that $g \simeq^{2\pi}_{SO(3)} g'$ iff $Ind_{3,k}(g) = Ind_{3,k}(g')$.

To prove the second claim let $g \in \mathcal{C}_{per}(\mathbb{R}^k, SO_3(2))$. I first assume that $g \in \mathcal{C}_{per}(\mathbb{R}^k, SO_3(2))$ is 2π-nullhomotopic w.r.t. $SO(3)$. Thus, by Proposition C.18b, a constant function $f \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ exists such that $f \simeq^{2\pi}_{SO(3)} g$. Therefore Proposition C.18e gives me $Ind_{3,k}(f) = Ind_{3,k}(g)$. Since f is a constant function in $\mathcal{C}_{per}(\mathbb{R}^k, SO(3))$, it follows from Definition C.14 that $Ind_{3,k}(f)$ is the identity whence $Ind_{3,k}(g)$ is the identity. To prove the other direction, let $Ind_{3,k}(g)$ be the identity. By Definition C.14 the $SO(3)$-index of the constant function $f' \in \mathcal{C}_{per}(\mathbb{R}^k, SO_3(2))$ whose constant value is $I_{3\times3}$, is the identity. Thus by the first claim $g \simeq^{2\pi}_{SO(3)} f'$. Since f' is constant one concludes from Proposition C.18b that g is 2π-nullhomotopic w.r.t. $SO(3)$. This concludes the proof of the second claim.

The third claim follows from the second claim and Theorem C.15b. □

Lemma C.23 *a) Let (E,p,B) be a Hurewicz fibration. Let also $G \in \mathcal{C}(\mathbb{R}^k \times [0,1], B)$ be such that every $G(\cdot,t)$ is in $\mathcal{C}_{per}(\mathbb{R}^k, B)$ and let the function $G(\cdot,0) \in \mathcal{C}_{per}(\mathbb{R}^k, B)$ have a 2π-periodic lifting h w.r.t. (E,p,B). Then G has a lifting H w.r.t. (E,p,B) such that $H(\cdot,0) = h(\cdot)$ and such that every $H(\cdot,t)$ is in $\mathcal{C}_{per}(\mathbb{R}^k, E)$.*

b) Let (E,p,B) be a Hurewicz fibration. Then every $g \in \mathcal{C}_{per}(\mathbb{R}^k, B)$ which is 2π-nullhomotopic w.r.t. B has a 2π-periodic lifting w.r.t. (E,p,B).

c) Let (E, p, B) be a fiber structure and let k, k' be positive integers such that $k \leq k'$. Let $g \in \mathcal{C}_{per}(\mathbb{R}^k, B)$ have no 2π-periodic lifting w.r.t. (E, p, B). Then a $g' \in \mathcal{C}_{per}(\mathbb{R}^{k'}, B)$ exists which has no 2π-periodic lifting w.r.t. (E, p, B). If g is of class C^∞ then g' can be chosen such that it is of class C^∞.

Proof of Lemma C.23a: Let (E, p, B) be a Hurewicz fibration. Let also $G \in \mathcal{C}(\mathbb{R}^k \times [0,1], B)$ be such that every $G(\cdot, t)$ is in $\mathcal{C}_{per}(\mathbb{R}^k, B)$ and let the function $G(\cdot, 0) \in \mathcal{C}_{per}(\mathbb{R}^k, B)$ have a 2π-periodic lifting h w.r.t. (E, p, B). I abbreviate $f := FAC_k(h, E)$. By Proposition C.18a, the function $F : \mathbb{T}^k \times [0,1] \to B$, defined by $F(\cdot, t) := FAC_k(G(\cdot, t), B)$, is in $\mathcal{C}(\mathbb{T}^k \times [0,1], B)$. One concludes, for $\phi \in \mathbb{R}^k$, that $F(p_{4,k}(\phi), 0) = G(\phi, 0) = p \circ h(\phi) = p \circ f \circ p_{4,k}(\phi)$ whence $F(\cdot, 0) = FAC_k(F(p_{4,k}(\cdot), 0), B) = FAC_k(p \circ f \circ p_{4,k}, B) = p \circ f$. Thus $F(\cdot, 0)$ has the lifting f w.r.t. (E, p, B). Since (E, p, B) is a Hurewicz fibration we conclude from Definition C.5 that F has a lifting F' w.r.t. (E, p, B) such that $F'(\cdot, 0) = f(\cdot)$. Defining the function $H \in \mathcal{C}(\mathbb{R}^k \times [0,1], E)$ by $H(\phi, t) := F'(p_{4,k}(\phi), t)$ one concludes that $(p \circ H)(\phi, t) = p(F'(p_{4,k}(\phi), t)) = F(p_{4,k}(\phi), t)) = G(\phi, t)$ whence H is a lifting of G w.r.t. (E, p, B). Clearly $H(\cdot, t) \in \mathcal{C}_{per}(\mathbb{R}^k, E)$ and $H(\phi, 0) = F'(p_{4,k}(\phi), 0) = f(p_{4,k}(\phi)) = h(\phi)$. □

Proof of Lemma C.23b: Let (E, p, B) be a Hurewicz fibration and let $g \in \mathcal{C}_{per}(\mathbb{R}^k, B)$ be 2π-nullhomotopic w.r.t. B. It follows by Proposition C.18b that a function $G \in \mathcal{C}(\mathbb{R}^k \times [0,1], B)$ exists such that $G(\cdot, t) \in \mathcal{C}_{per}(\mathbb{R}^k, B)$ and such that $G(\cdot, 0)$ is constant and $G(\cdot, 1) = g(\cdot)$. Because p is onto B, a constant function $f \in \mathcal{C}_{per}(\mathbb{R}^k, E)$ exists such that $G(\cdot, 0) = p \circ f$. Applying Lemma C.23a one obtains a function $H \in \mathcal{C}(\mathbb{R}^k \times [0,1], E)$ such that $G = p \circ H$ and such that $H(\cdot, t) \in \mathcal{C}_{per}(\mathbb{R}^k, E)$. It follows that $H(\cdot, 1)$ is a 2π-periodic lifting of g w.r.t. (E, p, B). □

Proof of Lemma C.23c: Let (E, p, B) be a fiber structure and let k, k' be positive integers such that $k \leq k'$. Let $g \in \mathcal{C}_{per}(\mathbb{R}^k, B)$ have no 2π-periodic lifting w.r.t. (E, p, B). I define the function $g' \in \mathcal{C}_{per}(\mathbb{R}^{k'}, B)$ by $g'(\phi_1, ..., \phi_{k'}) := g(\phi_1, ..., \phi_k)$. I now show, by contraposition, that g' has no 2π-periodic lifting w.r.t. (E, p, B).

Assume that g' has a 2π-periodic lifting f' w.r.t. (E, p, B). It follows, for $\phi \in \mathbb{R}^{k'}$, that $p \circ f'(\phi_1, ..., \phi_{k'}) = g'(\phi_1, ..., \phi_{k'}) = g(\phi_1, ..., \phi_k)$. Note that $f' \in \mathcal{C}_{per}(\mathbb{R}^{k'}, E)$. The function $f \in \mathcal{C}_{per}(\mathbb{R}^k, E)$, defined by $f(\phi_1, ..., \phi_k) := f'(\phi_1, ..., \phi_k, 0, ..., 0)$, satisfies $p \circ f(\phi_1, ..., \phi_k) = p \circ f'(\phi_1, ..., \phi_k, 0, ..., 0) = g'(\phi_1, ..., \phi_k, 0, ..., 0) = g(\phi_1, ..., \phi_k)$. Therefore one is led to the wrong conclusion that g has the 2π-periodic lifting f w.r.t. (E, p, B).

This completes the proof that g' has no 2π-periodic lifting w.r.t. (E, p, B). Clearly if g is of class C^∞ then g' is of class C^∞. □

Theorem C.24 *a) Let $g \in \mathcal{C}_{per}(\mathbb{R}^k, \mathbb{S}^2)$. If g is 2π-nullhomotopic w.r.t. $\mathbb{S}^2$ then g has a 2π-periodic lifting f w.r.t. $(SO(3), p_3, \mathbb{S}^2)$, i.e., a $f \in \mathcal{C}_{per}(\mathbb{R}^k, SO(3))$ exists such that $g = p_3 \circ f = fe^3$.*

b) If $g \in \mathcal{C}_{per}(\mathbb{R}, \mathbb{S}^2)$, then g is 2π-nullhomotopic w.r.t. $\mathbb{S}^2$ and has a 2π-periodic lifting w.r.t. $(SO(3), p_3, \mathbb{S}^2)$. If $h \in \mathcal{C}_{per}(\mathbb{R}^2, \mathbb{S}^2)$, then it has a 2π-periodic lifting w.r.t. $(SO(3), p_3, \mathbb{S}^2)$ iff h is 2π-nullhomotopic w.r.t. $\mathbb{S}^2$.

c) If $k \geq 2$ is a positive integer, then there exists a function $g \in \mathcal{C}_{per}(\mathbb{R}^k, \mathbb{S}^2)$ of class C^∞ which has no 2π-periodic lifting w.r.t. $(SO(3), p_3, \mathbb{S}^2)$.

Proof of Theorem C.24a: We know from Corollary C.9 that $(SO(3), p_3, \mathbb{S}^2)$ is a Hurewicz fibration. The claim then follows from Lemma C.23b. □

Proof of Theorem C.24b: Let $g \in \mathcal{C}_{per}(\mathbb{R}, \mathbb{S}^2)$. I define $F := FAC_1(g, \mathbb{S}^2) \in \mathcal{C}(\mathbb{T}, \mathbb{S}^2)$. The topological space $\mathbb{T}$ carries the structure of a 1-dimensional C^∞ manifold without boundary. It thus follows that all functions in $\mathcal{C}(\mathbb{T}, \mathbb{S}^2)$ are nullhomotopic w.r.t. $\mathbb{S}^2$ [Bre, Section II.11]. Thus, by Definition C.17, g is 2π-nullhomotopic w.r.t. $\mathbb{S}^2$. This implies, by Theorem C.24a, that g has a 2π-periodic lifting w.r.t. $(SO(3), p_3, \mathbb{S}^2)$ which completes the proof of the first claim.

To prove the second claim, let $h \in \mathcal{C}_{per}(\mathbb{R}^2, \mathbb{S}^2)$. If h is 2π-nullhomotopic w.r.t. $\mathbb{S}^2$ then, by Theorem C.24a, h has a 2π-periodic lifting w.r.t. $(SO(3), p_3, \mathbb{S}^2)$.

To prove the other direction, let h have a 2π-periodic lifting f w.r.t. $(SO(3), p_3, \mathbb{S}^2)$. I define $s := Ind_{3,2}(f)$. By Theorem C.15c we have $Ind_{3,2}(g_2^{(s)}) = s$ whence, by Theorem C.22c, $g_2^{(s)} \simeq_{2\pi}^{SO(3)} f$ so that, by Proposition C.18d, $p_3 \circ g_2^{(s)} \simeq_{2\pi}^{\mathbb{S}^2} p_3 \circ f$. Clearly $p_3 \circ f(\phi) = h(\phi)$ and, by Definition C.14, $p_3 \circ g_2^{(s)}(\phi) = g_2^{(s)}(\phi)e^3 = e^3$ whence h is 2π-homotopic w.r.t. $\mathbb{S}^2$ to a constant function so that, by Proposition C.18b, h is 2π-nullhomotopic w.r.t. $\mathbb{S}^2$. □

Proof of Theorem C.24c: I first prove the claim for $k = 2$. I define the functions $g_i \in \mathcal{C}_{per}(\mathbb{R}, \mathbb{R}^3)$ by $g_1(t) := (1/2 + \cos(t), 0, \sin(t))^T$ and $g_2(t) := (-1/2 - \cos(t), -\sin(t), 0)^T$. Clearly g_1, g_2 are of class C^∞ and $g_1 - g_2$ has no zeros. I thus can define the function $g \in \mathcal{C}_{per}(\mathbb{R}^2, \mathbb{S}^2)$ by $g(\phi_1, \phi_2) := (g_1(\phi_1) - g_2(\phi_2))/|g_1(\phi_1) - g_2(\phi_2)|$. Clearly g is of class C^∞. Abbreviating $f := FAC_2(g, \mathbb{S}^2) \in \mathcal{C}(\mathbb{T}^2, \mathbb{S}^2)$ one knows (see [BG, Section 7.4]) that f is not nullhomotopic w.r.t. $\mathbb{S}^2$. Thus, by Definition C.17, g is not 2π-nullhomotopic w.r.t. $\mathbb{S}^2$. It follows by Theorem C.24b, that g has no 2π-periodic lifting w.r.t. $(SO(3), p_3, \mathbb{S}^2)$. This proves the claim for $k = 2$.

Let k' be a positive integer such that $k' \geq 2$. Since g is of class C^∞ and since g has no 2π-periodic lifting w.r.t. $(SO(3), p_3, \mathbb{S}^2)$ it follows from Lemma C.23c that there exists a function $g' \in \mathcal{C}_{per}(\mathbb{R}^{k'}, \mathbb{S}^2)$ of class C^∞ which has no 2π-periodic lifting w.r.t. $(SO(3), p_3, \mathbb{S}^2)$. □

Proposition C.25 *Let (E, p, B) be a fiber structure and let there be a positive integer k such that a $g \in \mathcal{C}_{per}(\mathbb{R}^k, B)$ exists which has no 2π-periodic lifting w.r.t. (E, p, B). Let me denote the smallest of those integers k by k_0. Then, for the fiber structures $(\mathbb{R}, p_1, SO_3(2))$, $(\mathbb{S}^3, p_2, SO(3))$,$(\mathbb{R}^m, p_{4,m}, \mathbb{T}^m)$, we have $k_0 = 1$ where m is a positive integer. Moreover, for the fiber structure $(SO(3), p_3, \mathbb{S}^2)$, we have $k_0 = 2$.*

Proof of Proposition C.25: I first consider the fiber structure $(\mathbb{R}, p_1, SO_3(2))$ and I will show, by contraposition, that the function $g_1^{(-1)} \in \mathcal{C}_{per}(\mathbb{R}, SO_3(2))$ has no 2π-periodic lifting w.r.t. $(\mathbb{R}, p_1, SO_3(2))$. In fact, lets assume that $g_1^{(-1)}$ has a 2π-periodic lifting f w.r.t. $(\mathbb{R}, p_1, SO_3(2))$. Then, by Theorem C.15b, $g_1^{(-1)}$ has a 2π-periodic lifting $\tilde{g}$ w.r.t. $(\mathbb{S}^3, p_2, SO(3))$ where $\tilde{g}$ is given by (C.5). Thus, by Definition C.12, $Ind_{1,1}(\tilde{g}) = 1$ whence, by Definition C.14, $Ind_{3,1}(g_1^{(-1)}) = 1$. However, by Theorem C.15c, $Ind_{3,1}(g_1^{(-1)}) = -1$ which poses a contradiction. One concludes that $g_1^{(-1)}$ has no 2π-periodic lifting w.r.t. $(\mathbb{R}, p_1, SO_3(2))$. Thus, for the fiber structure $(\mathbb{R}, p_1, SO_3(2))$, we have $k_0 = 1$.

I now consider the fiber structure $(\mathbb{S}^3, p_2, SO(3))$ and I will show that the function $g_1^{(-1)} \in \mathcal{C}_{per}(\mathbb{R}, SO_3(2))$ has no 2π-periodic lifting w.r.t. $(\mathbb{S}^3, p_2, SO(3))$. In fact, by Theorems C.13a,C.15c, $\pm\tilde{g}_1^{(-1)}$ are the liftings of $g_1^{(-1)}$ w.r.t. $(\mathbb{S}^3, p_2, SO(3))$. By Definition C.14 $Ind_{1,1}(\tilde{g}_1^{(-1)}) = -1$ whence, by Definition C.12, $\tilde{g}_1^{(-1)}$ is not 2π-periodic so that both liftings of $g_1^{(-1)}$ w.r.t. $(\mathbb{S}^3, p_2, SO(3))$ are not 2π-periodic. Thus, for the fiber structure $(\mathbb{S}^3, p_2, SO(3))$, we have $k_0 = 1$.

I now consider the fiber structure $(\mathbb{R}^m, p_{4,m}, \mathbb{T}^m)$ where m is a positive integer. I will show that the function $g \in \mathcal{C}_{per}(\mathbb{R}, \mathbb{T}^m)$, defined by $g(t) := (\exp(it), 1, ..., 1)^T$, has no 2π-periodic lifting w.r.t. $(\mathbb{R}^m, p_{4,m}, \mathbb{T}^m)$. In fact $f \in \mathcal{C}(\mathbb{R}, \mathbb{R}^m)$, defined by $f(t) := (t, 0, ..., 0)^T$, is a lifting of g w.r.t. $(\mathbb{R}^m, p_{4,m}, \mathbb{T}^m)$. Thus, by Theorem C.11d, every lifting of g w.r.t. $(\mathbb{R}^m, p_{4,m}, \mathbb{T}^m)$ is not 2π-periodic so that, for the fiber structure $(\mathbb{R}^m, p_{4,m}, \mathbb{T}^m)$, we have $k_0 = 1$.

I now consider the fiber structure $(SO(3), p_3, \mathbb{S}^2)$. Clearly, by Theorem C.24b, every $g \in \mathcal{C}_{per}(\mathbb{R}, \mathbb{S}^2)$ has a 2π-periodic lifting w.r.t. $(SO(3), p_3, \mathbb{S}^2)$ whence either $k_0 > 1$ or k_0 does not exist. However by Theorem C.24c, a function $g \in \mathcal{C}_{per}(\mathbb{R}^2, \mathbb{S}^2)$ exists which has no 2π-periodic lifting w.r.t. $(SO(3), p_3, \mathbb{S}^2)$. Thus, for the fiber structure $(SO(3), p_3, \mathbb{S}^2)$, we have $k_0 = 2$. □

Appendix D

Fourier analytic concepts and facts

D.1 Quasiperiodic functions

Definition D.1 *Let* $f \in \mathcal{C}_{per}(\mathbb{R}^d, X)$ *with* $X = \mathbb{C}^j$ *or* $X = \mathbb{C}^{j\times j}$ *for some positive integer* j*. If* $\chi \in \mathbb{R}^d$ *then* f *is called the '*χ*-generator' of the function* $F : \mathbb{Z} \to X$ *defined by* $F(n) = f(2\pi n\chi)$*. A function* $F : \mathbb{Z} \to X$ *is called '*χ*–quasiperiodic' if it has a* χ*-generator and it is called 'quasiperiodic' if it has a* χ*-generator for some* χ*.*

With $\chi \in \mathbb{R}^k$ *I define*

$$Y_\chi := \{m^T\chi + n : m \in \mathbb{Z}^k, n \in \mathbb{Z}\} \ . \tag{D.1}$$

A $\chi \in \mathbb{R}^k$ *is said to be 'nonresonant' if the equation* $m^T\chi = 0$*, together with the condition* $m \in \mathbb{Z}^k$*, can only be fulfilled for* $m = 0$ *(whenever I write* $\mathbb{Z}^k$*, this implies that* k *is a positive integer). A spin-orbit torus* (ω, A) *is said to be 'off orbital resonance' if* $(1, \omega)$ *is nonresonant. Otherwise the spin-orbit torus is 'on orbital resonance'.* $\Box$

Remark:

(1) I choose the sets $\mathbb{R}$ and $\mathbb{C}$ such that $\mathbb{R} = \{x \in \mathbb{C} : \Im m\{x\} = 0\}$, i.e. $\mathbb{R} \subset \mathbb{C}$. Thus if F is a quasiperiodic function whose components are real then it has a generator f whose components are real (just take the real part of a given generator!). □

A χ-generator f of a χ–quasiperiodic function F fulfills three conditions: $F(n) = f(2\pi n\chi)$, the 2π–periodicity of f and the 'regularity' condition that f is continuous. Unlike the former two conditions, the third condition is a matter of choice. Thus the regularity condition determines the quasiperiodicity properties one has to deal with. The regularity of f can basically vary between the extremes 'f being continuous' and 'f being analytic'. In this paper I choose f to be continuous because it is convenient and because the emphasis in this work is on continuity.

Since $A(\phi_0+2\pi n\omega)$ is a ω–quasiperiodic function of n, the dynamical system (6.8) has ω–quasiperiodic equations of motion. This circumstance makes the concept of quasiperiodicity relevant for spin motions.

While the trivial solution $S(n) = 0$ always exists and is ω–quasiperiodic it is a natural question of whether nonzero ω–quasiperiodic spin trajectories exist. However I must leave this interesting question open. Nevertheless, experience with explicitly solvable models indicates that the answer is positive (for every ϕ_0).

D.2 A dense subset of $\mathbb{R}^k$

Theorem D.2 *Let $\phi_0, \omega \in \mathbb{R}^k$ and let $(1, \omega)$ be nonresonant. Then the set $\{\phi_0 + 2\pi n\omega + 2\pi m : m \in \mathbb{Z}^k, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}^k$.*

Proof of Theorem D.2: Let $\phi_0, \omega \in \mathbb{R}^k$ and let $(1, \omega)$ be nonresonant. I define

$$A := \{\phi_0 + 2\pi n\omega + 2\pi m : m \in \mathbb{Z}^k, n \in \mathbb{Z}\} , \qquad A' := p_{4,k}(A) ,$$
$$A'' := \mathbb{R}^k \setminus \bar{A} , \qquad A''' := \mathbb{T}^k \setminus A' .$$

Since the aim is to show that A is dense in $\mathbb{R}^k$ I have to show that A'' is empty. I first note (see for example [HK2, Section 1.4]) that A' is dense in $\mathbb{T}^k$, i.e.,

$$\overline{A'} = \mathbb{T}^k . \tag{D.2}$$

The second observation is that, by the special form of A,

$$p_{4,k}^{-1}(A') = p_{4,k}^{-1}(p_{4,k}(A)) = A . \tag{D.3}$$

It is now easy to prove the claim. One concludes from (D.2) and (D.3) that

$$p_{4,k}^{-1}(A''') = p_{4,k}^{-1}(\mathbb{T}^k \setminus A') = \mathbb{R}^k \setminus p_{4,k}^{-1}(A') = \mathbb{R}^k \setminus A \supset \mathbb{R}^k \setminus \bar{A} = A'' ,$$

whence

$$A''' = p_{4,k}(p_{4,k}^{-1}(A''')) \supset p_{4,k}(A'') . \tag{D.4}$$

Recalling Proposition C.8a, $p_{4,k}$ is a covering map whence it is open. Thus $p_{4,k}(A'')$ is open in $\mathbb{T}^k$ whence $p_{4,k}(A'')$ is open and a subset of the complement A''' of A'. However, by (D.2) the only open set in the complement of A' is the empty set whence $p_{4,k}(A'') = \emptyset$ which implies that $A'' = \emptyset$. □

Corollary D.3 *a) Let $f \in \mathcal{C}_{per}(\mathbb{R}^k, \mathbb{R})$ and let χ be in $\mathbb{R}^k$ such that $(1, \chi)$ is nonresonant. If, for all $\phi \in \mathbb{R}^k$, $f(\phi + 2\pi\chi) = f(\phi)$ then f is constant, i.e., $f(\phi) = f(0)$ for all $\phi \in \mathbb{R}^k$.*

b) Let $\chi \in \mathbb{R}^k$ such that $(1, \chi)$ is nonresonant and let j be a positive integer. If $F : \mathbb{Z} \to \mathbb{R}^j$ is a χ–quasiperiodic function then it has exactly one χ-generator and this χ-generator is $\mathbb{R}^j$-valued. If $F : \mathbb{Z} \to \mathbb{R}^{j\times j}$ is a χ–quasiperiodic function then it has exactly one χ-generator and this χ-generator is $\mathbb{R}^{j\times j}$-valued.

Proof of Corollary D.3a: Let $f \in \mathcal{C}_{per}(\mathbb{R}^k, \mathbb{R})$. Let χ be in $\mathbb{R}^k$ such that $(1, \chi)$ is nonresonant and let, for all $\phi \in \mathbb{R}^k$, $f(\phi + 2\pi\chi) = f(\phi)$.

By induction in n one obtains that, for all integers n, $f(2\pi n\chi) = f(0)$. Defining

$$A := \{2\pi n\chi + 2\pi m : m \in \mathbb{Z}^k, n \in \mathbb{Z}\} , \qquad A' := \{\phi \in \mathbb{R}^k : f(\phi) = f(0)\} ,$$

one obtains that $A \subset A'$ whence $\bar{A} \subset \overline{A'} = A'$ where I used the fact that A' is closed. Using Theorem D.2 we have $\bar{A} = \mathbb{R}^k$ whence $A' = \mathbb{R}^k$. □

Proof of Corollary D.3b: Let $\chi \in \mathbb{R}^k$ such that $(1, \chi)$ is nonresonant and let $F : \mathbb{Z} \to \mathbb{R}$ be a χ–quasiperiodic function. By Definition D.1, F has a χ-generator which is a function $f \in \mathcal{C}_{per}(\mathbb{R}^k, \mathbb{C})$ such that, for $n \in \mathbb{Z}$, $F(n) = f(2\pi n\chi)$. To show that f is the only χ-generator of F let g be an arbitrary χ-generator of F, i.e., $g \in \mathcal{C}_{per}(\mathbb{R}^k, \mathbb{C})$ such that, for $n \in \mathbb{Z}$, $F(n) = g(2\pi n\chi)$. Since f and g are 2π-periodic we have for $m \in \mathbb{Z}^k, n \in \mathbb{Z}$ that $f(2\pi n\chi + 2\pi m) = g(2\pi n\chi + 2\pi m)$. Thus, defining the set $A := \{2\pi n\chi + 2\pi m : m \in \mathbb{Z}^k, n \in \mathbb{Z}\}$, we see that $f(\phi) = g(\phi)$ for all $\phi \in A$. Since $(1, \chi)$ is nonresonant, one concludes from Theorem D.2 that the set A is dense in $\mathbb{R}^k$. Since A is dense in $\mathbb{R}^k$ and since f and g are continuous, it thus follows that $f = g$ whence f is the unique χ-generator of F.

To show that f is $\mathbb{R}$-valued, I define $h \in \mathcal{C}_{per}(\mathbb{R}^k, \mathbb{R})$ by $h := (f + f^*)/2$ where $f^*(\phi)$ denotes the complex conjugate of $f(\phi)$. Clearly, for $n \in \mathbb{Z}$, we have that $2h(2\pi n\chi) = f(2\pi n\chi) + f^*(2\pi n\chi) = F(n) + F^*(n) = 2F(n)$ whence h is a χ-generator of F. However since f is the unique χ-generator of F we have $h = f$ whence f is real valued.

Let j be a positive integer and $F : \mathbb{Z} \to X$ be a χ–quasiperiodic function, where either $X = \mathbb{R}^j$ or $X = \mathbb{R}^{j\times j}$. Then each component of F is a real valued χ–quasiperiodic function. Thus, having already proven the claims for $\mathbb{R}$-valued F, one concludes that each component of F has a unique χ-generator and that this χ-generator is real valued. I thus define the function $f \in \mathcal{C}_{per}(\mathbb{R}^k, X)$ such that each of its components is the unique χ-generator of the corresponding component of F. Clearly f is the unique χ-generator of F. Of course all components of f are real valued which completes the proof. □

D.3 Applying Fejér's multivariate theorem

In this section I first present (see Lemma D.4a) Fejér's multivariate theorem and then derive from that several facts needed in this work.

If $F : \mathbb{Z} \to \mathbb{C}$ is a function and $\lambda \in [0, 1), N \in \mathbb{Z}_+$, I define

$$a_N(F, \lambda) := (N + 1)^{-1} \sum_{n=0}^{N} F(n) \exp(-2\pi i n \lambda) \ ,$$

where $\mathbb{Z}_+$ denotes the set of nonnegative integers. I denote by $\Lambda_{tot}(F)$ the set of those $\lambda \in [0, 1)$ for which $a_N(F, \lambda)$ converges as $N \to \infty$. If $\lambda \in \Lambda_{tot}(F)$ I denote the limit of $a_N(F, \lambda)$ by $a(F, \lambda)$ and I define the 'spectrum $\Lambda(F)$ of F' by $\Lambda(F) := \{\lambda \in \Lambda_{tot}(F) : a(F, \lambda) \neq 0\}$.

I define the function $E_c : \mathbb{Z} \to \mathbb{C}$ by $E_c(n) := \exp(i2\pi nc)$ where $n \in \mathbb{Z}$ and where c is an arbitrary real number. Clearly, we have $\Lambda_{tot}(E_c) = [0, 1)$ and, for $\lambda \in [0, 1)$,

$$a(E_c, \lambda) = \begin{cases} 1 & \text{if } \lambda = \lfloor c \rfloor \\ 0 & \text{if } \lambda \neq \lfloor c \rfloor \ , \end{cases}$$

whence

$$\Lambda(E_c) = \{\lfloor c \rfloor\} \ . \tag{D.5}$$

Let $f : \mathbb{R}^k \to \mathbb{C}$ be a continuous and 2π–periodic function. Then for $m \in \mathbb{R}^k$ the 'm-th Fourier coefficient' of f is defined by

$$f_m := \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} f(\phi) \exp(-i m^T \phi) d\phi_1 \cdots d\phi_k \ . \tag{D.6}$$

If $m \in \mathbb{R}^k, N \in \mathbb{Z}_+$ I define

$$A^k_{N,m} := \prod_{n=1}^{k} \frac{N+1-|m_n|}{N+1} \, , \qquad ||m|| := max(|m_1|, ..., ||m_k|) \, . \qquad \text{(D.7)}$$

Lemma D.4 *a) (Fejér's multivariate theorem) Let $f : \mathbb{R}^k \to \mathbb{C}$ be a continuous and 2π–periodic function. Defining for $N \in \mathbb{Z}_+$ the continuous and 2π–periodic function $f^N : \mathbb{R}^k \to \mathbb{C}$ by*

$$f^N(\phi) := \sum_{\substack{m \in \mathbb{Z}^k \\ ||m|| \le N}} A^k_{N,m} f_m \exp(im^T\phi) \, , \qquad \text{(D.8)}$$

the sequence f^N converges uniformly on $\mathbb{R}^k$ to f as $N \to \infty$.

b) Let $F : \mathbb{Z} \to \mathbb{C}$ be a χ–quasiperiodic function where $\chi \in \mathbb{R}^k$ and let f be a χ-generator of F, i.e., $F(n) = f(2\pi n\chi)$. Defining for $N \in \mathbb{Z}_+$ the function $F^N : \mathbb{Z} \to \mathbb{C}$ by

$$F^N(n) := \sum_{\substack{m \in \mathbb{Z}^k \\ ||m|| \le N}} A^k_{N,m} f_m \exp(i2\pi n m^T\chi) \, , \qquad \text{(D.9)}$$

where f_m is the m-th Fourier coefficient of f, then the sequence F^N converges uniformly on $\mathbb{Z}$ to F as $N \to \infty$. Furthermore $\Lambda_{tot}(F^N) = [0,1)$ and $\Lambda(F^N) \subset Y_\chi$.

c) Let $F : \mathbb{Z} \to \mathbb{C}$ be a χ–quasiperiodic function where $\chi \in \mathbb{R}^k$ such that $(1,\chi)$ is nonresonant. Let f be a χ-generator of F, i.e., $F(n) = f(2\pi n\chi)$ and let me define for $N \in \mathbb{Z}_+$ the function $F^N : \mathbb{Z} \to \mathbb{C}$ by (D.9), where f_m is the m-th Fourier coefficient of f. Then $Y_\chi \subset \Lambda_{tot}(F)$ and, for every $m \in \mathbb{Z}^k$, $f_m = a(F, m^T\chi)$.

d) Let $F : \mathbb{Z} \to \mathbb{C}$ be a χ–quasiperiodic function and let $\Lambda_{tot}(F) = [0,1)$. Then $\Lambda(F) \subset Y_\chi$.

Proof of Lemma D.4a: Let $f : \mathbb{R}^k \to \mathbb{C}$ be a continuous and 2π–periodic function. That the sequence f^N converges uniformly on $\mathbb{R}^k$ to f, is the generalization of Fejér's univariate theorem from $k = 1$ to arbitrary k (see for example [Maa, Sec. III.22],[Ko, Sec. 79]). □

Proof of Lemma D.4b: Let $F : \mathbb{Z} \to \mathbb{C}$ be a χ–quasiperiodic function where $\chi \in \mathbb{R}^k$ and let f be a χ-generator of F.

Defining for $N \in \mathbb{Z}_+$ the continuous and 2π–periodic function $f^N : \mathbb{R}^k \to \mathbb{C}$ by (D.8), it follows from Lemma D.4a that the sequence f^N converges uniformly on $\mathbb{R}^k$ to f as $N \to \infty$. Defining for $N \in \mathbb{Z}_+$ the function $F^N : \mathbb{Z} \to \mathbb{C}$ by (D.9), it is clear that $F^N(n) = f^N(2\pi n\chi)$. By the uniform convergence of f^N I conclude that the sequence F^N converges uniformly on $\mathbb{Z}$ to F as $N \to \infty$.

That $\Lambda_{tot}(F^N) = [0, 1)$ follows from the facts that F^N is a finite sum of exponential functions E_c and that $\Lambda_{tot}(E_c) = [0, 1)$.

To prove the last claim let $\lambda \in \Lambda(F^N)$. Then $a(F^N, \lambda) \neq 0$ whence there exists an $m \in \mathbb{Z}^k$ such that λ belongs to the spectrum of the exponential function $\exp(i2\pi n m^T\chi)$, i.e., $a(E_c, \lambda) \neq 0$ for $c = m^T\chi$. It thus follows from (D.5) that $\lambda = \lfloor m^T\chi \rfloor$ whence $\lambda \in Y_\chi$. I thus have shown that $\Lambda(F^N) \subset Y_\chi$. □

Proof of Lemma D.4c: Let $F : \mathbb{Z} \to \mathbb{C}$ be a χ–quasiperiodic function where $\chi \in \mathbb{R}^k$ such that $(1, \chi)$ is nonresonant. Let f be a χ-generator of F and let me define for $N \in \mathbb{Z}_+$ the function $F^N : \mathbb{Z} \to \mathbb{C}$ by (D.9), where f_m is the m-th Fourier coefficient of f.

By using a 'map' version of Weyl's equidistribution theorem ([CFS, Chapter 3]), one obtains, for $m \in \mathbb{Z}^k$, that $m^T\chi \in \Lambda_{tot}(F)$ and that $f_m = a(F, m^T\chi)$. Since, for $N \in \mathbb{Z}_+, n \in \mathbb{Z}$ we have $a_N(F, m^T\chi + n) = a_N(F, m^T\chi)$ one concludes that $Y_\chi \subset \Lambda_{tot}(F)$. □

Proof of Lemma D.4d: Let $F : \mathbb{Z} \to \mathbb{C}$ be a χ–quasiperiodic function where $\chi \in \mathbb{R}^k$ and let $\Lambda_{tot}(F) = [0, 1)$. Let λ be in $[0, 1)$.

It follows from Lemma D.4b that a sequence of functions $F^N : \mathbb{Z} \to \mathbb{C}$ exists which converges uniformly on $\mathbb{Z}$ to F as $N \to \infty$ and such that $\Lambda_{tot}(F^N) = [0, 1)$, $\Lambda(F^N) \subset Y_\chi$. Thus since $a(F^N, \lambda)$ and $a(F, \lambda)$ exist, we have

$$\begin{aligned} &|a(F^N, \lambda) - a(F, \lambda)| = |a(F^N - F, \lambda)| \\ &= |\lim_{T\to\infty} \frac{1}{T+1} \sum_{n=0}^{T} (F^N(n) - F(n)) \exp(-2\pi i \lambda n)| \leq \sup_n |F^N(n) - F(n)| \ , \end{aligned}$$

where I also used the fact that F^N and F are bounded functions. It follows that

$$\lim_{N\to\infty} a(F^N, \lambda) = a(F, \lambda) \ , \tag{D.10}$$

since F^N converges uniformly on $\mathbb{Z}$ to F as $N \to \infty$. Note that (D.10) holds for every $\lambda \in [0, 1)$. If $\lambda \in [0, 1) \setminus Y_\chi$, then, since $\Lambda(F^N) \subset Y_\chi$ and $\Lambda_{tot}(F^N) = [0, 1)$, we have that $\lambda \in \Lambda_{tot}(F^N) \setminus \Lambda(F^N)$. Thus $a(F^N, \lambda) = 0$ and (D.10) gives $a(F, \lambda) = 0$ whence $\lambda \in [0, 1) \setminus \Lambda(F)$. Thus $[0, 1) \setminus Y_\chi \subset [0, 1) \setminus \Lambda(F)$ whence $\Lambda(F) \subset Y_\chi$. □

Remark:

(1) One can show that every quasiperiodic function $F : \mathbb{Z} \to \mathbb{C}$ has the property $\Lambda_{tot}(F) = [0, 1)$. Thus the assumption in Lemma D.4d, that $\Lambda_{tot}(F) = [0, 1)$, is redundant. However since it would be tedious to prove that this assumption is redundant in Lemma D.4d and since I apply Lemma D.4d only to functions F where we know that $\Lambda_{tot}(F) = [0, 1)$, we see that Lemma D.4d is convenient for our purposes. Note also that my only application of Lemma D.4d is the proof of Theorem D.5. □

While it is obvious that E_c is c–quasiperiodic, it is a natural but not quite trivial question of whether there are other vectors χ for which the function E_c is χ–quasiperiodic (obviously $\chi = c$ is one of these vectors). The answer to this question is given by the following theorem.

Theorem D.5 *Let c be a real number and let $E_c : \mathbb{Z} \to \mathbb{C}$ be the c–quasiperiodic function, defined by $E_c(n) := \exp(i2\pi nc)$. Let also $\chi \in \mathbb{R}^k$. Then E_c is χ–quasiperiodic iff $c \in Y_\chi$.*

Proof of Theorem D.5: I first consider the case that E_c is χ–quasiperiodic. Recalling that $\Lambda_{tot}(E_c) = [0, 1)$, I can apply Lemma D.4d and thus obtain $\Lambda(E_c) \subset Y_\chi$. It thus follows from (D.5) that $\{\lfloor c \rfloor\} \subset Y_\chi$, i.e., that $\lfloor c \rfloor \in Y_\chi$ whence (recall (D.1)) there exist $m \in \mathbb{Z}^k, n \in \mathbb{Z}$ such that $\lfloor c \rfloor = m^T\chi + n$. It follows that $c \in Y_\chi$.

I now consider the case that $c \in Y_\chi$. Then $m \in \mathbb{Z}^k, n \in \mathbb{Z}$ exist such that $c = m^T\chi + n$ whence $E_c(n) = \exp(i2\pi nc) = \exp(i2\pi nm^T\chi)$. It follows that E_c is χ–quasiperiodic. □

Remark:

(2) The claim of Theorem D.5 is obvious if one makes the assumption that the χ–quasiperiodic function E_c has a χ-generator which is a trigonometric polynomial. In fact, under that assumption the proof of Theorem D.5 would be trivial whence Lemma D.4 would be superfluous in the proof of Theorem D.5. However, it is of course not allowed to assume that every χ-generator of E_c is a trigonometric polynomial whence Lemma D.4 is crucial for the proof of Theorem D.5. □

Appendix E

Principal bundles and their associated bundles

In this section I provide those concepts and facts from the theory of principal bundles which are needed for Section 9.3. I follow the elegant treatment of Husemoller's book [Hus] avoiding the sometimes clumsy machinery of coordinate bundles (the latter is covered for example in [St]). Note that the principal bundles defined in [Hus] are sometimes (for example in: [Mac]) called 'Cartan principal bundles'. Since principal bundles are bundles refined by group actions, the present section builds up on Appendices B and C. Adhering to the philosophy practiced in Appendices B-D I present the material in such detail that it is essentially self contained. Most of the material of the present section is an elaboration on material from Sections 1-6 in [Hus].

This section is structured as follows. In the basic Sections E.1-E.5 I provide facts and concepts about principal bundles and their associated bundles and in Section E.6 I reconsider Sections E.1-E.5 in the special case of the product principal bundle since this will be applied in Section 9.3 of this work.

In Section E.1 I introduce, in descending order of generality, G-prebundles, G-bundles and principal G-bundles where G is an arbitrary topological group. Furthermore the category $Bun(G)$ of principal G-bundles is introduced and the automorphism group $\mathfrak{A}ut_{Bun(G)}(\lambda)$ of a principal G-bundle λ is defined. Proposition E.1 is proved which gives a necessary and sufficient condition for a G-prebundle to be a G-bundle and which is applied in Section E.6.1 to prove that the product principal G-bundle is indeed a principal bundle.

In Section E.2 an arbitrary associated bundle $\lambda[F, L]$ of a principal G-bundle λ is considered and properties are derived which are essential for Sections E.3 and E.5.

Section E.3 introduces the left $\mathfrak{A}ut_{Bun(G)}(\lambda)$ actions L', L''. In Section E.3.1 I introduce L' which acts on the total space of the associated bundle $\lambda[F, L]$ and I show that L' is based on fibre morphisms of the associated bundle. In Section E.3.2 I introduce L'' which acts on the cross sections of $\lambda[F, L]$ and which builds up on L'.

In Section E.5 I introduce the H-reductions of principal G-bundles where H is a closed topological subgroup. The H-reductions are at the heart of the Feres machinery since they are the vehicles for the reductions theorems.

In Section E.6 I reconsider Sections E.1-E.5 in the special case of the product principal G-bundle which in fact is the principal bundle that is eventually applied in Section 9.3 of this work.

E.1 Principal G-bundles

Let

$$\xi = (E, p, B) \tag{E.1}$$

be a bundle. Bundles form a category, *Bun*, and I denote the set of morphisms from ξ to itself by $\mathfrak{Mor}_{Bun}(\xi)$. Note that, by definition, $\mathfrak{Mor}_{Bun}(\xi)$ consists of the pairs $(\varphi, \bar{\varphi})$ for which $\varphi \in \mathcal{C}(E, E)$ and $\bar{\varphi} \in \mathcal{C}(B, B)$ such that

$$\bar{\varphi} \circ p = p \circ \varphi \ . \tag{E.2}$$

The identity morphism in $\mathfrak{Mor}_{Bun}(\xi)$ is (id_E, id_B) and the composition law in *Bun* reads for $(\varphi_i, \bar{\varphi}_i) \in \mathfrak{Mor}_{Bun}(\xi)$ and $i = 1, 2$ as $(\varphi_2, \bar{\varphi}_2)(\varphi_1, \bar{\varphi}_1) = (\varphi_2 \circ \varphi_1, \bar{\varphi}_2 \circ \bar{\varphi}_1)$. Analogously the composition law of *Bun* is defined for morphisms which connect different bundles and so Category Theory provides the concepts of isomorphism and automorphism in *Bun*.

Let G be a topological group and R be a right G-action on E such that (E, R) is a topological right G-space. Let the quadruple λ be defined by

$$\lambda := (\xi, R) = (E, p, B, R) \ . \tag{E.3}$$

I call λ a 'G-prebundle' if p is a G-map from the right G-space (E, R) to the trivial right G-space over B, i.e., if for $x \in E, g \in G$

$$p(R(g; x)) = p(x) \ . \tag{E.4}$$

Thus λ in (E.3) is a G-prebundle iff for all $x \in E, g \in G$ the set $p^{-1}(p(x))$ is invariant under $R(g; \cdot)$. This implies that if λ in (E.3) is a G-prebundle then for every $x \in E$ the function $R_x : G \times p^{-1}(p(x)) \to p^{-1}(p(x))$, defined as the restriction of R to $G \times p^{-1}(p(x))$, is a right G-action on $p^{-1}(p(x))$.

Recalling the orbit space E/R and the canonical surjection $p_R : E \to E/R$ from Appendix B one observes that if λ in (E.3) is a G-prebundle and if $x, x' \in E$ satisfy $x' \in p_R(x)$ then a $g \in G$ exists such that $x' = R(g; x)$ whence, by (E.4), $p(x') = p(x)$ so that $x' \in p^{-1}(p(x))$. Thus if λ is a G-prebundle then for every $x \in E$ I get the inclusion

$$p_R(x) \subset p^{-1}(p(x)) \ , \tag{E.5}$$

which plays a major role in the proof of Proposition E.1. I define the fiber structure

$$\alpha(E, R) := (E, p_R, E/R) , \tag{E.6}$$

where the α-notation is taken from [Hus]. Defining also the quadruple

$$\lambda_R := (\alpha(E, R), R) = (E, p_R, E/R, R) , \tag{E.7}$$

one observes, since p_R is a G-map from the right G-space (E, R) to the trivial right G-space over E/R, that λ_R is a G-prebundle.

I now consider the problem of finding, under the assumption that λ a G-prebundle, a function f on E/R which satisfies

$$f \circ p_R = p . \tag{E.8}$$

Note that since p_R is onto E/R there exists at most one such f. Since p is continuous and p_R is onto E/R and identifying, one observes [Hu, Section II.6] that f, if it exists, is continuous. Furthermore if p is onto B, then f is onto B if it exists. To show that f exists I define the function $\pi_\lambda : E/R \to B$ for $x \in E$ by

$$\pi_\lambda(p_R(x)) := p(x) . \tag{E.9}$$

Note that π_λ is defined by (E.9) *for all* $z \in E/R$ since p_R is onto E/R. Note also that π_λ is single valued since if $x, x' \in E$ and $p_R(x) = p_R(x')$ then, recalling Appendix B, a $g \in G$ exists such that $R(g; x) = x'$ whence one gets by (E.9)

$$\pi_\lambda(p_R(x')) = p(x') = p(R(g; x)) = p(x) = \pi_\lambda(p_R(x)) , \tag{E.10}$$

where in the third equality of (E.10) I used the fact that λ is a G-prebundle. With (E.10) I have completed the proof that π_λ is a function: $E/R \to B$ if λ is a G-prebundle. It is clear by (E.9) that $f = \pi_\lambda$ satisfies (E.8) so that, by the remarks after (E.8), $f = \pi_\lambda$ is the unique solution of (E.8) and

$$\pi_\lambda \circ p_R = p . \tag{E.11}$$

I thus conclude by the remarks after (E.8) that π_λ is continuous and, if p is onto B, π_λ is onto B. I call π_λ the 'prebundle function' of the G-prebundle λ. Therefore by the remarks after (E.8) a prebundle function is always continuous (of course, w.r.t. the topological spaces $E/R, B$). Note that since $id_{E/R} \circ p_R = p_R$ the prebundle function of the G-prebundle λ_R is $id_{E/R}$. Note also that since p_R is onto E/R it follows from [Du, Section VI.3] and (E.11) that the prebundle function is identifying iff p is identifying.

If λ in (E.3) is a G-prebundle then it is called a 'G-bundle' if its prebundle function is a homeomorphism onto B. If λ in (E.3) is a G-bundle and if the topological right G-space (E, R) is principal then λ is called a 'principal G-bundle'. These definitions of G-bundle and principal G-bundle are the distinguishing features of the elegant treatment in Husemoller's book [Hus] (I added, since it is convenient, the definition of G-prebundle). Note that these definitions don't involve local triviality (in particular no coordinate bundles are involved). Note also that G is called the 'structure group' of λ and that the principal bundles defined in this way are sometimes called 'Cartan principal bundles'.

Of course if λ in (E.3) is a G-bundle or even a principal G-bundle then, due to (E.11), p is onto B, i.e., ξ is a fiber structure. The standard example of a G-bundle is the G-prebundle λ_R since, as mentioned above, its prebundle function is $id_{E/R}$. Thus λ_R is a principal G-bundle iff the topological right G-space (E, R) is principal.

The principal G-bundles form a category, $Bun(G)$, and in this category I denote the set of morphisms from λ to itself by $\mathfrak{Mor}_{Bun(G)}(\lambda)$. Note that, by definition, $\mathfrak{Mor}_{Bun(G)}(\lambda)$ consists of those elements $(\varphi, \bar{\varphi})$ of $\mathfrak{Mor}_{Bun}(\xi)$ for which φ is a G-map on the right G-space (E, R). The identity morphism, (id_E, id_B), in $\mathfrak{Mor}_{Bun(G)}(\lambda)$ is the same as in $\mathfrak{Mor}_{Bun}(\xi)$ and the composition law in $Bun(G)$ is the same as in Bun. Category Theory provides the concepts of isomorphism and automorphism in $Bun(G)$. In particular the automorphisms on λ are those elements $(\varphi, \bar{\varphi})$ of $\mathfrak{Mor}_{Bun(G)}(\lambda)$ for which a $(\tilde{\varphi}, \bar{\tilde{\varphi}}) \in \mathfrak{Mor}_{Bun(G)}(\lambda)$ exists such that $(\varphi, \bar{\varphi})(\tilde{\varphi}, \bar{\tilde{\varphi}}) = (id_E, id_B) = (\tilde{\varphi}, \bar{\tilde{\varphi}})(\varphi, \bar{\varphi})$ and I denote the set of these automorphisms by $\mathfrak{Aut}_{Bun(G)}(\lambda)$. Note that $\mathfrak{Aut}_{Bun(G)}(\lambda)$ is a group under the composition law of $Bun(G)$ with neutral element (id_E, id_B). Clearly if $(\varphi, \bar{\varphi}) \in \mathfrak{Aut}_{Bun(G)}(\lambda)$ then $\varphi \in HOMEO(E, E)$ and $\bar{\varphi} \in HOMEO(B, B)$. I define

$$\mathfrak{Gau}_{Bun(G)}(\lambda) := \{\varphi \in \mathcal{C}(E, E) : (\varphi, id_B) \in \mathfrak{Aut}_{Bun(G)}(\lambda)\} . \tag{E.12}$$

Clearly the (φ, id_B) with $\varphi \in \mathfrak{Gau}_{Bun(G)}(\lambda)$ form a subgroup of $\mathfrak{Aut}_{Bun(G)}(\lambda)$ whence $\mathfrak{Gau}_{Bun(G)}(\lambda)$ is a group under the composition of functions. One calls $\mathfrak{Gau}_{Bun(G)}(\lambda)$ the 'gauge group of λ' [Hus].

For the following proposition 'transitivity on fibres' is an important criterion. If λ in (E.3) is a G-prebundle then, as mentioned above, we have for every $x \in E$ the right G-action R_x on $p^{-1}(p(x))$ and I call R 'transitive on the fibres of p' if all R_x are transitive.

Part c) of the following proposition will be applied in Section E.6.1.

Proposition E.1 *Let G be a topological group and let the quadruple λ in (E.3) be a G-prebundle. Denoting the prebundle function of λ by π_λ the following hold:*

a) π_λ is one-one iff for every $x \in E$

$$p_R(x) \supset p^{-1}(p(x)) . \tag{E.13}$$

b) R is transitive on all fibres of p iff (E.13) holds for every $x \in E$.

c) λ is a G-bundle iff p is onto B and identifying and R is transitive on all fibres of p.

Proof of Proposition E.1a: I first consider the case where π_λ is one-one so let $x \in E, x' \in p^{-1}(p(x))$ whence $p(x') = p(x)$ so that by (E.11) I obtain $\pi_\lambda(p_R(x')) =$

$\pi_\lambda(p_R(x))$. Since π_λ is one-one this entails $p_R(x') = p_R(x)$ whence $x' \in p_R(x)$ so that (E.13) holds.

I now consider the case where (E.13) holds for every $x \in E$. To show that π_λ is one-one let $x, x' \in E$ such that $\pi_\lambda(p_R(x')) = \pi_\lambda(p_R(x))$. Thus I am done when $p_R(x') = p_R(x)$. Note that since p_R is onto E/R, every element of the domain of π_λ belongs to the image of p_R. By (E.11) we have $p(x') = p(x)$ whence by (E.5),(E.13) I obtain $p_R(x) = p^{-1}(p(x)) = p^{-1}(p(x')) = p_R(x')$. □

Proof of Proposition E.1b: I first consider the case where R is transitive on all fibres of p so let $x \in E$ and $x' \in p^{-1}(p(x))$. Thus $x, x' \in p^{-1}(p(x))$ whence, by the transitivity of R_x, a $g \in G$ exists such that $x' = R_x(g; x) = R(g; x)$ which entails $x' \in p_R(x)$.

I now consider the case where (E.13) holds for every $x \in E$. Thus by Proposition E.1a π_λ is one-one. Let $x \in E$ so I am done when I show that R_x is transitive. Let therefore $x', x'' \in p^{-1}(p(x))$ whence by (E.11) $\pi_\lambda(p_R(x)) = p(x) = p(x') = \pi_\lambda(p_R(x'))$ so that, since π_λ is one-one, $p_R(x) = p_R(x')$ and, analogously, $p_R(x) = p_R(x'')$. Thus $g', g'' \in G$ exist such that $x' = R(g'; x)$ and $x'' = R(g''; x)$ whence $x'' = R(g'^{-1}g''; x') = R_x(g'^{-1}g''; x')$ which proves the transitivity of R_x. □

Proof of Proposition E.1c: I first consider the case where λ is a G-bundle, i.e., $\pi_\lambda \in HOMEO(E/R, B)$. Thus π_λ is onto B whence, by (E.11), p is onto B. Since p_R is onto E/R and π_λ, p_R are identifying I obtain from (E.11) that p is identifying [Du, Section VI.3]. Moreover since π_λ is one-one one concludes from Propositions E.1a-b that R is transitive on all fibres of p.

I now consider the case where p is onto B and identifying and R is transitive on all fibres of p. The latter entails by Propositions E.1a-b that π_λ is one-one. Since p_R is onto E/R and p, p_R are identifying I obtain from [Du, Section VI.3] and (E.11) that π_λ is identifying. Also since p is onto B we have by (E.11) that π_λ is onto B. I thus have shown that π_λ is one-one, onto B and identifying. Therefore

$\pi_\lambda \in HOMEO(E/R, B)$ whence λ is a G-bundle. □

I will apply Proposition E.1 time and again in the ensuing sections and in this paragraph I give a first example of that by showing that, if λ in (E.3) is a principal G-bundle, then each fibre of p is homeomorphic to G. Let therefore λ be a principal G-bundle and $b \in B$. Picking an $x \in p^{-1}(b)$ I define the function $u : G \to p^{-1}(b)$, for $g \in G$, by $u(g) := R(g; x)$. Clearly u is continuous and, due to Proposition E.1c, u is onto $p^{-1}(b)$. To show that u is a homeomorphism onto $p^{-1}(b)$ I define the function $u' : p^{-1}(b) \to G$ for $g \in G$ by $u'(R(g; x)) := g$. Note that, due to Proposition E.1c, u' is defined for the whole domain $p^{-1}(b)$. Moreover, u' is single valued since the right G-action R is free. On the other hand we have, for $g \in G$, $u'(u(g)) = u'(R(g; x)) = g$, and $u(u'(R(g; x))) = u(g) = R(g; x)$, whence u' is the inverse of u. Furthermore, recalling Appendix B, we have, for $x' \in p^{-1}(b)$, that $u'(x') = \tau_R(x, x')$. Since λ is a principal G-bundle, the translation function τ_R of R is continuous whence u' is continuous which entails that u is a homeomorphism onto $p^{-1}(b)$ as was to be shown. It is interesting to observe that the above proof uses the continuity of τ_R, shedding thus a first glimpse of light on a property of τ_R which at first sight may look artificial. In fact, in the ensuing sections the continuity of τ_R will bear further fruits.

E.2 Bundles associated with principal G-bundles

As in Section E.1 I consider the quadruple λ in (E.3) and I here assume that it is a principal G-bundle. Thus λ is a G-bundle whence, by Proposition E.1c, p is onto B.

E.2.1 Defining associated bundles

Let (F, L) be a topological left G-space. To come to the corresponding associated bundle $\lambda[F, L]$ one defines the topological space

$$E' := E \times F \;, \tag{E.14}$$

and the function $R' : G \times E' \to E'$ by

$$R'(g; x, y) := (R(g; x), L(g^{-1}; y)) \;, \tag{E.15}$$

and observes that (E', R') is a topological right G-space. To define the bundle $\lambda[F, L]$, one considers the problem of finding a function $q' : E'/R' \to B$ which satisfies

$$q' \circ p_{R'} = p \circ q \;, \tag{E.16}$$

where the function $q : E' \to E$ is defined for $x \in E, y \in F$ by

$$q(x, y) := x \;. \tag{E.17}$$

Note that since $p_{R'}$ is onto E'/R' there is at most one such q'. Since p is onto B and since q is onto E one observes from (E.16) that q', if it exists, is onto B. Furthermore since $p \circ q$ is continuous and $p_{R'}$ is onto E'/R' and identifying, one observes (see for example [Hu, Section II.6]) from (E.16) that q', if it exists, is continuous.

To show that q' exists I define the function $p' : E'/R' \to B$ for $(x, y) \in E'$ by

$$p'(p_{R'}(x, y)) := p(x) \;. \tag{E.18}$$

Note that p' is defined by (E.18) *for all* $z \in E'/R'$ since $p_{R'}$ is onto E'/R'. Note also that p', defined by (E.18), is single valued since if $(x, y), (x', y') \in E'$ and $p_{R'}(x, y) = p_{R'}(x', y')$ then, recalling Appendix B, a $g \in G$ exists such that $R'(g; x, y) = (x', y')$, i.e., by (E.15)

$$(x', y') = R'(g; x, y) = (R(g; x), L(g^{-1}; y)) \;, \tag{E.19}$$

so that

$$p(x') = p(R(g;x)) = p(x) \ , \tag{E.20}$$

where in the second equality of (E.20) I used the fact that p is a G-map (which follows from the fact that λ is a G-prebundle). With (E.20) I have completed the proof that p' is a function: $E'/R' \to B$. Clearly we have by (E.18) that $q' = p'$ satisfies (E.16) so that I conclude by the remarks after (E.17) that $q' = p'$ is the unique function: $E'/R' \to B$ which satisfies (E.16) whence I got

$$p' \circ p_{R'} = p \circ q \ . \tag{E.21}$$

I conclude from (E.21) and the remarks after (E.17) that p' is onto B and continuous. Furthermore, since $p \circ q$ is continuous and q is onto E and identifying and since p is identifying I obtain from [Du, Section VI.3] that $p \circ q$ is identifying whence, by (E.21), $p' \circ p_{R'}$ is identifying. Thus and since $p_{R'}$ is onto E'/R' and identifying I obtain from [Du, Section VI.3] and (E.21) that p' is identifying. Note also that, recalling Definition C.1, it follows from (E.21) that p' is a factor of $p \circ q$ w.r.t. the fiber structure $\alpha(E', R') = (E', p_{R'}, E'/R')$. Equipped with p' one defines ξ' by

$$\xi' := \lambda[F, L] := (E'/R', p', B) \ . \tag{E.22}$$

Note that ξ' is called the 'associated bundle', or more precisely, the bundle 'associated with λ via the topological left G-space (F, L)'. Clearly ξ' is a fiber structure.

E.2.2 Correspondence between cross sections and pseudo cross sections of an associated bundle

Let, as in Section E.2.1, (F, L) be a topological left G-space. In the theory of reductions of the principal bundle λ the cross sections of $\lambda[F, L]$ play an important role (see Section E.6.6 and recall the definition of cross sections in Section C.1).

On the other hand, working with $\Gamma(\lambda[F,L])$ is facilitated by using a correspondence between cross sections and pseudo cross sections which I introduce now. I denote the set of pseudo cross sections associated with λ via (F,L) by the symbol $\tilde{\Gamma}_{\lambda,F,L}$. The set $\tilde{\Gamma}_{\lambda,F,L}$ consists of those functions ψ in $\mathcal{C}(E,F)$ which satisfy, for $g \in G, x \in E$,

$$\psi(R(g;x)) = L(g^{-1};\psi(x)) \ . \tag{E.23}$$

The correspondence between $\Gamma(\lambda[F,L])$ and $\tilde{\Gamma}_{\lambda,F,L}$ is established by the function $\gamma_{\lambda,F,L} : \tilde{\Gamma}_{\lambda,F,L} \to \Gamma(\lambda[F,L])$ which is defined for $\psi \in \tilde{\Gamma}_{\lambda,F,L}$ by

$$\gamma_{\lambda,F,L}(\psi) := \sigma \ , \tag{E.24}$$

where the function $\sigma : B \to E'/R'$ is defined for $x \in E$ by

$$\sigma(p(x)) := p_{R'}(x,\psi(x)) \ . \tag{E.25}$$

Note that σ is defined by (E.25) on the whole set B since p is onto B. To show that σ is single valued let $x, x' \in E$ such that $p(x') = p(x)$ whence, by Proposition E.1c, a $g \in G$ exists such that $x' = R(g;x)$ so that one concludes from (E.15),(E.23),(E.25)

$$\begin{aligned}\sigma(p(x')) &= p_{R'}(x',\psi(x')) = p_{R'}(R(g;x),\psi(R(g;x))) = p_{R'}(R(g;x),L(g^{-1};\psi(x))) \\ &= p_{R'}(R'(g;x,\psi(x))) = p_{R'}(x,\psi(x)) = \sigma(p(x)) \ . \end{aligned} \tag{E.26}$$

Thus indeed σ is a function: $B \to E'/R'$. Since, by (E.25), $\sigma \circ p$ is continuous and since p is onto B and identifying it follows (see, e.g., [Hu, Section II.6]) that σ is continuous. Furthermore I conclude from (E.18),(E.25) that, for $x \in E$, $(p' \circ \sigma)(p(x)) = p'(p_{R'}(x,\psi(x))) = p(x)$, whence, since p is onto B,

$$p' \circ \sigma = id_B \ . \tag{E.27}$$

Since $\sigma \in \mathcal{C}(B, E'/R')$ it follows from (E.22), (E.27) that $\sigma \in \Gamma(\lambda[F,L])$. This completes the proof that $\gamma_{\lambda,F,L}$ is a function: $\tilde{\Gamma}_{\lambda,F,L} \to \Gamma(\lambda[F,L])$. Note that $\gamma_{\lambda,F,L}$ is one-one. In fact let $\psi, \psi' \in \tilde{\Gamma}_{\lambda,F,L}$ such that $\gamma_{\lambda,F,L}(\psi') = \gamma_{\lambda,F,L}(\psi)$. Thus by (E.24), (E.25) we have, for $x \in E$,

$$p_{R'}(x, \psi'(x)) = p_{R'}(x, \psi(x)) \ ,$$

whence a $g \in G$ exists such that $(x, \psi'(x)) = R'(g; x, \psi(x))$ which entails by (E.15)

$$(x, \psi'(x)) = R'(g; x, \psi(x)) = (R(g; x), L(g^{-1}; \psi(x))) \ . \tag{E.28}$$

Since λ is a principal G-bundle, the right G-action R is free so that, by (E.28), $g = e_G$ whence, by (E.28), $\psi' = \psi$. Thus $\gamma_{\lambda,F,L}$ is one-one. Under mild conditions on λ, F, L one can even show that $\gamma_{\lambda,F,L}$ is a bijection onto $\Gamma(\lambda[F, L])$ and this property makes $\gamma_{\lambda,F,L}$ a useful tool. In fact in the case of the product principal bundle I will prove the bijection property of $\gamma_{\lambda,F,L}$ (see Section E.6.4).

E.3 Two canonical left actions of the automorphism group of a principal G-bundle

I here assume that the quadruple λ in (E.3) is a principal G-bundle. I here apply the Feres machinery by showing how $\mathfrak{Aut}_{Bun(G)}(\lambda)$ acts from the left in two canonical ways. A pivotal role is played by those morphisms in $\mathfrak{Mor}_{Bun}(\xi')$ which are fibre morphisms.

E.3.1 The canonical left action on the total space of an associated bundle

Since λ is a principal G-bundle I can apply the tools of Section E.2.1 to construct the left $\mathfrak{Aut}_{Bun(G)}(\lambda)$-action L' on the total space E'/R' of the associated bundle $\xi' = \lambda[F, L]$ so let $(\varphi, \bar{\varphi}) \in \mathfrak{Aut}_{Bun(G)}(\lambda)$. Note that by Section E.1 φ is a homeomorphism onto E and a G-map on the right G-space (E, R). I define the function $\varphi' : E' \to E'$ for $(x, y) \in E'$ by

$$\varphi'(x, y) := (\varphi(x), y) \tag{E.29}$$

and observe by (E.2) that for $(x, y) \in E'$

$$(p \circ q)(\varphi'(x, y)) = (p \circ q)(\varphi(x), y) = p(\varphi(x)) = \bar{\varphi}(p(x)) = \bar{\varphi}(p(q(x, y))) \ ,$$

i.e.,

$$p \circ q \circ \varphi' = \bar{\varphi} \circ p \circ q \ . \tag{E.30}$$

Basic to the construction of the group action on E'/R' is the consideration of the problem of finding a function $\varphi'' : E'/R' \to E'/R'$ which satisfies

$$\varphi'' \circ p_{R'} = p_{R'} \circ \varphi' \ . \tag{E.31}$$

Note that since $p_{R'}$ is onto E'/R' there is at most one such φ''. Moreover since φ is onto E one observes from (E.29) that φ' is onto E' whence, since $p_{R'}$ is onto E'/R', (E.31) entails that φ'', if it exists, is onto E'/R'. Furthermore since $p_{R'} \circ \varphi'$ is continuous and $p_{R'}$ is onto E'/R' and identifying, one observes [Hu, Section II.6] that φ'', if it exists, is continuous. Also, if φ'' exists, then by (E.21),(E.30),(E.31),

$$p' \circ \varphi'' \circ p_{R'} = p' \circ p_{R'} \circ \varphi' = p \circ q \circ \varphi' = \bar{\varphi} \circ p \circ q = \bar{\varphi} \circ p' \circ p_{R'} \ . \tag{E.32}$$

Since $p_{R'}$ is onto E'/R' it follows from (E.32) that, if φ'' exists, then

$$p' \circ \varphi'' = \bar{\varphi} \circ p' \ , \tag{E.33}$$

whence $(\varphi'', \bar{\varphi}) \in \mathfrak{Mor}_{Bun}(\xi')$. To show that φ'' exists I define the function $\tilde{\varphi} : E'/R' \to E'/R'$ for $(x, y) \in E'$ by

$$\tilde{\varphi}(p_{R'}(x, y)) := (p_{R'} \circ \varphi')(x, y) = p_{R'}(\varphi(x), y) \ , \tag{E.34}$$

where in the second equality I used (E.29). Note that $\tilde{\varphi}$ is defined *for all* $z \in E'/R'$ by (E.34) since $p_{R'}$ is onto E'/R'. Note also that $\tilde{\varphi}$, defined by (E.34), is single valued since if $(x,y),(x',y') \in E'$ and $p_{R'}(x,y) = p_{R'}(x',y')$ then, recalling Section E.2.1, a $g \in G$ exists such that (E.19) holds which implies by (E.15),(E.34)

$$\begin{aligned} &\tilde{\varphi}(p_{R'}(x',y')) = p_{R'}(\varphi(x'),y') = p_{R'}(\varphi(R(g;x)),L(g^{-1};y)) \\ &= p_{R'}(R(g;\varphi(x)),L(g^{-1};y)) = p_{R'}(R'(g;\varphi(x),y)) = p_{R'}(\varphi(x),y) \\ &= \tilde{\varphi}(p_{R'}(x,y)) \ , \end{aligned} \tag{E.35}$$

where in the third equality I used the fact that φ is a G-map on (E,R). With (E.35) I have completed the proof that $\tilde{\varphi}$ is a function: $E'/R' \to E'/R'$. Clearly we have by (E.34) that $\varphi'' = \tilde{\varphi}$ satisfies (E.31) so that one concludes by the remarks after (E.31) that $\varphi'' = \tilde{\varphi}$ is the unique function: $E'/R' \to E'/R'$ which satisfies (E.31) whence I got

$$\tilde{\varphi} \circ p_{R'} = p_{R'} \circ \varphi' \ . \tag{E.36}$$

I define the function $L' : \mathfrak{Aut}_{Bun(G)}(\lambda) \times E'/R' \to E'/R'$ for $(\varphi,\bar{\varphi}) \in \mathfrak{Aut}_{Bun(G)}(\lambda)$ and $z \in E'/R'$ by

$$L'(\varphi,\bar{\varphi};z) := \tilde{\varphi}(z) \ . \tag{E.37}$$

Thus $\varphi'' = L'(\varphi,\bar{\varphi};\cdot)$ is the unique function: $E'/R' \to E'/R'$ which satisfies (E.31) whence I got

$$L'(\varphi,\bar{\varphi};\cdot) \circ p_{R'} = p_{R'} \circ \varphi' \ . \tag{E.38}$$

By the remarks after (E.31) I also have that $L'(\varphi,\bar{\varphi};\cdot)$ is onto E'/R', is continuous and satisfies

$$p' \circ L'(\varphi,\bar{\varphi};\cdot) = \bar{\varphi} \circ p' \ , \tag{E.39}$$

whence, by recalling the definition of $\mathfrak{Mor}_{Bun}$ in Section E.1,

$$(L'(\varphi,\bar{\varphi};\cdot),\bar{\varphi}) \in \mathfrak{Mor}_{Bun}(\xi') \ . \tag{E.40}$$

Clearly by (E.29),(E.38) we have for $(x, y) \in E'$

$$L'(\varphi, \bar{\varphi}; p_{R'}(x, y)) = (p_{R'} \circ \varphi')(x, y) = p_{R'}(\varphi(x), y) \; . \tag{E.41}$$

To prove that L' is a group action I compute for $(x, y) \in E'$ by (E.41)

$$L'(id_E, id_B; p_{R'}(x, y)) = p_{R'}(x, y) \; , \tag{E.42}$$

and for $(\varphi_1, \bar{\varphi}_1), (\varphi_2, \bar{\varphi}_2) \in \mathfrak{Aut}_{Bun(G)}(\lambda)$ and $(x, y) \in E'$ by using again (E.41)

$$\begin{aligned}&\Big(L'(\varphi_2, \bar{\varphi}_2; \cdot) \circ L'(\varphi_1, \bar{\varphi}_1; \cdot)\Big)(p_{R'}(x, y)) = L'(\varphi_2, \bar{\varphi}_2; L'(\varphi_1, \bar{\varphi}_1; p_{R'}(x, y)))\\ &= L'(\varphi_2, \bar{\varphi}_2; p_{R'}(\varphi_1(x), y)) = p_{R'}(\varphi_2(\varphi_1(x)), y) = p_{R'}((\varphi_2 \circ \varphi_1)(x), y)\\ &= L'(\varphi_2 \circ \varphi_1, \bar{\varphi}_2 \circ \bar{\varphi}_1; p_{R'}(x, y)) \; ,\end{aligned} \tag{E.43}$$

where in the fifth equality I used the fact that $\mathfrak{Aut}_{Bun(G)}(\lambda)$ is a group under the composition law in $Bun(G)$. Because $p_{R'}$ is onto E'/R' it follows from (E.42),(E.43) that L' is a left $\mathfrak{Aut}_{Bun(G)}(\lambda)$-action on E'/R'. The following remark puts L' into perspective.

Remark:

(1) A 'fibre morphism' on the associated bundle $\xi' = \lambda[F, L]$ is an element $(f, \bar{f})$ of $\mathfrak{Mor}_{Bun}(\xi')$ for which a continuous G-map f' exists on the topological right G-space (E', R') such that $f \circ p_{R'} = p_{R'} \circ f'$ [Hus, Section 4.6]. Thus by (E.40) the question arises of whether $(L'(\varphi, \bar{\varphi}; \cdot), \bar{\varphi})$ is a fibre morphism on ξ'. In fact it follows from (E.38) that if φ' is a G-map on (E', R') then $(L'(\varphi, \bar{\varphi}; \cdot), \bar{\varphi})$ is a fibre morphism on ξ'. I thus compute by (E.15), (E.29) for $(\varphi, \bar{\varphi}) \in \mathfrak{Aut}_{Bun(G)}(\lambda)$ and $g \in G, x \in E, y \in F$,

$$\begin{aligned}&\varphi'(R'(g; x, y)) = \varphi'(R(g; x), L(g^{-1}; y)) = (\varphi(R(g; x)), L(g^{-1}; y))\\ &= (R(g; \varphi(x)), L(g^{-1}; y)) = R'(g; \varphi(x), y) = R'(g; \varphi'(x, y)) \; ,\end{aligned} \tag{E.44}$$

where in the third equality I used the fact that φ is a G-map on (E, R). It follows from (E.44) that the continuous function φ' is a G-map on (E', R') whence $(L'(\varphi, \bar{\varphi}; \cdot), \bar{\varphi})$ is a fibre morphism on ξ'.

Note also that since L' is a left $\mathfrak{Aut}_{Bun(G)}(\lambda)$-action on E'/R' and $L'(\varphi, \bar{\varphi}; \cdot)$ is continuous I conclude that each $L'(\varphi, \bar{\varphi}; \cdot)$ is a homeomorphism onto E'/R' whence $(L'(\varphi, \bar{\varphi}; \cdot), \bar{\varphi})$ is an automorphism in Bun. □

E.3.2 The canonical left action on the cross sections of an associated bundle

The Feres machinery provides me also with a canonical left $\mathfrak{Aut}_{Bun(G)}(\lambda)$-action, L'', on the set $\Gamma(\xi')$ of cross sections of the associated bundle $\xi' = \lambda[F, L]$ and this goes as follows. One defines the function $L'' : \mathfrak{Aut}_{Bun(G)}(\lambda) \times \Gamma(\xi') \to \Gamma(\xi')$ for $(\varphi, \bar{\varphi}) \in \mathfrak{Aut}_{Bun(G)}(\lambda)$ and $\sigma \in \Gamma(\xi')$ by

$$L''(\varphi, \bar{\varphi}; \sigma) := L'(\varphi, \bar{\varphi}; \cdot) \circ \sigma \circ \bar{\varphi}^{-1} , \tag{E.45}$$

i.e., for $z \in B$,

$$(L''(\varphi, \bar{\varphi}; \sigma))(z) = L'(\varphi, \bar{\varphi}; \sigma(\bar{\varphi}^{-1}(z))) . \tag{E.46}$$

Since $L'(\varphi, \bar{\varphi}; \cdot)$, σ, and $\bar{\varphi}^{-1}$ are continuous functions it follows from (E.45) that $L''(\varphi, \bar{\varphi}; \sigma) \in \mathcal{C}(B, E'/R')$. Furthermore by Definition C.1 and (E.22) we have for $\sigma \in \Gamma(\xi')$ that $p' \circ \sigma = id_B$ whence we obtain from (E.39),(E.45) that for $(\varphi, \bar{\varphi}) \in \mathfrak{Aut}_{Bun(G)}(\lambda)$ and $\sigma \in \Gamma(\xi')$

$$p' \circ L''(\varphi, \bar{\varphi}; \sigma) = p' \circ L'(\varphi, \bar{\varphi}; \cdot) \circ \sigma \circ \bar{\varphi}^{-1} = \bar{\varphi} \circ p' \circ \sigma \circ \bar{\varphi}^{-1} = \bar{\varphi} \circ id_B \circ \bar{\varphi}^{-1} = id_B ,$$

so that, by Definition C.1, $L''(\varphi, \bar{\varphi}; \sigma) \in \Gamma(\xi')$ which completes the proof that L'' is a function: $\mathfrak{Aut}_{Bun(G)}(\lambda) \times \Gamma(\xi') \to \Gamma(\xi')$. To show that L'' is a left $\mathfrak{Aut}_{Bun(G)}(\lambda)$-action on $\Gamma(\xi')$ let $(\varphi_1, \bar{\varphi}_1), (\varphi_2, \bar{\varphi}_2) \in \mathfrak{Aut}_{Bun(G)}(\lambda)$ and $\sigma \in \Gamma(\xi')$ and let me define

$\sigma' \in \Gamma(\xi')$ by

$$\sigma' := L''(\varphi_1, \bar{\varphi_1}; \sigma) \ . \tag{E.47}$$

Note that for $z \in B$ we have by (E.46),(E.47)

$$\sigma'(z) = L'(\varphi_1, \bar{\varphi_1}; \sigma(\bar{\varphi_1}^{-1}(z))) \ . \tag{E.48}$$

Since L' is a left $\mathfrak{Aut}_{Bun(G)}(\lambda)$-action on E'/R' it follows from (E.46),(E.47),(E.48) that for $(\varphi_1, \bar{\varphi_1}), (\varphi_2, \bar{\varphi_2}) \in \mathfrak{Aut}_{Bun(G)}(\lambda)$ and $z \in B$

$$(L''(id_E, id_B; \sigma))(z) = L'(id_E, id_B; \sigma(z)) = \sigma(z) \ , \tag{E.49}$$

$$\begin{aligned}
&\Big(L''(\varphi_2 \circ \varphi_1, \bar{\varphi_2} \circ \bar{\varphi_1}; \sigma)\Big)(z) = L'(\varphi_2 \circ \varphi_1, \bar{\varphi_2} \circ \bar{\varphi_1}; (\sigma \circ \bar{\varphi_1}^{-1} \circ \bar{\varphi_2}^{-1})(z)) \\
&= L'(\varphi_2, \bar{\varphi_2}; L'(\varphi_1, \bar{\varphi_1}; (\sigma \circ \bar{\varphi_1}^{-1} \circ \bar{\varphi_2}^{-1})(z))) = L'(\varphi_2, \bar{\varphi_2}; \sigma'(\bar{\varphi_2}^{-1}(z))) \\
&= (L''(\varphi_2, \bar{\varphi_2}; \sigma'))(z) = (L''(\varphi_2, \bar{\varphi_2}; L''(\varphi_1, \bar{\varphi_1}; \sigma)))(z) \ .
\end{aligned} \tag{E.50}$$

I conclude from (E.49),(E.50) that L'' is a left $\mathfrak{Aut}_{Bun(G)}(\lambda)$-action on $\Gamma(\xi')$.

E.4 Group homomorphisms into the automorphism group of a principal G-bundle

Let the quadruple λ in (E.3) be a principal G-bundle. If K is a group then I denote the set of group homomorphisms from K into $\mathfrak{Aut}_{Bun(G)}(\lambda)$ by $HOM_K(\lambda)$. If $\Phi \in HOM_K(\lambda)$ then $\Phi(K)$ is a subgroup of $\mathfrak{Aut}_{Bun(G)}(\lambda)$ and, for $k \in K$, I write

$$\Phi(k) = (\varphi(k; \cdot), \bar{\varphi}(k; \cdot)) \ , \tag{E.51}$$

where $(\varphi(k; \cdot), \bar{\varphi}(k; \cdot)) \in \mathfrak{Aut}_{Bun(G)}(\lambda)$. Let $\tilde{\varphi} \in \mathfrak{Gau}_{Bun(G)}(\lambda)$, i.e., by (E.12), $\tilde{\Phi} := (\tilde{\varphi}, id_B)$ is in $\mathfrak{Aut}_{Bun(G)}(\lambda)$. If $\Phi \in HOM_K(\lambda)$ then I define the function $\Phi' : K \to \mathfrak{Aut}_{Bun(G)}(\lambda)$ for $k \in K$ by

$$\Phi'(k) := \tilde{\Phi}^{-1}\Phi(k)\tilde{\Phi} = (\tilde{\varphi}, id_B)^{-1}\Phi(k)(\tilde{\varphi}, id_B) = (\tilde{\varphi}^{-1} \circ \varphi(k; \cdot) \circ \tilde{\varphi}, \bar{\varphi}(k; \cdot)) \ , \tag{E.52}$$

where I used the notation of (E.51). Clearly $\Phi' \in HOM_K(\lambda)$ and $\Phi'(K)$ is a subgroup of $\mathfrak{Aut}_{Bun(G)}(\lambda)$. In fact, the groups $\Phi(K), \Phi'(K)$ are conjugate via $\tilde{\Phi}$.

E.5 Reducing the structure group G

Let G be a topological group and let H be a closed topological subgroup of G. Let also λ in (E.3) be a principal G-bundle and $\hat{\lambda}$ be a principal H-bundle where I write

$$\hat{\lambda} = (\hat{E}, \hat{p}, B, \hat{R}) \ . \tag{E.53}$$

If $f \in \mathcal{C}(\hat{E}, E)$ exists such that, for $x \in \hat{E}, h \in H$,

$$f(\hat{R}(h; x)) = R(h; f(x)) \ , \tag{E.54}$$

then I call $\hat{\lambda}$ a 'H-quasireduction of λ'. With f I can define the function $\bar{f} : B \to B$ for $x \in \hat{E}$ by

$$\bar{f}(\hat{p}(x)) := (p \circ f)(x) \ . \tag{E.55}$$

Note that $\bar{f}$ is defined by (E.55) for all $b \in B$ since, by Proposition E.1c, $\hat{p}$ is onto B. To show that $\bar{f}$ is single valued let $x, x' \in \hat{E}$ such that $\hat{p}(x') = \hat{p}(x)$ whence, by Proposition E.1c, a $h \in H$ exists such that $x' = \hat{R}(h; x)$ so that I conclude from (E.54),(E.55)

$$\bar{f}(\hat{p}(x')) = (p \circ f)(x') = (p \circ f)(\hat{R}(h; x)) = p(R(h; f(x)) = p(f(x)) = \bar{f}(\hat{p}(x)) \ .$$

Thus indeed $\bar{f}$ is a function: $B \to B$. Since $p \circ f$ is continuous and since, by Proposition E.1c, $\hat{p}$ is onto B and identifying, it follows (see, e.g., [Hu, Section II.6]) from (E.55) that $\bar{f}$ is continuous. I call the pair $(f, \bar{f})$ a 'quasihomomorphism from $\hat{\lambda}$ to λ'. Clearly a principal H-bundle $\hat{\lambda}$ is a H-quasireduction of λ iff a quasihomomorphism from $\hat{\lambda}$ to λ exists. Note by (E.55) that, since $\hat{p}$ is onto B, the only function

$g : B \to B$ which satisfies $g \circ \hat{p} = p \circ f$ is given by $g = \bar{f}$. If $\hat{\lambda}$ is a H-quasireduction of λ and if, in the notation of (E.53), its total space $\hat{E}$ is a closed topological subspace of E then I call $\hat{\lambda}$ a 'H-reduction of λ' if a quasihomomorphism from $\hat{\lambda}$ to λ exists which has the form (f, id_B) where f is the natural injection: $\hat{E} \to E$. Of course if $\hat{\lambda}$ is a H-reduction of λ then by (E.55)

$$\hat{p} = p\Big|\hat{E} \ , \tag{E.56}$$

and, by (E.54), $\hat{R}$ is the restriction of R to $H \times \hat{E}$, i.e.,

$$\hat{R} = R\Big|(H \times \hat{E}) \ . \tag{E.57}$$

Clearly the H-reductions of λ form a set and I denote this set by $RED_H(\lambda)$. I also note that, in the notation of (E.53), a principal H-bundle $\hat{\lambda}$ is a H-reduction of λ iff the following hold: $\hat{E}$ is a closed topological subspace of E and (E.56),(E.57) hold. Moreover it is clear by (E.56),(E.57) that if $\hat{\lambda}$ and $\hat{\lambda}'$ are principal H-bundles in $RED_H(\lambda)$ which have the same total space then $\hat{\lambda} = \hat{\lambda}'$. In particular a H-reduction of λ is completely determined by its total space. In other words, if $\hat{E}$ is a closed subspace of E then a H-reduction of λ with total space $\hat{E}$ is, if it exists at all, given by $(\hat{E}, p\Big|\hat{E}, B, R\Big|(H \times \hat{E}))$.

If $(\varphi, \bar{\varphi}) \in \mathfrak{A}ut_{Bun(G)}(\lambda)$ then I call a H-reduction $\hat{\lambda}$ of λ 'invariant under $(\varphi, \bar{\varphi})$' if, in the notation of (E.53), $\hat{E}$ is invariant under φ, i.e., $\varphi(\hat{E}) = \hat{E}$. Analogously, using the notation of Section E.4, if K is a group and $\Phi \in HOM_K(\lambda)$ then I call a H-reduction $\hat{\lambda}$ of λ 'invariant under the group $\Phi(K)$' if, in the notation of (E.53), $\hat{E}$ is invariant under $\Phi(k)$ for every $k \in K$. This concept of invariant H-reduction is very important since it underlies the so-called reduction theorems (see Section E.6.6).

To study H-reductions it is, as will become clear in Section E.6.6, very useful to introduce the topological space G/H and I first define the function $R_{G/H} : H \times G \to G$

for $h \in H, g \in G$ by

$$R_{G/H}(h; g) := gh \ . \tag{E.58}$$

Clearly $(E, R_{G/H})$ is a topological right H-space. I denote the orbit of a $g \in G$ under $R_{G/H}$ by gH, i.e.,

$$gH := \{R_{G/H}(h; g) : h \in H\} = \{gh : h \in H\} \ . \tag{E.59}$$

The orbit space will be denoted by G/H, i.e.,

$$G/H := \{gH : g \in G\} \ . \tag{E.60}$$

Following Appendix B, I define the function $p_{R_{G/H}} : G \to G/H$ for $g \in G$ by

$$p_{R_{G/H}}(g) = gH \ , \tag{E.61}$$

and I equip G/H with the identifying topology w.r.t. $p_{R_{G/H}}$. Thus $p_{R_{G/H}}$ is identifying and even open. I now define the function $L_{G/H} : G \times G/H \to G/H$ for $g, g' \in G$ by

$$L_{G/H}(g'; gH) := (g'g)H \ . \tag{E.62}$$

Clearly $L_{G/H}$ is a transitive left G-action on G/H. To show that $L_{G/H}$ is continuous it is now helpful to have $R_{G/H}$ at hand. In fact, defining the auxiliary function $j \in \mathcal{C}(G \times G, G \times (G/H))$ for $g, g' \in G$ by

$$j(g', g) := (g', p_{R_{G/H}}(g)) = (g', gH) \ , \tag{E.63}$$

we have by (E.61),(E.62) for $g, g' \in G$

$$(L_{G/H} \circ j)(g', g) = L_{G/H}(g'; gH) = (g'g)H = p_{R_{G/H}}(g'g) \ . \tag{E.64}$$

Since id_G and $p_{R_{G/H}}$ are open functions and j is the cartesian product of id_G and $p_{R_{G/H}}$, one concludes that j is an open function so that, by [Hu, Section II.6], j is identifying. Because j is onto $G \times (G/H)$ and identifying and since, due to (E.64), $L_{G/H} \circ j$ is continuous one concludes by [Hu, Section II.6] that $L_{G/H}$ is continuous. Thus $(G/H, L_{G/H})$ is a topological left G-space. The importance of $(G/H, L_{G/H})$ lies in the fact that the associated bundle $\lambda[G/H, L_{G/H}]$ is a tool for studying the H-reductions of λ (see Section E.6.6). I now draw an important conclusion from my assumption that H is closed in G. I observe by (E.59),(E.61) that $p^{-1}_{R_{G/H}}(e_G H) = H$. Since H is closed in G and $p_{R_{G/H}}$ is identifying I conclude that the singleton $e_G H$ is closed in G/H. However since the continuous left G-action $L_{G/H}$ is transitive, it follows that every singleton in G/H is closed, i.e., G/H is a T_1 space.

E.6 The special case of the product principal G-bundles

I here reconsider Sections E.1-E.5 in the special case where the quadruple λ in (E.3) is a product principal G-bundle. The product principal G-bundles are important for this work because Section 9.3 is based on a product principal $SO(3)$-bundle.

To define the product principal G-bundle I first define

$$E := B \times G \ , \tag{E.65}$$

whence by (E.1),(E.3)

$$\xi = (B \times G, p, B) \ , \tag{E.66}$$

$$\lambda = (\xi, R) = (B \times G, p, B, R) \ . \tag{E.67}$$

Furthermore $p : E \to B$ is defined for $b \in B, g \in G$ by

$$p(b, g) := b \ , \tag{E.68}$$

and $R : G \times E \to E$ is defined for $g, g' \in G, b \in B$ by

$$R(g'; b, g) := (b, gg') \, . \tag{E.69}$$

Of course (E, R) given by (E.65),(E.69) is a topological right G-space and p is, due to (E.68), onto B. In the following section I will show that λ, defined by (E.67),(E.68),(E.69), is a principal G-bundle.

E.6.1 The automorphism group of a product principal G-bundle

In the present section I show that λ, defined by (E.67),(E.68),(E.69), is a principal G-bundle and that $\mathfrak{Aut}_{Bun(G)}(\lambda)$ has a simple structure (the latter will pay off in Section E.6.3). To show that λ is a principal G-bundle we have to remind us of Appendix B and Section E.1 and I first note that for $g, g' \in G, b \in B$ we have by (E.68),(E.69)

$$p(R(g'; b, g)) = p(b, gg') = b = p(b, g) \, , \tag{E.70}$$

whence λ is a G-prebundle. I next use Proposition E.1 to show that λ is a G-bundle. Firstly I note by (E.68) that p is onto B and identifying since it is the projection onto the first argument. Secondly, for $b \in B$, the fibre of p over b reads by (E.68) as

$$p^{-1}(b) = \{b\} \times G \, , \tag{E.71}$$

whence, for $(b', g'), (b'', g'') \in p^{-1}(b)$, we have $b = b' = b''$ and $R(g'^{-1}g''; b', g') = (b', g'g'^{-1}g'') = (b', g'') = (b'', g'')$ so that R is transitive on all fibres of p. With these two properties of λ one concludes from Proposition E.1c that λ is a G-bundle. To show that λ is a principal G-bundle it remains to be shown that (E, R) is principal. First of all if for $g, g' \in G, b \in B$ I impose the condition $R(g'; b, g) = (b, g)$ then by (E.69) $(b, gg') = (b, g)$ whence $g' = e_G$ which entails that the right G-action R is free.

Recalling Appendix B I define $E^* := \{(b, g, R(g'; b, g)) : b \in B, g, g' \in G\}$ whence by (E.69)

$$E^* = \{(b, g, b, gg') : b \in B, g, g' \in G\} = \{(b, g, b, g') : b \in B, g, g' \in G\} . \quad \text{(E.72)}$$

I define the function $\tau_R : E^* \to G$ for $(b, g, b, g') \in E^*$ by

$$\tau_R(b, g, b, g') := g^{-1}g' , \quad \text{(E.73)}$$

and observe for $(b, g, b, g') \in E^*$ that by (E.69)

$$R(\tau_R(b, g, b, g'); b, g) = R(g^{-1}g'; b, g) = (b, gg^{-1}g') = (b, g') , \quad \text{(E.74)}$$

so that τ_R is the translation function of R. Clearly τ_R is continuous whence the topological right G-space (E, R) is principal which completes the proof that λ is a principal G-bundle. Note also that λ is called a 'product principal G-bundle'.

Most importantly, since in the present context λ is a product principal G-bundle, its automorphism group, which is defined in Section E.1, has quite a simple structure as I will now demonstrate. Defining the function $r : E \to G$ for $b \in B, g \in G$ by $r(b, g) := g$, every $\varphi \in \mathcal{C}(E, E)$ reads as $\varphi = (p \circ \varphi, r \circ \varphi)$ and we have $p \circ \varphi \in \mathcal{C}(E, B)$, $r \circ \varphi \in \mathcal{C}(E, G)$. If $(\varphi, \bar{\varphi}) \in \mathfrak{Mor}_{Bun}(\xi)$ then for $b \in B, g \in G$ we have by (E.2),(E.68)

$$\bar{\varphi}(b) = (\bar{\varphi} \circ p)(b, g) = (p \circ \varphi)(b, g) , \quad \text{(E.75)}$$

whence

$$\varphi(b, g) = (\bar{\varphi}(b), (r \circ \varphi)(b, g)) . \quad \text{(E.76)}$$

If $(\varphi, \bar{\varphi}) \in \mathfrak{Mor}_{Bun(G)}(\lambda)$ then for $b \in B, g, g' \in G$ we have by (E.69),(E.76) and by recalling Section E.1

$$(\bar{\varphi}(b), (r \circ \varphi)(b, g)g') = R(g'; \bar{\varphi}(b), (r \circ \varphi)(b, g)) = R(g'; \varphi(b, g)) = \varphi(R(g'; b, g))$$
$$= \varphi(b, gg') = (\bar{\varphi}(b), (r \circ \varphi)(b, gg')) , \quad \text{(E.77)}$$

where in the third equality I used the fact that φ is a G-map on (E, R). Of course by (E.77) we have for $b \in B, g \in G$ that $(r \circ \varphi)(b, e_G)g = (r \circ \varphi)(b, g)$ so that by (E.76) $\varphi(b, g) = (\bar{\varphi}(b), (r \circ \varphi)(b, e_G)g)$ whence

$$\mathfrak{Mor}_{Bun(G)}(\lambda) \subset \{(\varphi, \bar{\varphi}) \in \mathcal{C}(E, E) \times \mathcal{C}(B, B) : \\ [(\forall\, b \in B, g \in G)\varphi(b, g) = (\bar{\varphi}(b), f(b)g)], f \in \mathcal{C}(B, G)\} . \tag{E.78}$$

Furthermore if $(\varphi, \bar{\varphi})$ is an element of the set on the rhs of (E.78) then for $b \in B, g \in G$ we have $\varphi(b, g) = (\bar{\varphi}(b), f(b)g)$ where $\bar{\varphi} \in \mathcal{C}(B, B)$ and $f \in \mathcal{C}(B, G)$. Note also that $f(b) = r(\varphi(b, e_G))$. This $(\varphi, \bar{\varphi})$ satisfies (E.2), whence $(\varphi, \bar{\varphi}) \in \mathfrak{Mor}_{Bun}(\xi)$, and for $b \in B, g, g' \in G$ this $(\varphi, \bar{\varphi})$ satisfies by (E.69)

$$R(g'; \varphi(b, g)) = R(g'; \bar{\varphi}(b), f(b)g) = (\bar{\varphi}(b), f(b)gg') = \varphi(b, gg') = \varphi(R(g'; b, g)) ,$$

so that φ is a G-map on (E, R). Thus I have shown that every element of the set on the rhs of (E.78) belongs to $\mathfrak{Mor}_{Bun(G)}(\lambda)$ whence by (E.78) I got

$$\mathfrak{Mor}_{Bun(G)}(\lambda) = \{(\varphi, \bar{\varphi}) \in \mathcal{C}(E, E) \times \mathcal{C}(B, B) : \\ [(\forall\, b \in B, g \in G)\varphi(b, g) = (\bar{\varphi}(b), f(b)g)], f \in \mathcal{C}(B, G)\} . \tag{E.79}$$

To determine $\mathfrak{Aut}_{Bun(G)}(\lambda)$ I recall from Section E.1 that if $(\varphi, \bar{\varphi}) \in \mathfrak{Aut}_{Bun(G)}(\lambda)$ then $(\varphi, \bar{\varphi}) \in \mathfrak{Mor}_{Bun(G)}(\lambda)$ and $\bar{\varphi} \in HOMEO(B, B)$ so that by (E.79)

$$\mathfrak{Aut}_{Bun(G)}(\lambda) \subset \{(\varphi, \bar{\varphi}) \in \mathcal{C}(E, E) \times HOMEO(B, B) : \\ [(\forall\, b \in B, g \in G)\varphi(b, g) = (\bar{\varphi}(b), f(b)g)], f \in \mathcal{C}(B, G)\} . \tag{E.80}$$

To show that equality holds in (E.80) let $(\varphi, \bar{\varphi})$ be an element of the set on the rhs of (E.80), i.e., let $\bar{\varphi} \in HOMEO(B, B)$ and $f \in \mathcal{C}(B, G)$ such that for $b \in B, g \in G$ we have $\varphi(b, g) = (\bar{\varphi}(b), f(b)g)$. I now define the function $\tilde{\varphi} \in \mathcal{C}(E, E)$ for $b \in B, g \in G$ by

$$\tilde{\varphi}(b, g) := (\bar{\varphi}^{-1}(b), (f(\bar{\varphi}^{-1}(b)))^{-1}g) . \tag{E.81}$$

Since $\bar{\varphi} \in HOMEO(B,B)$ I have $\bar{\varphi}^{-1} \in \mathcal{C}(B,B)$ whence, by (E.79),(E.81), $(\tilde{\varphi}, \bar{\varphi}^{-1}) \in \mathfrak{Mor}_{Bun(G)}(\lambda)$. I now compute by (E.81) for $b \in B, g \in G$

$$\begin{aligned} \varphi(\tilde{\varphi}(b,g)) &= \varphi(\bar{\varphi}^{-1}(b), (f(\bar{\varphi}^{-1}(b)))^{-1}g) = (\bar{\varphi}(\bar{\varphi}^{-1}(b)), f(\bar{\varphi}^{-1}(b))(f(\bar{\varphi}^{-1}(b)))^{-1}g) \\ &= (b,g) \ , \\ \tilde{\varphi}(\varphi(b,g)) &= \tilde{\varphi}(\bar{\varphi}(b), f(b)g) = (\bar{\varphi}^{-1}(\bar{\varphi}(b)), (f(\bar{\varphi}^{-1}(\bar{\varphi}(b))))^{-1}f(b)g) \\ &= (b, (f(b))^{-1}f(b)g) = (b,g) \ , \end{aligned}$$

whence by the composition rule in $Bun(G)$ (recall Section E.1)

$$(\varphi, \bar{\varphi})(\tilde{\varphi}, \bar{\varphi}^{-1}) = (\varphi \circ \tilde{\varphi}, \bar{\varphi} \circ \bar{\varphi}^{-1}) = (id_E, id_B) = (\tilde{\varphi} \circ \varphi, \bar{\varphi}^{-1} \circ \bar{\varphi}) = (\tilde{\varphi}, \bar{\varphi}^{-1})(\varphi, \bar{\varphi}) \ ,$$

which entails that $(\varphi, \bar{\varphi}) \in \mathfrak{Aut}_{Bun(G)}(\lambda)$ so that by (E.80)

$$\begin{aligned} \mathfrak{Aut}_{Bun(G)}(\lambda) = \{(\varphi, \bar{\varphi}) \in \mathcal{C}(E,E) \times HOMEO(B,B) : \\ [(\forall\, b \in B, g \in G)\varphi(b,g) = (\bar{\varphi}(b), f(b)g)], f \in \mathcal{C}(B,G)\} \ . \end{aligned} \tag{E.82}$$

This simple formula becomes important in Section E.6.3 where I consider the canonical left $\mathfrak{Aut}_{Bun(G)}(\lambda)$-actions L', L''.

E.6.2 The triviality of the associated bundles of a product principal G-bundle

Since the motto of Section E.6 is to reconsider Sections E.1-E.5 in the case when λ is the product principal G-bundle, defined by (E.67),(E.68),(E.69), I now reconsider Section E.2.1, i.e., I study the bundle $\xi' = \lambda[F,L]$ in (E.22) which is the bundle associated with λ via the topological left G-space (F,L).

In fact in the present case ξ' is remarkably simple since, as I now show, it is trivial. Thus the task of this section is to construct an appropriate isomorphism from ξ' to the product bundle ξ'' which is defined by

$$\xi'' = (B \times F, p'', B) \ , \tag{E.83}$$

where the function $p'' : B \times F \to B$ is defined for $(b, y) \in B \times F$ by $p''(b, y) := b$. The main burden of my task is to find an appropriate homeomorphism, r'', from E'/R' onto $B \times F$. With (E.14),(E.65) we have $E' = E \times F = B \times G \times F$ and I define the function $r' : E' \to B \times F$ for $b \in B, g \in G, y \in F$ by

$$r'(b, g, y) := (b, L(g; y)) . \tag{E.84}$$

Note that r' is onto $B \times F$ and continuous. I will see below that finding an appropriate homeomorphism boils down to the problem of finding a function $h : E'/R' \to B \times F$ which satisfies

$$h \circ p_{R'} = r' . \tag{E.85}$$

Note that since $p_{R'}$ is onto E'/R' there is at most one such h. Moreover since r' is onto $B \times F$ one observes that h, if it exists, is onto $B \times F$. Furthermore since r' is continuous and $p_{R'}$ is onto E'/R' and identifying, one observes [Hu, Section II.6] that h, if it exists, is continuous. To show that h exists I define the function $r'' : E'/R' \to B \times F$ for $(b, g, y) \in E'$ by

$$r''(p_{R'}(b, g, y)) := r'(b, g, y) = (b, L(g; y)) . \tag{E.86}$$

Note that r'' is defined *for all* $z \in E'/R'$ by (E.86) since $p_{R'}$ is onto E'/R'. To show that r'', defined by (E.86), is single valued, let $(b, g, y), (b', g', y') \in E'$ and $p_{R'}(b, g, y) = p_{R'}(b', g', y')$ whence, recalling Appendix B, a $g'' \in G$ exists such that $R'(g''; b, g, y) = (b', g', y')$, i.e., by (E.15),(E.69) I obtain

$$(b', g', y') = R'(g''; b, g, y) = (R(g''; b, g), L(g''^{-1}; y)) = (b, gg'', L(g''^{-1}; y)) . \tag{E.87}$$

It follows from (E.84),(E.86),(E.87) that r'' is single valued since I compute:

$$\begin{aligned} r''(p_{R'}(b', g', y')) &= r'(b', g', y') = r'(b, gg'', L(g''^{-1}; y)) = (b, L(gg''; L(g''^{-1}; y))) \\ &= (b, L(g; y)) = r'(b, g, y) = r''(p_{R'}(b, g, y)) , \end{aligned} \tag{E.88}$$

where in the fourth equality I used the fact that L is a left G-action on F. With (E.88) I have completed the proof that r'' is a function: $E'/R' \to B \times F$.

To establish r'' as the main stepping stone for an isomorphism from ξ' to ξ'' I first show that it is a homeomorphism onto $B \times F$. Clearly we have by (E.86) that $h = r''$ satisfies (E.85) so that one concludes by the remarks after (E.85) that $h = r''$ is the unique function: $E'/R' \to B \times F$ which satisfies (E.85) whence I got

$$r'' \circ p_{R'} = r' \ . \tag{E.89}$$

It also follows from the remarks after (E.85) that r'' is continuous and onto $B \times F$. To show that r'' is a homeomorphism onto $B \times F$ I first demonstrate that r' is identifying. Defining the functions $r'_1 : E' \to E'$, $r'_2 : E' \to E'$, $r'_3 : E' \to B \times F$ for $b \in B, g \in G, y \in F$ by

$$r'_1(b, g, y) := (b, g, L(g; y)) \ , \ r'_2(b, g, y) := (b, g, L(g^{-1}; y)) \ , \ r'_3(b, g, y) := (b, y) \ ,$$

I observe by (E.84) that

$$r' = r'_3 \circ r'_1 \ , \tag{E.90}$$

$$r'_1 \circ r'_2 = r'_2 \circ r'_1 = id_{E'} \ . \tag{E.91}$$

Moreover r'_1, r'_2, r'_3 are continuous and r'_3, being the projection onto the first and third component, is identifying. Since r'_1, r'_2 are continuous we have by (E.91) that $r'_1 \in HOMEO(E', E')$ whence r'_1 is identifying. Since r'_1, r'_3 are identifying and r'_1 is onto E' it follows from (E.90) and [Du, Section VI.3] that r' is identifying.

To finish the proof that r'' is a homeomorphism onto $B \times F$ I define the function $\tilde{r}'' : B \times F \to E'/R'$ for $(b, g, y) \in E'$ by

$$\tilde{r}''(r'(b, g, y)) := p_{R'}(b, g, y) \ , \tag{E.92}$$

and show that it is a continuous inverse of r''. Note that $\tilde{r}''$ is defined *for all* $z \in B \times F$ by (E.92) since r' is onto $B \times F$. Note also that $\tilde{r}''$, defined by (E.92), is single valued since if $(b,g,y),(b',g',y') \in E'$ and $r'(b,g,y) = r'(b',g',y')$ then, by (E.84), $(b, L(g;y)) = (b', L(g';y'))$ so that

$$b = b' \ , \qquad L(g'^{-1}g;y) = y' \ . \tag{E.93}$$

Thus by (E.15),(E.69),(E.92),(E.93)

$$\begin{aligned}
&\tilde{r}''(r'(b',g',y')) = p_{R'}(b',g',y') = p_{R'}(b,g',L(g'^{-1}g;y)) \\
&= p_{R'}\Big(R'(g'^{-1}g;b,g',L(g'^{-1}g;y))\Big) = p_{R'}\Big(R(g'^{-1}g;b,g'),L(g^{-1}g';L(g'^{-1}g;y))\Big) \\
&= p_{R'}(b,g,y) = \tilde{r}''(r'(b,g,y)) \ ,
\end{aligned} \tag{E.94}$$

where in the fifth equality I used the fact that L is a left G-action on F. This completes the proof that $\tilde{r}''$ is a function: $B \times F \to E'/R'$. Since $p_{R'}$ is continuous and r' is onto $B \times F$ and identifying, I conclude [Hu, Section II.6] from (E.92) that $\tilde{r}''$ is continuous. It follows from (E.89),(E.92) that

$$r' = r'' \circ p_{R'} = r'' \circ \tilde{r}'' \circ r' \ , \tag{E.95}$$

$$\tilde{r}'' \circ r'' \circ p_{R'} = \tilde{r}'' \circ r' = p_{R'} \ . \tag{E.96}$$

Since r' is onto $B \times F$ it follows from (E.95) that

$$r'' \circ \tilde{r}'' = id_{B\times F} \ , \tag{E.97}$$

and since $p_{R'}$ is onto E'/R' it follows from (E.96) that

$$\tilde{r}'' \circ r'' = id_{E'/R'} \ . \tag{E.98}$$

I conclude from (E.97),(E.98) that the continuous function $\tilde{r}''$ is the inverse of the continuous function r'' whence $r'' \in HOMEO(E'/R', B \times F)$. To construct an isomorphism from ξ' to ξ'' I compute by (E.84),(E.89) for $(b,g,y) \in E'$

$$(p'' \circ r'' \circ p_{R'})(b,g,y) = (p'' \circ r')(b,g,y) = p''(b, L(g;y)) = b \ , \tag{E.99}$$

and by (E.21),(E.68) for $(b,g,y) \in E'$

$$(p' \circ p_{R'})(b,g,y) = (p \circ q)(b,g,y) = p(b,g) = b \ , \tag{E.100}$$

where q is defined in Section E.2.1. I conclude from (E.99),(E.100) that $p'' \circ r'' \circ p_{R'} = p' \circ p_{R'}$, whence, since $p_{R'}$ is onto E'/R',

$$p'' \circ r'' = p' \ . \tag{E.101}$$

Since r'' is a homeomorphism onto $B \times F$ it follows from (E.101) that (r'', id_B) is an isomorphism from ξ' to ξ'' in the category Bun of bundles whence the bundle ξ' is trivial [Hus, Section 2.3]. Note also that (E.101) entails that r'' is an isomorphism from ξ' to ξ'' in the category Bun_B of bundles over B.

E.6.3 The two canonical left actions of the automorphism group of a product principal G-bundle

Since the motto of Section E.6 is to reconsider Sections E.1-E.5 in the case when λ is the product principal G-bundle, defined by (E.67),(E.68),(E.69), I now reconsider Section E.3, i.e., I study the left $\mathfrak{Aut}_{Bun(G)}(\lambda)$-actions L' and L''. The isomorphism (r'', id_B) from ξ' to ξ'', which I derived in Section E.6.2, is now the key tool.

I first consider L'. I define the function $\tilde{L}' : \mathfrak{Aut}_{Bun(G)}(\lambda) \times B \times F \to B \times F$ for $(\varphi, \bar{\varphi}) \in \mathfrak{Aut}_{Bun(G)}(\lambda)$ and $z \in E'/R'$ by

$$\tilde{L}'(\varphi, \bar{\varphi}; r''(z)) := r''(L'(\varphi, \bar{\varphi}; z)) \ . \tag{E.102}$$

Note that since r'' is a bijection onto $B \times F$, (E.102) indeed defines a function: $\mathfrak{Aut}_{Bun(G)}(\lambda) \times B \times F \to B \times F$. Note also that by (E.102) we have for $(\varphi, \bar{\varphi}) \in \mathfrak{Aut}_{Bun(G)}(\lambda)$

$$\tilde{L}'(\varphi, \bar{\varphi}; \cdot) \circ r'' = r'' \circ L'(\varphi, \bar{\varphi}; \cdot) \ . \tag{E.103}$$

Since, as shown in Section E.3.1, L' is a left $\mathfrak{Aut}_{Bun(G)}(\lambda)$-action on E'/R' and r'' is a bijection onto $B \times F$, it follows from (E.103) that $\tilde{L}'$ is a left $\mathfrak{Aut}_{Bun(G)}(\lambda)$-action on

$B\times F$ and that, most importantly, the left $\mathfrak{Aut}_{Bun(G)}(\lambda)$-spaces $(E'/R', L')$, $(B\times F, \tilde{L}')$ are conjugate. I will now see that $\tilde{L}'$ has a very simple structure. It follows from (E.41),(E.84),(E.89), (E.102) that for $(\varphi, \bar{\varphi}) \in \mathfrak{Aut}_{Bun(G)}(\lambda)$ and $(b, g, y) \in E'$

$$\begin{aligned}&\tilde{L}'(\varphi, \bar{\varphi}; b, L(g; y)) = \tilde{L}'(\varphi, \bar{\varphi}; r'(b, g, y)) = \tilde{L}'(\varphi, \bar{\varphi}; r''(p_{R'}(b, g, y)))\\&= r''\Big(L'(\varphi, \bar{\varphi}; p_{R'}(b, g, y))\Big) = r''(p_{R'}(\varphi(b, g), y)) = r'(\varphi(b, g), y)\ ,\end{aligned}$$

whence for $(\varphi, \bar{\varphi}) \in \mathfrak{Aut}_{Bun(G)}(\lambda)$ and $(b, g, y) \in E'$

$$\tilde{L}'(\varphi, \bar{\varphi}; b, y) = r'(\varphi(b, g), L(g^{-1}; y))\ . \tag{E.104}$$

If $(\varphi, \bar{\varphi}) \in \mathfrak{Aut}_{Bun(G)}(\lambda)$ then, by (E.82), we have for $(b, g) \in E$

$$\varphi(b, g) = (\bar{\varphi}(b), f(b)g)\ , \tag{E.105}$$

where $f \in \mathcal{C}(B, G)$ is determined by φ via $f(b) := (r \circ \varphi)(b, e_G)$ with r being defined in Section E.6.1. By (E.84),(E.104),(E.105) we have for $(b, g, y) \in E'$

$$\begin{aligned}&\tilde{L}'(\varphi, \bar{\varphi}; b, y) = r'(\bar{\varphi}(b), f(b)g, L(g^{-1}; y)) = (\bar{\varphi}(b), L(f(b)g; L(g^{-1}; y)))\\&= (\bar{\varphi}(b), L(f(b); y))\ ,\end{aligned} \tag{E.106}$$

which indeed is remarkably simple.

I now consider L''. I define the function $r''' : \Gamma(\xi') \to \Gamma(\xi'')$ for $\sigma \in \Gamma(\xi')$ by

$$r'''(\sigma) := r'' \circ \sigma\ . \tag{E.107}$$

Clearly $r'''(\sigma) \in \mathcal{C}(B, B \times F)$ and by (E.101),(E.107) and Definition C.1 we have

$$p'' \circ r'''(\sigma) = p'' \circ r'' \circ \sigma = p' \circ \sigma = id_B\ , \tag{E.108}$$

so that indeed r''' is a function: $\Gamma(\xi') \to \Gamma(\xi'')$. If $\tilde{\sigma} \in \Gamma(\xi'')$ then, since $r'' \in HOMEO(E'/R', B \times F)$, we have $(r''^{-1} \circ \tilde{\sigma}) \in \mathcal{C}(B, E'/R')$ whence by (E.101)

$$p' \circ r''^{-1} \circ \tilde{\sigma} = p'' \circ \tilde{\sigma} = id_B\ , \tag{E.109}$$

which entails $(r''^{-1} \circ \tilde{\sigma}) \in \Gamma(\xi')$ so that, since $r'''(r''^{-1} \circ \tilde{\sigma}) = r'' \circ r''^{-1} \circ \tilde{\sigma} = \tilde{\sigma}$, I conclude that r''' is onto $\Gamma(\xi'')$. Furthermore it is clear by (E.107) that r''' is one-one whence r''' is a bijection onto $\Gamma(\xi'')$. I define the function $\tilde{L}'' : \mathfrak{Aut}_{Bun(G)}(\lambda) \times \Gamma(\xi'') \to \Gamma(\xi'')$ for $(\varphi, \bar{\varphi}) \in \mathfrak{Aut}_{Bun(G)}(\lambda)$ and $\sigma \in \Gamma(\xi')$ by

$$\tilde{L}''(\varphi, \bar{\varphi}; r'''(\sigma)) := r'''(L''(\varphi, \bar{\varphi}; \sigma)) \ . \tag{E.110}$$

Note that since r''' is a bijection onto $\Gamma(\xi'')$, (E.110) indeed defines a function: $\mathfrak{Aut}_{Bun(G)}(\lambda) \times \Gamma(\xi'') \to \Gamma(\xi'')$. Note that by (E.110)

$$\tilde{L}''(\varphi, \bar{\varphi}; \cdot) \circ r''' = r''' \circ L''(\varphi, \bar{\varphi}; \cdot) \ . \tag{E.111}$$

Since, as shown in Section E.3.2, L'' is a left $\mathfrak{Aut}_{Bun(G)}(\lambda)$-action on $\Gamma(\xi')$ and r''' is a bijection onto $\Gamma(\xi'')$, it follows from (E.111) that $\tilde{L}''$ is a left $\mathfrak{Aut}_{Bun(G)}(\lambda)$-action on $\Gamma(\xi'')$ and, most importantly, that the left $\mathfrak{Aut}_{Bun(G)}(\lambda)$-spaces $(\Gamma(\xi'), L'')$, $(\Gamma(\xi''), \tilde{L}'')$ are conjugate. I will now see that $\tilde{L}''$ has a very simple structure. I compute for $(\varphi, \bar{\varphi}) \in \mathfrak{Aut}_{Bun(G)}(\lambda)$ and $\sigma \in \Gamma(\xi'')$ by (E.45),(E.103),(E.107),(E.110)

$$\begin{aligned} \tilde{L}''(\varphi, \bar{\varphi}; \sigma) &= r'''\Big(L''(\varphi, \bar{\varphi}; r'''^{-1}(\sigma))\Big) = r'''\Big(L'(\varphi, \bar{\varphi}; \cdot) \circ r'''^{-1}(\sigma) \circ \bar{\varphi}^{-1}\Big) \\ &= r'' \circ L'(\varphi, \bar{\varphi}; \cdot) \circ r'''^{-1}(\sigma) \circ \bar{\varphi}^{-1} = \tilde{L}'(\varphi, \bar{\varphi}; \cdot) \circ r'' \circ r'''^{-1}(\sigma) \circ \bar{\varphi}^{-1} \\ &= \tilde{L}'(\varphi, \bar{\varphi}; \cdot) \circ r'''(r'''^{-1}(\sigma)) \circ \bar{\varphi}^{-1} = \tilde{L}'(\varphi, \bar{\varphi}; \cdot) \circ \sigma \circ \bar{\varphi}^{-1} \ , \end{aligned} \tag{E.112}$$

whence for $(\varphi, \bar{\varphi}) \in \mathfrak{Aut}_{Bun(G)}(\lambda)$ and $\sigma \in \Gamma(\xi''), b \in B$

$$\Big(\tilde{L}''(\varphi, \bar{\varphi}; \sigma)\Big)(b) = \tilde{L}'(\varphi, \bar{\varphi}; \sigma(\bar{\varphi}^{-1}(b))) \ . \tag{E.113}$$

Recalling Definition C.1 we have for $\sigma \in \Gamma(\xi'')$ that $p'' \circ \sigma = id_B$ whence for $b \in B$ we have $\sigma(b) = (b, \hat{\sigma}(b))$ where $\hat{\sigma}$ can be any elemment of $\mathcal{C}(B, F)$. I thus obtain from (E.106),(E.113) for $(\varphi, \bar{\varphi}) \in \mathfrak{Aut}_{Bun(G)}(\lambda)$ and $\sigma \in \Gamma(\xi''), b \in B$ that

$$\begin{aligned} \Big(\tilde{L}''(\varphi, \bar{\varphi}; \sigma)\Big)(b) &= \tilde{L}'(\varphi, \bar{\varphi}; \sigma(\bar{\varphi}^{-1}(b))) = \tilde{L}'\Big(\varphi, \bar{\varphi}; \bar{\varphi}^{-1}(b), \hat{\sigma}(\bar{\varphi}^{-1}(b))\Big) \\ &= \Big(b, L(f(\bar{\varphi}^{-1}(b)); \hat{\sigma}(\bar{\varphi}^{-1}(b)))\Big) \ , \end{aligned} \tag{E.114}$$

where φ is given by (E.105) with $f \in \mathcal{C}(B, G)$ being determined by φ via $f(b) := (r \circ \varphi)(b, e_G)$. Eq. (E.114) is indeed remarkably simple. Formulas (E.106),(E.114) are important in Section 9.3 where they provide the link between spin-orbit tori and a product principal $SO(3)$-bundle.

E.6.4 Correspondence between cross sections and pseudo cross sections of an associated bundle

Since the motto of Section E.6 is to reconsider Sections E.1-E.5 in the case when λ is the product principal G-bundle, defined by (E.67),(E.68),(E.69), I now reconsider Section E.2.2, i.e., I reconsider the correspondence $\gamma = \gamma_{\lambda,F,L}$ between $\Gamma(\lambda[F, L])$ and $\tilde{\Gamma}_{\lambda,F,L}$. In fact I here show that, in the present case, $\gamma_{\lambda,F,L}$ is a bijection from $\tilde{\Gamma}_{\lambda,F,L}$ onto $\Gamma(\lambda[F, L])$. The bijection property of $\gamma_{\lambda,F,L}$ becomes very important in the context of H-reductions (see Section E.6.6).

Recall that we already know from Section E.2.2 that γ is one-one. Since r''', defined in (E.107), is a bijection from $\Gamma(\lambda[F, L])$ onto $\Gamma(\xi'')$ I am done if I show that the function $\tilde{\gamma} : \tilde{\Gamma}_{\lambda,F,L} \to \Gamma(\xi'')$, defined by

$$\tilde{\gamma}_{\lambda,F,L} = \tilde{\gamma} := r''' \circ \gamma \ , \tag{E.115}$$

is a bijection onto $\Gamma(\xi'')$. I first observe from (E.107),(E.115) that for $\psi \in \tilde{\Gamma}_{\lambda,F,L}$

$$\tilde{\sigma} = (r''' \circ \gamma)(\psi) = r'' \circ \sigma \ , \tag{E.116}$$

where $\sigma \in \Gamma(\lambda[F, L])$ and $\tilde{\sigma} \in \Gamma(\xi'')$ are defined by

$$\sigma := \gamma(\psi) \ , \qquad \tilde{\sigma} := \tilde{\gamma}(\psi) \ . \tag{E.117}$$

Of course by (E.25),(E.68) we have, for $b \in B, g \in G$,

$$\sigma(b) = \sigma(p(b, g)) = p_{R'}(b, g, \psi(b, g)) \ . \tag{E.118}$$

It follows from (E.89),(E.116),(E.118) that, for $b \in B, g \in G$,

$$\tilde{\sigma}(b) = (r'' \circ \sigma)(b) = (r'' \circ p_{R'})(b, g, \psi(b, g)) = r'(b, g, \psi(b, g)) \,. \quad \text{(E.119)}$$

On the other hand, recalling Section E.2.2, $\tilde{\Gamma}_{\lambda,F,L}$ consists of those functions ψ in $\mathcal{C}(B \times G, F)$ which satisfy, for $g, g' \in G, b \in B$,

$$\psi(b, gg') = \psi(R(g'; b, g)) = L(g'^{-1}; \psi(b, g)) \,, \quad \text{(E.120)}$$

where in the first equality I used (E.69). Thus $\tilde{\Gamma}_{\lambda,F,L}$ consists of those functions ψ in $\mathcal{C}(E, F)$ which satisfy, for $g \in G, b \in B$,

$$\psi(b, g) = L(g^{-1}; \psi(b, e_G)) \,. \quad \text{(E.121)}$$

In other words, $\tilde{\Gamma}_{\lambda,F,L}$ consists of those functions $\psi : B \times G \to F$ which read for $g \in G, b \in B$ as

$$\psi(b, g) = L(g^{-1}; f(b)) \,, \quad \text{(E.122)}$$

where f is an arbitrary function in $\mathcal{C}(B, F)$. I thus define the function $\hat{\gamma}_{\lambda,F,L} : \mathcal{C}(B, F) \to \tilde{\Gamma}_{\lambda,F,L}$ for $b \in B, g \in G, f \in \mathcal{C}(B, F)$ by

$$\hat{\gamma}_{\lambda,F,L}(f) := \psi \,, \qquad \psi(b, g) := L(g^{-1}; f(b)) \,. \quad \text{(E.123)}$$

Clearly $\hat{\gamma}_{\lambda,F,L}$ is a bijection onto $\tilde{\Gamma}_{\lambda,F,L}$. One also sees by (E.122) that the pseudo cross sections have, in the present case, a remarkably simple structure. Returning to the computation of $\tilde{\sigma}$ I conclude from (E.84),(E.117), (E.119),(E.121) for $g \in G, b \in B$

$$\begin{aligned}(\tilde{\gamma}(\psi))(b) &= \tilde{\sigma}(b) = r'(b, g, \psi(b, g)) = (b, L(g; \psi(b, g)) = (b, L(g; L(g^{-1}; \psi(b, e_G)))) \\ &= (b, \psi(b, e_G)) \,. \end{aligned} \quad \text{(E.124)}$$

Since γ, r''' are one-one we observe by (E.115) that $\tilde{\gamma}$ is one-one. To show that $\tilde{\gamma}$ is onto $\Gamma(\xi'')$ let $\sigma' \in \Gamma(\xi'')$. Thus by the remarks after (E.113) we have for $b \in B$ that $\sigma'(b) = (b, f'(b))$ where $f' \in \mathcal{C}(B, F)$. To show that σ' belongs to the image of $\tilde{\gamma}$ I define the function $\psi' \in \mathcal{C}(B \times G, F)$ for $g \in G, b \in B$ by $\psi'(b,g) := L(g^{-1}; f'(b))$. It follows from the remarks after (E.121) that $\psi' \in \tilde{\Gamma}_{\lambda,F,L}$ and from (E.122),(E.124) that for $b \in B$

$$(\tilde{\gamma}(\psi'))(b) = (b, \psi'(b, e_G)) = (b, L(e_G; f'(b)) = (b, f'(b)) = \sigma'(b) \ , \quad \text{(E.125)}$$

whence $\tilde{\gamma}$ is onto $\Gamma(\xi'')$ which completes the proof that $\tilde{\gamma}$ is a bijection onto $\Gamma(\xi'')$. Clearly $\tilde{\gamma}_{\lambda,F,L} \circ \hat{\gamma}_{\lambda,F,L}$ is a bijection onto $\Gamma(\xi'')$.

E.6.5 Group homomorphisms into the automorphism group of a principal G-bundle

Since the motto of Section E.6 is to reconsider Sections E.1-E.5 in the case when λ is the product principal G-bundle, defined by (E.67),(E.68),(E.69), I now reconsider Section E.4.

If K is a group and $\Phi \in HOM_K(\lambda)$ then by (E.51),(E.82) I can write for $k \in K, b \in B, g \in G$

$$(\Phi(k))(b,g) = (\varphi(k;b,g), \bar{\varphi}(k;b)) \ , \quad \text{(E.126)}$$

where $(\varphi(k;\cdot), \bar{\varphi}(k;\cdot)) \in \mathfrak{Aut}_{Bun(G)}(\lambda)$, i.e., $\bar{\varphi}(k,\cdot) \in HOMEO(B,B)$ and

$$\varphi(k;b,g) = (\bar{\varphi}(k;b), \hat{\varphi}(k;b)g) \ , \quad \text{(E.127)}$$

with $\hat{\varphi}(k;\cdot) \in \mathcal{C}(B,G)$ being uniquely determined by φ via $\varphi(k;\cdot,e_G) = (\bar{\varphi}(k;\cdot), \hat{\varphi}(k;\cdot))$. If K is a topological group then the product principal G-bundle λ provides me with a correspondence between G-cocycles and group homomorphisms from K into $\mathfrak{Aut}_{Bun(G)}(\lambda)$ (recall the definition of cocycles in Appendix B). More precisely, this correspondence is established by the function $\rho_{B,K,G} : COC(B,K,G) \to HOM_K(\lambda)$ which is defined for $(l, \hat{\varphi}) \in COC(B,K,G)$ by

$$\rho_{B,K,G}(l, \hat{\varphi}) := \Phi \ , \quad \text{(E.128)}$$

where $\Phi(k)$ is given, for $k \in K$, by (E.126),(E.127) with

$$\bar{\varphi} := l \; . \tag{E.129}$$

To show that Φ, as defined by (E.126),(E.129), is in $HOM_K(\lambda)$ I first note by (E.129) that $\bar{\varphi}(k;\cdot) \in HOMEO(B,B)$ whence, since $\hat{\varphi}(k;\cdot) \in \mathcal{C}(B,G)$ I obtain from (E.82),(E.126) that $\Phi(k) \in \mathfrak{Aut}_{Bun(G)}(\lambda)$. Moreover it follows from (B.8),(E.127), (E.129) that for $k,k' \in K, b \in B, g \in G$,

$$\bar{\varphi}(k'k;b) = l(k'k;b) = l(k';l(k;b)) = \bar{\varphi}(k';\bar{\varphi}(k;b)) = (\bar{\varphi}(k';\cdot)\circ\bar{\varphi}(k;\cdot))(b) \; , \tag{E.130}$$

$$\begin{aligned} \varphi(k'k;b,g) &= (\bar{\varphi}(k'k;b), \hat{\varphi}(k'k;b)g) = (\bar{\varphi}(k';\bar{\varphi}(k;b)), \hat{\varphi}(k';l(k;b))\hat{\varphi}(k;b)g) \\ &= (\bar{\varphi}(k';\bar{\varphi}(k;b)), \hat{\varphi}(k';\bar{\varphi}(k;b))\hat{\varphi}(k;b)g) = \varphi(k';\bar{\varphi}(k;b), \hat{\varphi}(k;b)g) \\ &= \varphi(k';\varphi(k;b,g)) = (\varphi(k';\cdot)\circ\varphi(k;\cdot))(b,g) \; , \end{aligned} \tag{E.131}$$

where I also used the fact that l is a left K-action. It follows from (E.126),(E.130) and the composition law of $Bun(G)$

$$\begin{aligned} \Phi(k'k) &= (\varphi(k'k;\cdot), \bar{\varphi}(k'k;\cdot)) = (\varphi(k';\cdot)\circ\varphi(k;\cdot), \bar{\varphi}(k';\cdot)\circ\bar{\varphi}(k;\cdot)) \\ &= (\varphi(k';\cdot), \bar{\varphi}(k';\cdot))(\varphi(k;\cdot), \bar{\varphi}(k;\cdot)) = \Phi(k')\Phi(k) \; , \end{aligned} \tag{E.132}$$

which completes the proof that $\Phi \in HOM_K(\lambda)$. Thus indeed $\rho_{B,K,G}$ is a function: $COC(B,K,G) \to HOM_K(\lambda)$. To show that $\rho_{B,K,G}$ is one-one let $(l',\hat{\varphi}') \in COC(B,K,G)$ such that

$$\rho_{B,K,G}(l,\hat{\varphi}) = \rho_{B,K,G}(l',\hat{\varphi}') \; . \tag{E.133}$$

Clearly

$$\rho_{B,K,G}(l',\hat{\varphi}') = \Phi' \; , \tag{E.134}$$

where $\Phi'(k)$ is given, for $k \in K, b \in B, g \in G$, by

$$(\Phi'(k))(b,g) = (\varphi'(k;b,g), l'(k;g)) \; , \tag{E.135}$$

with $\bar{\varphi}'(k,\cdot) \in HOMEO(B,B)$ and

$$\varphi'(k;b,g) = (l'(k;b), \hat{\varphi}'(k;b)g) \ . \tag{E.136}$$

Since $\Phi = \Phi'$ we have, by (E.126),(E.135), that $\varphi = \varphi'$ whence, by (E.127),(E.129), (E.136), $l = l'$ and $\hat{\varphi} = \hat{\varphi}'$ so that $\rho_{B,K,G}$ is one-one. The function $\rho_{B,K,G}$ thus allows to store information about G-cocycles, in a 'lossless' way, in the automorphism group of the product principal G-bundle λ. I will apply this technique in Section 9.3 to spin-orbit tori (in that case, $(B,K,G) = (\mathbb{R}^d, \mathbb{Z}, SO(3))$). The following remark puts $\rho_{B,K,G}$ into perspective.

Remark:

(1) I define $\sigma \in \Gamma(B \times G, p, B)$ for $b \in B$ by $\sigma(b) := (b, e_G)$. Let $(l,f) \in COC(B,K,G)$ and let $\rho_{B,K,G}(l,f) =: \Phi$. Using the notation of (E.126) I obtain $\bar{\varphi} = l, \hat{\varphi} = f$ and from (E.69), (E.127) that, for $k \in K, b \in B$,

$$\begin{aligned}\varphi(k;\sigma(b)) = \varphi(k;b,e_G) = (l(k;b), f(k;b)) = R(f(k;b); l(k;b), e_G) \\ = R(f(k;b); \sigma(l(k;b))) \ ,\end{aligned}$$

i.e.,

$$\varphi(k;\sigma(b)) = R(f(k;b); \sigma(l(k;b))) \ . \tag{E.137}$$

One can easily show that (E.126),(E.129), (E.137) fix Φ for every (l,f) in $COC(B,K,G)$. In other words, the injection $\rho_{B,K,G}$ is induced by the cross section σ. The point to be made here is that one can even show that for *every* $\sigma \in \Gamma(B \times G, p, B)$, an injection from $COC(B,K,G)$ into $HOM_K(\lambda)$ is induced by σ via (E.137). □

If the topological group K is discrete (e.g., if $K = \mathbb{Z}$) then one has the stronger result that $\rho_{B,K,G}$ is a bijection onto $HOM_K(\lambda)$. To prove this, let K be discrete and $\Phi \in$

$HOM_K(\lambda)$ so I am looking for a $(l,f) \in COC(B,K,G)$ such that $\rho_{B,K,G}(l,f) = \Phi$. Since Φ is a group homomorphism, we have

$$\Phi(e_K) = (id_{B\times G}, id_B) \ , \tag{E.138}$$

and, for $k, k' \in K$, by using the notation of (E.126)

$$\begin{aligned}(\varphi(k';\cdot)\circ\varphi(k;\cdot), \bar{\varphi}(k';\cdot)\circ\bar{\varphi}(k;\cdot)) &= (\varphi(k';\cdot), \bar{\varphi}(k';\cdot))(\varphi(k;\cdot), \bar{\varphi}(k;\cdot)) \\ = \Phi(k')\Phi(k) = \Phi(k'k) &= (\varphi(k'k;\cdot), \bar{\varphi}(k'k;\cdot)) \ . \end{aligned} \tag{E.139}$$

Defining l by (E.129), one observes by (E.126),(E.129), (E.138),(E.139) that, for $k, k' \in K, b \in B$,

$$l(e_K; b) = \bar{\varphi}(e_K; b) = b \ , \qquad l(k'; l(k;b)) = \bar{\varphi}(k'; \bar{\varphi}(k;b)) = \bar{\varphi}(k'k; b) = l(k'k; b) \ ,$$

whence (B, l) is a left K-space. Moreover since $\bar{\varphi}(k;\cdot) \in \mathcal{C}(B,B)$ and since K is discrete, we have $\bar{\varphi} \in \mathcal{C}(K\times B, B)$ whence, by (E.129), $l \in \mathcal{C}(K\times B, B)$ so that (B,l) is a topological left K-space. Using the notation of (E.127), where $\hat{\varphi}(k;\cdot) \in \mathcal{C}(B,G)$ is uniquely determined by φ via $\varphi(k;\cdot,e_G) = (\bar{\varphi}(k;\cdot), \hat{\varphi}(k;\cdot))$, I define f by

$$f := \hat{\varphi} \ . \tag{E.140}$$

Since K is discrete and $f(k;\cdot) \in \mathcal{C}(B,G)$ I conclude that $f \in \mathcal{C}(K\times B, G)$. To show that f is a G-cocycle over (B,l), I conclude from (E.127),(E.129), (E.139),(E.140) that, for $k, k' \in K, b \in B, g \in G$,

$$\begin{aligned}(l(k'k;b), f(k'k;b)g) &= (\bar{\varphi}(k'k;b), \hat{\varphi}(k'k;b)g) = \varphi(k'k;b,g) = \varphi(k';\varphi(k;b,g)) \\ &= \varphi(k';\bar{\varphi}(k;b), \hat{\varphi}(k;b)g) = (\bar{\varphi}(k';\bar{\varphi}(k;b)), \hat{\varphi}(k';\bar{\varphi}(k;b))\hat{\varphi}(k;b)g) \\ &= (l(k';l(k;b)), f(k';l(k;b))f(k;b)g) \ ,\end{aligned}$$

whence $f(k'k;b) = f(k';l(k;b))f(k;b)$, which completes the proof that f is a G-cocycle over (B,l). Thus $\rho_{B,K,G}(l,f)$ is well defined and I obtain from (E.126),(E.127), (E.128),(E.129),(E.140) that $\rho_{B,K,G}(l,f) = \Phi$ which completes the proof that $\rho_{B,K,G}$

is onto $HOM_K(\lambda)$. Since $\rho_{B,K,G}$ is one-one, this completes the proof that $\rho_{B,K,G}$ is a bijection onto $HOM_K(\lambda)$, if K is discrete.

Due to (E.12), (E.82), the gauge group of λ has the simple form:

$$\mathfrak{Gau}_{Bun(G)}(\lambda) = \{\varphi \in \mathcal{C}(E,E) : [(\forall\, b \in B, g \in G)\varphi(b,g) = (b, f(b)g)], f \in \mathcal{C}(B,G)\} \ . \tag{E.141}$$

Let $\varphi' \in \mathfrak{Gau}_{Bun(G)}(\lambda)$, i.e., by (E.12), $\Phi' := (\varphi', id_B)$ is in $\mathfrak{Aut}_{Bun(G)}(\lambda)$ whence, by (E.141), we have, for $b \in B, g \in G$,

$$\varphi'(b,g) = (b, f'(b)g) \ , \tag{E.142}$$

where $f' \in \mathcal{C}(B,G)$. Note by (E.142) that, for $b \in B, g \in G$, the inverse of φ' in $\mathfrak{Gau}_{Bun(G)}(\lambda)$ satisfies, for $b \in B, g \in G$,

$$\varphi'^{-1}(b,g) = (b, (f'(b))^{-1}g) \ . \tag{E.143}$$

Let $\Phi \in HOM_K(\lambda)$ and let me define $\Phi'' \in HOM_K(\lambda)$ for $k \in K$ by

$$\begin{aligned}\Phi''(k) &:= \Phi'^{-1}\Phi(k)\Phi' = (\varphi', id_B)^{-1}\Phi(k)(\varphi', id_B)\\ &= (\varphi'^{-1} \circ \varphi(k;\cdot) \circ \varphi', \bar{\varphi}(k;\cdot)) \ .\end{aligned} \tag{E.144}$$

where I also used the notation of (E.126). I conclude from (E.127),(E.142),(E.143) that for $b \in B, g \in G, k \in K$

$$\begin{aligned}&(\varphi'^{-1} \circ \varphi(k;\cdot) \circ \varphi')(b,g) = (\varphi'^{-1} \circ \varphi(k;\cdot))(b, f'(b)g) = \varphi'^{-1}(\varphi(k;b,f'(b)g))\\ &= \varphi'^{-1}(\bar{\varphi}(k;b), f(k,b)f'(b)g) = (\bar{\varphi}(k;b), (f'(\bar{\varphi}(k;b)))^{-1}f(k,b)f'(b)g) \ ,\end{aligned} \tag{E.145}$$

whence by (E.144)

$$(\Phi''(k))(b,g) = \Big(\bar{\varphi}(k;b), (f'(\bar{\varphi}(k;b)))^{-1}f(k,b)f'(b)g, \bar{\varphi}(k;b)\Big) \ . \tag{E.146}$$

E.6.6 Reducing the structure group G

Since the motto of Section E.6 is to reconsider Sections E.1-E.5 in the case when λ is the product principal G-bundle, defined by (E.67),(E.68),(E.69), I now reconsider Section E.5, i.e., I study the H-reductions of λ. As in Section E.5, H is assumed to be a closed subgroup of G. In addition I here assume that G is compact since it will allow me to prove, rather easily, Theorem E.3 which completely characterizes the H-reductions of λ in terms of the cross sections of the associated bundle $\lambda[G/H, L_{G/H}]$.

I now outline how I proceed in this section. To be able to state Theorem E.3 I will construct, after stating and proving Lemma E.2, the functions $\widetilde{MAIN}_{\lambda,H}$, $MAIN_{\lambda,H}$, $\widehat{MAIN}_{\lambda,H}$ into $RED_H(\lambda)$. The theorem is followed by Corollary E.4 which states a special case of Zimmer's reduction theorem [Fe].

I first need:

Lemma E.2 *Let G be a compact topological group and let (X, R) be a topological right G-space. Let also $Y \subset X$ and A be a closed subset of X. Abbreviating $GY := \{R(g; x) : g \in G, x \in Y\}$ and $\mathcal{O} := (G \times X) \setminus R^{-1}(A)$, the following hold.*

a) $Y \cap GA = \emptyset \Leftrightarrow G \times Y \cap R^{-1}(A) = \emptyset$.

b) $\mathcal{O}$ is open in $G \times X$ and, if $x \in X \setminus GA$, then $G \times \{x\} \subset \mathcal{O}$.

c) For every $g \in G$ and $x \in X \setminus GA$ there exists an open neighborhood $U_x(g)$ of g and an open neighborhood $V_g(x)$ of x such that $U_x(g) \times V_g(x) \subset \mathcal{O}$.

d) Let, for every $g \in G$ and $x \in X \setminus GA$, the open sets $U_x(g)$ and $V_g(x)$ as in Lemma E.2c. Then, for every $x \in X \setminus GA$, there exists a positive integer $n(x)$ and $g(1, x), ..., g(n(x), x) \in G$ such that

$$G = \bigcup_{i=1}^{n(x)} U_x(g(i, x)) \ . \tag{E.147}$$

Moreover if $x \in X \setminus GA$ *and* $g(1,x), ..., g(n(x),x) \in G$ *satisfy (E.147) then* $V(x) := \bigcap_{i=1}^{n(x)} V_{g(i,x)}(x)$ *is an open neighborhood of* x *with* $G \times V(x) \subset \mathcal{O}$ *and* $V(x) \subset X \setminus GA$.

e) GA *is a closed subset of* X *and* p_R *is a closed function.*

Remark: The idea of the proof of Lemma E.2 is taken from Sections 1.4 and 1.6 in [Ka].

Proof of Lemma E.2a: If $Y \cap GA \neq \emptyset$ then $a \in A, g \in G, y \in Y$ exist such that $y = R(g; a)$. Thus $a = R(g^{-1}; y)$ whence $(g^{-1}, y) \in R^{-1}(A)$ so that $G \times Y \cap R^{-1}(A) \neq \emptyset$. If $G \times Y \cap R^{-1}(A) \neq \emptyset$ then $a \in A, g \in G, y \in Y$ exist such that $a = R(g; y)$ whence $y = R(g^{-1}; a)$ so that $Y \cap GA \neq \emptyset$. I thus have shown that $Y \cap GA \neq \emptyset \Leftrightarrow G \times Y \cap R^{-1}(A) \neq \emptyset$. This implies the claim. □

Proof of Lemma E.2b: Since A is closed in X and R is continuous, $R^{-1}(A)$ is closed in $G \times X$ whence $\mathcal{O}$ is open in $G \times X$.

Let $x \in X \setminus GA$. Then setting $Y = \{x\}$ we have $Y \cap GA = \emptyset$ whence, by Lemma E.2a, $G \times Y \cap R^{-1}(A) = \emptyset$. Thus $G \times \{x\} \cap R^{-1}(A) = \emptyset$ whence $G \times \{x\} \subset \mathcal{O}$. □

Proof of Lemma E.2c: Let $g \in G$ and $x \in X \setminus GA$. Thus, by Lemma E.2b, $(g, x) \in \mathcal{O}$. Since, by Lemma E.2b, $\mathcal{O}$ is open w.r.t. the product topology on $G \times X$, $\mathcal{O}$ is the union of sets $U \times V$ where U is open in G and V is open in X. Since $(g, x) \in \mathcal{O}$, I conclude that there exists an open set $U_x(g)$ in G and an open set $V_g(x)$ in X such that $(g, x) \in U_x(g) \times V_g(x) \subset \mathcal{O}$. Clearly $U_x(g)$ is an open neighborhood of g and $V_g(x)$ is an open neighborhood of x. □

Proof of Lemma E.2d: Let $x \in X \setminus GA$. It follows from Lemma E.2c that $G = \bigcup_{g \in G} U_x(g)$ whence, since the $U_x(g)$ are open and G is compact, a positive integer $n(x)$ and $g(1,x), ..., g(n(x),x) \in G$ exist such that (E.147) holds. Since the $V_g(x)$ are open neighborhoods of x one obtains that $V(x) := \bigcap_{i=1}^{n(x)} V_{g(i,x)}(x)$ is an open neighborhood of x. Thus if $(g, x') \in G \times V(x)$ then, by (E.147), a positive integer $1 \leq$

$k(g,x') \leq n(x)$ exists such that $g \in U_x(g(k(g,x'),x))$ whence, since $V(x)$ contains x', we have $(g,x') \in U_x(g(k(g,x'),x)) \times V(x) \subset U_x(g(k(g,x'),x)) \times V_{g(k(g,x'),x)}(x)$. However, by Lemma E.2c, $U_x(g(k(g,x'),x)) \times V_{g(k(g,x'),x)}(x) \subset \mathcal{O}$ whence $(g,x') \in \mathcal{O}$ which proves that $G \times V(x) \subset \mathcal{O}$. Since $G \times V(x) \subset \mathcal{O}$ and $(G \times X) \setminus \mathcal{O} = R^{-1}(A)$, it follows that $G \times V(x) \cap R^{-1}(A) = \emptyset$. Setting $Y = V(x)$ in Lemma E.2a, one obtains $V(x) \cap GA = \emptyset$ whence $V(x) \subset X \setminus GA$. □

Proof of Lemma E.2e: It follows from Lemma E.2d that $X \setminus GA = \bigcup_{x \in X \setminus GA} V(x)$. Since, by Lemma E.2d, $V(x)$ is open, one obtains that $X \setminus GA$ is open whence GA is closed. On the other hand one concludes from (B.13) that

$$p_R^{-1}(p_R(A)) = \bigcup_{g \in G} \bigcup_{x \in A} \{R(g;x)\} = GA \,. \tag{E.148}$$

Also we have $p_R^{-1}(X \setminus p_R(A)) = X \setminus (p_R^{-1}(p_R(A)))$ whence $p_R^{-1}(p_R(A)) = X \setminus (p_R^{-1}(X \setminus p_R(A)))$ so that, by (E.148),

$$GA = X \setminus (p_R^{-1}(X \setminus p_R(A))) \,. \tag{E.149}$$

Since GA is closed, it follows from (E.149) that $p_R^{-1}(X \setminus p_R(A))$ is open. Since $p_R^{-1}(X \setminus p_R(A))$ is open and p_R is identifying I obtain that $X \setminus p_R(A)$ is open whence $p_R(A)$ is closed. This proves that p_R is a closed function. □

I now begin my constructions and I first take a look at the pseudo cross sections associated with λ via $(G/H, L_{G/H})$ (recall that in this section G is compact!). In fact, due to (E.121), the set $\tilde{\Gamma}_{\lambda,G/H,L_{G/H}}$ consists of those functions $\psi : B \times G \to G/H$ which read for $g \in G, b \in B$ as

$$\psi(b,g) = L_{G/H}(g^{-1}; \psi(b,e_G)) \,, \tag{E.150}$$

where $\psi(\cdot, e_G)$ is an arbitrary function in $\mathcal{C}(B, G/H)$. Of course each ψ in $\tilde{\Gamma}_{\lambda,G/H,L_{G/H}}$ is continuous. Furthermore, since $L_{G/H}$ is transitive, each ψ in $\tilde{\Gamma}_{\lambda,G/H,L_{G/H}}$ is onto G/H. To construct the function $\widetilde{MAIN}_{\lambda,H} : \tilde{\Gamma}_{\lambda,G/H,L_{G/H}} \to RED_H(\lambda)$ let $\psi \in$

$\tilde{\Gamma}_{\lambda,G/H,L_{G/H}}$. I define the subspace $\hat{E}_\psi$ of $B \times G$ by

$$\hat{E}_\psi := \psi^{-1}(e_G H) \;, \tag{E.151}$$

whence by (E.62),(E.150)

$$\begin{aligned} \hat{E}_\psi &= \{(b,g) \in B \times G : \psi(b,g) = e_G H\} \\ &= \{(b,g) \in B \times G : L_{G/H}(g^{-1}; \psi(b, e_G)) = e_G H\} \\ &= \{(b,g) \in B \times G : \psi(b, e_G) = gH\} \;. \end{aligned} \tag{E.152}$$

Note that since H is closed in G I know from the remarks after (E.64) that the singleton $e_G H$ is closed in G/H whence, by (E.151), $\hat{E}_\psi$ is closed in $B \times G$. The aim now is to construct a H-reduction of λ whose total space is $\hat{E}_\psi$. It is clear by Section E.5 that, if such a H-reduction of λ exists at all, then it reads as

$$\hat{\lambda}_\psi = (\hat{E}_\psi, \hat{p}_\psi, B, \hat{R}_\psi) \;, \tag{E.153}$$

where

$$\hat{p}_\psi := p\Big|\hat{E}_\psi \;, \qquad \hat{R}_\psi := R\Big|(H \times \hat{E}_\psi) \;. \tag{E.154}$$

Since $\hat{E}_\psi$ is closed in $B \times G$, it is clear by Section E.5 and (E.153),(E.154) that, if $\hat{\lambda}_\psi$ is a principal H-bundle, then $\hat{\lambda}_\psi \in RED_H(\lambda)$. Thus my aim is to show that $\hat{\lambda}_\psi$ is a principal H-bundle and I first show that it is a H-prebundle. Clearly $\hat{p}_\psi$ is continuous. To show that $\hat{R}_\psi$ is a right H-action on $\hat{E}_\psi$, let $(b,g) \in \hat{E}_\psi$ whence, by (E.152), $\psi(b, e_G) = gH$. Picking a $h \in H$ and defining $g' := gh \in G$ I observe that $g'H = (gh)H = gH = \psi(b, e_G)$ whence, by (E.152), $(b, g') \in \hat{E}_\psi$. On the other hand I obtain from (E.69),(E.154) that $\hat{R}_\psi(h; b, g) = R(h; b, g) = (b, gh) = (b, g')$ whence $\hat{R}_\psi(h; b, g) \in \hat{E}_\psi$. I thus have shown that the image of $\hat{R}_\psi$ is a subset of $\hat{E}_\psi$ whence $\hat{R}_\psi$ is a right H-action on $\hat{E}_\psi$ (and $\hat{E}_\psi$ is the image of $\hat{R}_\psi$). Clearly $(\hat{E}_\psi, \hat{R}_\psi)$ is topological right H-space. Since λ is a principal H-bundle, p is a G-map from the right G-space $(B \times G, R)$ to the trivial right G-space over B, whence one concludes from (E.154) that $\hat{p}_\psi$ is a H-map from the right H-space $(\hat{E}_\psi, \hat{R}_\psi)$ to the trivial right H-space over B which entails that $\hat{\lambda}_\psi$ is a H-prebundle. I will now use Proposition E.1 to show that $\hat{\lambda}_\psi$ is a H-bundle. If $b \in B$ then, choosing $g \in G$ such that $\psi(b, e_G) = gH$, I obtain from (E.152) that $(b, g) \in \hat{E}_\psi$ and by (E.68),(E.154) that $\hat{p}_\psi(b, g) = p(b, g) = b$ whence $\hat{p}_\psi$ is onto B. To show that $\hat{R}_\psi$ is transitive on the fibres of $\hat{p}_\psi$ let $(b, g) \in \hat{E}_\psi$ and let $(b', g') \in \hat{p}_\psi^{-1}(\hat{p}_\psi(b, g))$. Thus $(b', g') \in \hat{E}_\psi$ and $\hat{p}_\psi(b', g') = \hat{p}_\psi(b, g)$ whence, by (E.152), $\psi(b, e_G) = gH, \psi(b', e_G) = g'H$ and, by (E.68),(E.154), $b' = p(b', g') = \hat{p}_\psi(b', g') = \hat{p}_\psi(b, g) = p(b, g) = b$ so that $gH = g'H$ which entails that a $h \in H$ exists such that $g' = gh$. Thus (E.69),(E.154) give me

$$(b', g') = (b, gh) = R(h; b, g) = \hat{R}_\psi(h; b, g) \ . \tag{E.155}$$

It follows from (E.155) that $(b', g') \in p_{\hat{R}_\psi}(b, g)$ whence I have shown that $\hat{p}_\psi^{-1}(\hat{p}_\psi(b, g)) \subset p_{\hat{R}_\psi}(b, g)$ so that, by Proposition E.1b, $\hat{R}_\psi$ is transitive on the fibres of $\hat{p}_\psi$ where I also use the fact that $\hat{\lambda}_\psi$ is a H-prebundle. To show that $\hat{p}_\psi$ is identifying I first note by Lemma E.2 that the function p_R is closed. On the other hand, since λ is a G-bundle, its prebundle function, π_λ, is a homeomorphism onto B whence π_λ is closed. Thus, by (E.11), p is the composition of closed functions which entails that p is closed. Since $\hat{E}_\psi$ is closed in $B \times G$ and p is closed it follows from (E.154) that $\hat{p}_\psi$ is closed whence (see [Hu, Section II.6]) $\hat{p}_\psi$ is identifying. I thus have completed the proof that $\hat{p}_\psi$ is onto B and identifying and that $\hat{R}_\psi$ is transitive on the fibres of $\hat{p}_\psi$. Thus, by Proposition E.1c, $\hat{\lambda}_\psi$ is a H-bundle. To finish the proof that $\hat{\lambda}_\psi$ is a principal H-bundle it remains to be shown that the topological right H-space $(\hat{E}_\psi, \hat{R}_\psi)$ is principal. Since R is free, it follows from (E.154) that $\hat{R}_\psi$ is free. To find the translation function of $\hat{R}_\psi$ I define, as suggested by Section E.6.1, the topological space $\hat{E}^*_\psi$ by

$$\begin{aligned} \hat{E}^*_\psi &:= \{(b, g, \hat{R}_\psi(h; b, g)) : (b, g) \in \hat{E}_\psi, h \in H\} \\ &= \{(b, g, R(h; b, g)) : (b, g) \in \hat{E}_\psi, h \in H\} \\ &= \{(b, g, b, gh) : (b, g) \in \hat{E}_\psi, h \in H\} \subset E^* \ , \end{aligned} \tag{E.156}$$

where I also used (E.69),(E.72), (E.154). I recall from Section E.6.1 that the continuous function $\tau_R : E^* \to G$, defined by (E.73), is the translation function of R, i.e., it satisfies for $(b, g, b, g') \in E^*$

$$R(\tau_R(b, g, b, g'); b, g) = (b, g') \ . \tag{E.157}$$

I define the function $\hat{\tau}_\psi : \hat{E}^*_\psi \to H$ by

$$\hat{\tau}_\psi := \tau_R \Big| \hat{E}^*_\psi \ , \tag{E.158}$$

i.e., for $(b, g, b, g') \in \hat{E}^*_\psi$ we have, by (E.73),

$$\hat{\tau}_\psi(b, g, b, g') = \tau_R(b, g, b, g') = g^{-1}g' \ . \tag{E.159}$$

Note that the image of $\hat{\tau}_\psi$ is a subset of H since, if $(b, g, b, g') \in \hat{E}^*_\psi$, then by (E.156) a $h \in H$ exists such that $g' = gh$ whence, by (E.159), $\hat{\tau}_\psi(b, g, b, g') = g^{-1}g' = g^{-1}gh = h \in H$. Thus $\hat{\tau}_\psi$ is indeed a function: $\hat{E}^*_\psi \to H$. Of course we have by (E.69), (E.154),(E.158) for $(b, g, b, g') \in \hat{E}^*_\psi$

$$\hat{R}_\psi(\hat{\tau}_\psi(b, g, b, g'); b, g) = \hat{R}_\psi(g^{-1}g'; b, g) = R(g^{-1}g'; b, g) = (b, g') \ , \tag{E.160}$$

whence $\hat{\tau}_\psi$ is the translation function of $\hat{R}_\psi$. Since λ is a principal bundle, the function τ_R is continuous so it follows from (E.158) that $\hat{\tau}_\psi$ is continuous which completes the proof that the right H-space $(\hat{E}_\psi, \hat{R}_\psi)$ is principal. This completes the proof that $\hat{\lambda}_\psi$ is a principal H-bundle.

Before I proceed it is worthwile to mention that the above argument, which proved that p is a closed function, can be immediately generalized to the following statement: If X is a topological space then the function $\tilde{p} \in \mathcal{C}(X \times G, X)$, defined for $x \in X, g \in G$ by $\tilde{p}(x, g) := x$, is a closed function.

From the remarks after (E.154) it is thus clear that $\hat{\lambda}_\psi \in RED_H(\lambda)$ whence I can define the function $\widetilde{MAIN}_{\lambda,H} : \tilde{\Gamma}_{\lambda,G/H,L_{G/H}} \to RED_H(\lambda)$ for $\psi \in \tilde{\Gamma}_{\lambda,G/H,L_{G/H}}$ by

$$\widetilde{MAIN}_{\lambda,H}(\psi) := \hat{\lambda}_\psi = (\hat{E}_\psi, \hat{p}_\psi, B, \hat{R}_\psi) \ , \tag{E.161}$$

where in the second equality I used (E.153). I also define the function $MAIN_{\lambda,H}$: $\Gamma(\lambda[G/H, L_{G/H}]) \to RED_H(\lambda)$ by

$$MAIN_{\lambda,H} := \widetilde{MAIN}_{\lambda,H} \circ \gamma^{-1}_{\lambda,G/H,L_{G/H}} \ . \tag{E.162}$$

As was shown in Section E.6.4, the function $\gamma_{\lambda,G/H,L_{G/H}} : \tilde{\Gamma}_{\lambda,G/H,L_{G/H}} \to \Gamma(\lambda[G/H, L_{G/H}])$ is a bijection onto $\Gamma(\lambda[G/H, L_{G/H}])$ whence $MAIN_{\lambda,H}$ is indeed a function: $\Gamma(\lambda[G/H, L_{G/H}]) \to RED_H(\lambda)$. Recalling Section E.6.4 I define the function $\widehat{MAIN}_{\lambda,H}$: $\mathcal{C}(B, G/H) \to RED_H(\lambda)$ by

$$\widehat{MAIN}_{\lambda,H} := \widetilde{MAIN}_{\lambda,H} \circ \hat{\gamma}_{\lambda,G/H,L_{G/H}} \ . \tag{E.163}$$

Note that by (E.123) and writing $\hat{\gamma}_{\lambda,G/H,L_{G/H}} = \hat{\gamma}$ we have, for $b \in B$ and $f \in \mathcal{C}(B, G/H)$ that $(\hat{\gamma}(f))(b, e_G) = f(b)$ whence, by (E.152), I obtain the convenient expression

$$\hat{E}_{\hat{\gamma}(f)} = \{(b, g) \in B \times G : f(b) = gH\} \ . \tag{E.164}$$

I can now formulate the theorem.

Theorem E.3 *Let G be a compact topological group and let H be a closed subgroup of G. Let λ be a product principal G-bundle in the notation of (E.67),(E.68),(E.69). Then the following hold:*

a) The function $\widetilde{MAIN}_{\lambda,H}$, defined by (E.161), is a bijection onto $RED_H(\lambda)$.

b) The function $MAIN_{\lambda,H}$, defined by (E.162), is a bijection onto $RED_H(\lambda)$.

c) The function $\widehat{MAIN}_{\lambda,H}$, defined by (E.163), is a bijection onto $RED_H(\lambda)$.

Proof of Theorem E.3a: To show that $\widetilde{MAIN}_{\lambda,H}$ is one-one, let $\psi, \psi' \in \tilde{\Gamma}_{\lambda,G/H,L_{G/H}}$ such that $\widetilde{MAIN}_{\lambda,H}(\psi) = \widetilde{MAIN}_{\lambda,H}(\psi')$. Thus, by (E.161), $\hat{E}_\psi = \hat{E}_{\psi'}$ so that, by (E.152),

$$\begin{aligned} &\{(b, g) \in B \times G : \psi(b, e_G) = gH\} = \hat{E}_\psi = \hat{E}_{\psi'} \\ &= \{(b, g) \in B \times G : \psi'(b, e_G) = gH\} \ . \end{aligned} \tag{E.165}$$

If $b \in B$ then I pick a $g \in G$ such that $\psi(b, e_G) = gH$ whence, by (E.165), $(b, g) \in \hat{E}_\psi = \hat{E}_{\psi'}$ and $\psi'(b, e_G) = gH$. I conclude that $\psi(\cdot, e_G) = \psi'(\cdot, e_G)$ whence, by (E.150), for $b \in B, g \in G$,

$$\psi(b, g) = L_{G/H}(g^{-1}; \psi(b, e_G)) = L_{G/H}(g^{-1}; \psi'(b, e_G)) = \psi'(b, g) \ , \quad \text{(E.166)}$$

so that $\psi = \psi'$ which completes the proof that $\widetilde{MAIN}_{\lambda,H}$ is one-one.

To show that $\widetilde{MAIN}_{\lambda,H}$ is onto $RED_H(\lambda)$, let $\hat{\lambda}$ be a H-reduction of λ, i.e., let $\hat{\lambda} \in RED_H(\lambda)$ so I am looking for a $\psi \in \tilde{\Gamma}_{\lambda,G/H,L_{G/H}}$ such that $\widetilde{MAIN}_{\lambda,H}(\psi) = \hat{\lambda}$. Using the notation of (E.53) I write $\hat{\lambda} = (\hat{E}, \hat{p}, B, \hat{R})$ and I define

$$\begin{aligned}\hat{E}' &:= \{(b, g, b', g') \in B \times G \times \hat{E} : p(b, g) = \hat{p}(b', g')\} \\ &= \{(b, g, b', g') \in B \times G \times \hat{E} : b = b'\} = \{(b, g, b, g') : g \in G, (b, g') \in \hat{E}\} \\ &= \{(b, g, b, g') \in E^* : (b, g') \in \hat{E}\} \subset E^* \ , \end{aligned} \quad \text{(E.167)}$$

where I also used (E.56), (E.68),(E.72). To construct ψ I first have to define the auxiliary functions f_1 and f_2. I define the function $f_1 : \hat{E}' \to G$ by $f_1 := \tau_R\Big|\hat{E}'$, i.e., for $(b, g, b, g') \in \hat{E}'$ we have by (E.73)

$$f_1(b, g, b, g') = g^{-1}g' \ . \quad \text{(E.168)}$$

Since τ_R is continuous, so is f_1. I define the function $f_2 \in \mathcal{C}(\hat{E}', B \times G)$ for $(b, g, b, g') \in \hat{E}'$ by

$$f_2(b, g, b, g') := (b, g) \ . \quad \text{(E.169)}$$

Since $\hat{\lambda}$ is a principal H-bundle one observes by Proposition E.1c that $\hat{p}$ is onto B whence, since $\hat{E} \subset B \times G$ and due to (E.56),(E.68), we have the fact that for every $b \in B$ a $g' \in G$ exists such that $(b, g') \in \hat{E}$. It thus follows from (E.167),(E.169) that f_2 is onto $B \times G$. I now define the function $\psi : B \times G \to G/H$ for $(b, g, b, g') \in \hat{E}'$ by

$$\psi(f_2(b, g, b, g')) := p_{R_{G/H}}(f_1(b, g, b, g')) = f_1(b, g, b, g')H = g^{-1}g'H \ , \quad \text{(E.170)}$$

where in the second equality I used (E.61) and in the third equality I used (E.168). Note that ψ is defined by (E.170) on the whole set $B \times G$ since f_2 is onto $B \times G$. To show that ψ is single valued one observes that if $(b, g, b, g'), (b'', g'', b'', g''') \in \hat{E}'$ with $f_2(b, g, b, g') = f_2(b'', g'', b'', g''')$ then, by (E.169), $b = b''$, $g'' = g$ whence by (E.170)

$$\psi(f_2(b'', g'', b'', g''')) = g''^{-1} g''' H = g^{-1} g''' H \ . \tag{E.171}$$

Since $\hat{\lambda}$ is a H-bundle, Proposition E.1c gives me the transitivity of $\hat{R}$ on the fibres of $\hat{p}$. Moreover since, by (E.167), $(b, g'), (b'', g''') \in \hat{E}$ and $b = b''$ one observes by (E.56),(E.68) that $(b, g'), (b'', g''')$ belong to the same fibre of $\hat{p}$. Thus a $h \in H$ exists such that $(b'', g''') = \hat{R}(h; b, g')$ which entails by (E.57),(E.69)

$$(b'', g''') = \hat{R}(h; b, g') = R(h; b, g') = (b, g'h) \ . \tag{E.172}$$

It follows from (E.170),(E.171),(E.172) that

$$\psi(f_2(b'', g'', b'', g''')) = g^{-1} g' h H = g^{-1} g' H = \psi(f_2(b, g, b, g')) \ , \tag{E.173}$$

whence indeed ψ is single valued.

Having got the function ψ my aim is to show that $\widetilde{MAIN}_{\lambda,H}(\psi) = \hat{\lambda}$ so I first have to show that ψ belongs to the domain, $\tilde{\Gamma}_{\lambda,G/H,L_{G/H}}$, of $\widetilde{MAIN}_{\lambda,H}$. Let $(b, g) \in B \times G$. I already showed earlier in this proof that I can pick a $g' \in G$ such that $(b, g') \in \hat{E}$ whence, by (E.167), $(b, e_G, b, g'), (b, g, b, g') \in \hat{E}'$. Thus one concludes from (E.169),(E.170) that

$$\psi(b, e_G) = \psi(f_2(b, e_G, b, g')) = g'H \ , \qquad \psi(b, g) = \psi(f_2(b, g, b, g')) = g^{-1} g' H \ ,$$

whence, by (E.62),

$$\psi(b, g) = g^{-1} g' H = L_{G/H}(g^{-1}; g'H) = L_{G/H}(g^{-1}; \psi(b, e_G)) \ , \tag{E.174}$$

so that (E.150) is satisfied. Thus, due to the remarks on (E.150), I will have established that $\psi \in \tilde{\Gamma}_{\lambda,G/H,L_{G/H}}$ if I can show that $\psi \in \mathcal{C}(B \times G, G/H)$. One observes by

(E.170) that

$$\psi \circ f_2 = p_{R_{G/H}} \circ f_1 \ . \tag{E.175}$$

I will show below that f_2 is identifying whence, since f_2 is onto $B \times G$ and $p_{R_{G/H}} \circ f_1$ is continuous, it follows from [Hu, Section II.6] and (E.175) that $\psi \in \mathcal{C}(B \times G, G/H)$. To show that f_2 is identifying I define the function $f_3 \in \mathcal{C}(\hat{E}', E^*)$ as the natural injection into E^* and the function $f_4 \in \mathcal{C}(E^*, B \times G)$ for $(b, g, b, g') \in E^*$ by $f_4(b, g, b, g') := (b, g)$. Note that, by (E.169), $f_2 = f_4 \circ f_3$. I will show below that f_3, f_4 are closed whence f_2 is closed which entails that f_2 is identifying [Hu, Section II.6]. To show that f_3 is closed I note by (E.167) that $\hat{E}' = E^* \cap (B \times G \times \hat{E})$. Since $\hat{\lambda}$ is a H-reduction of λ, $\hat{E}$ is closed in $B \times G$ whence $B \times G \times \hat{E}$ is closed in $B \times G \times B \times G$ so that $\hat{E}'$ is closed in E^*. Thus the natural injection f_3 is a closed function. To show that f_4 is closed I define $\tilde{E} := \{(b, g, b) : b \in B, g \in G\}$ and I define the function $f_5 \in \mathcal{C}(E^*, \tilde{E})$ for $b \in B, g, g' \in G$ by $f_5(b, g, b, g') := (b, g, b)$. I also define the function $f_6 \in \mathcal{C}(\tilde{E}, B \times G)$ for $b \in B, g \in G$ by $f_6(b, g, b) := (b, g)$. Clearly $f_4 = f_6 \circ f_5$. I will show below that f_5, f_6 are closed whence f_4 is closed. In fact, by (E.72), we have $E^* = \tilde{E} \times G$ whence $f_5 \in \mathcal{C}(\tilde{E} \times G, \tilde{E})$ so that, by a remark after (E.160), f_5 is a closed function. To show that f_6 is closed I define the function $f_7 \in \mathcal{C}(B \times G, \tilde{E})$ for $b \in B, g \in G$ by $f_7(b, g) := (b, g, b)$. Clearly $id_{\tilde{E}} = f_7 \circ f_6$ and $id_{B \times G} = f_6 \circ f_7$ whence f_7 is the inverse of f_6 so that $f_6 \in HOMEO(\tilde{E}, B \times G)$. Thus f_6 is closed which completes the proof that f_4 is closed. This completes the proof that f_2 is identifying which, in turn, completes the proof that ψ is continuous. This completes the proof that $\psi \in \tilde{\Gamma}_{\lambda, G/H, L_{G/H}}$. Thus $\widetilde{MAIN}_{\lambda,H}(\psi)$ is a well defined element of $RED_H(\lambda)$ so my remaining task is to show that $\widetilde{MAIN}_{\lambda,H}(\psi) = \hat{\lambda}$. It follows from (E.161) that $\hat{E}_\psi$ is the total space of $\widetilde{MAIN}_{\lambda,H}(\psi)$ whence one concludes from Section E.5 that if $\hat{E}_\psi$ is equal to the total space, $\hat{E}$, of $\hat{\lambda}$ then $\widetilde{MAIN}_{\lambda,H}(\psi) = \hat{\lambda}$. To show that

$\hat{E}_\psi = \hat{E}$ one concludes from (E.151),(E.167),(E.169), (E.170) that

$$\begin{aligned}\hat{E}_\psi &= \psi^{-1}(e_G H) = \{(b,g) \in B \times G : \psi(b,g) = e_G H\} \\ &= \{f_2(b,g,b,g') : (b,g,b,g') \in \hat{E}', \psi(f_2(b,g,b,g')) = e_G H\} \\ &= \{(b,g) \in B \times G : (b,g,b,g') \in \hat{E}', g^{-1}g'H = e_G H\} \\ &= \{(b,g) \in B \times G : (b,g,b,g') \in \hat{E}', g'H = gH\} \\ &= \{(b,g) \in B \times G : (b,g,b,g') \in B \times G \times \hat{E}, g'H = gH\} \\ &= \{(b,g) \in B \times G : (\exists g' \in G)(b,g') \in \hat{E}, g'H = gH\}\,, \end{aligned} \tag{E.176}$$

where I also used the fact that f_2 is onto $B \times G$. If $(b,g) \in \hat{E}$ then, trivially, $gH = gH$ whence, by (E.176), $(b,g) \in \hat{E}_\psi$ so that $\hat{E} \subset \hat{E}_\psi$. To show that $\hat{E} \supset \hat{E}_\psi$, let $(b,g) \in \hat{E}_\psi$ so that, by (E.176), a $g' \in G$ exists such that $(b,g') \in \hat{E}$ and $g'H = gH$. Thus a $h \in H$ exists such that $g = g'h$ whence, by (E.57),(E.69), $\hat{R}(h;b,g') = R(h;b,g') = (b,g'h) = (b,g)$ so that $(b,g) \in \hat{E}$ which completes the proof that $\hat{E} = \hat{E}_\psi$. This completes the proof that $\widetilde{MAIN}_{\lambda,H}(\psi) = \hat{\lambda}$ which in turn completes the proof that $\widetilde{MAIN}_{\lambda,H}$ is onto $RED_H(\lambda)$. This completes the proof that $\widetilde{MAIN}_{\lambda,H}$ is a bijection onto $RED_H(\lambda)$. □

Proof of Theorem E.3b: As mentioned after (E.162), the function $\gamma_{\lambda,G/H,L_{G/H}}$ is a bijection from $\tilde{\Gamma}_{\lambda,G/H,L_{G/H}}$ onto $\Gamma(\lambda[G/H, L_{G/H}])$. It thus follows from Theorem E.3a and (E.162) that $MAIN_{\lambda,H}$ is a bijection from $\Gamma(\lambda[G/H, L_{G/H}])$ onto $RED_H(\lambda)$. □

Proof of Theorem E.3c: As mentioned after (E.123), the function $\hat{\gamma}_{\lambda,G/H,L_{G/H}}$ is a bijection from $\mathcal{C}(B, G/H)$ onto $\tilde{\Gamma}_{\lambda,G/H,L_{G/H}}$. It thus follows from Theorem E.3a and (E.163) that $\widehat{MAIN}_{\lambda,H}$ is a bijection from $\mathcal{C}(B, G/H)$ onto $RED_H(\lambda)$. □

Note that the idea of the proof of Theorem E.3a is taken from the proof of Proposition 6.2.2 in [Fe].

I recall from Section E.5 that if $(\varphi, \bar{\varphi}) \in \mathfrak{Aut}_{Bun(G)}(\lambda)$ then a H-reduction, $\hat{\lambda}$, of λ is called 'invariant under $(\varphi, \bar{\varphi})$' if, in the notation of (E.53), $\hat{E}$ is invariant under

φ, i.e., $\varphi(\hat{E}) = \hat{E}$. I thus obtain the following immediate and important consequence of Theorem E.3.

Corollary E.4 *Let the conditions underlying Theorem E.3 be fulfilled, i.e., let G be a compact topological group, let H be a closed subgroup of G, and let λ be a product principal G-bundle in the notation of (E.67),(E.68),(E.69). Let $\hat{\lambda}$ be a H-reduction of λ and let $\psi \in \tilde{\Gamma}_{\lambda,G/H,L_{G/H}}$ be defined by $\psi := \overset{-1}{\widetilde{MAIN}}_{\lambda,H}(\hat{\lambda})$ and let me write, as in (E.161),*

$$\hat{\lambda} = \widetilde{MAIN}_{\lambda,H}(\psi) = (\hat{E}_\psi, \hat{p}_\psi, B, \hat{R}_\psi) \; , \tag{E.177}$$

where $\hat{E}_\psi, \hat{p}_\psi, \hat{R}_\psi$ are given by (E.151),(E.154). Let $(\varphi, \bar{\varphi}) \in \mathfrak{Aut}_{Bun(G)}(\lambda)$ and let me write φ as in (E.82), i.e., for $b \in B, g \in G$ I write

$$\varphi(b, g) = (\bar{\varphi}(b), f(b)g) \; , \tag{E.178}$$

where $f \in \mathcal{C}(B, G)$ is uniquely determined by φ via $\varphi(\cdot, e_G) = (\bar{\varphi}(\cdot), f(\cdot))$. Then the following hold.

a) $\hat{\lambda}$ is invariant under $(\varphi, \bar{\varphi})$ iff for every $b \in B$

$$\psi(\bar{\varphi}(b), e_G) = L_{G/H}(f(b); \psi(b, e_G)) \; . \tag{E.179}$$

b) Defining $\hat{f} \in \mathcal{C}(B, G/H)$ by

$$\hat{f} := \overset{-1}{\widehat{MAIN}}_{\lambda,H}(\hat{\lambda}) = \hat{\gamma}^{-1}_{\lambda,G/H,L_{G/H}}(\psi) \; , \tag{E.180}$$

we have that $\hat{\lambda}$ is invariant under $(\varphi, \bar{\varphi})$ iff for every $b \in B$

$$\hat{f}(\bar{\varphi}(b)) = L_{G/H}(f(b); \hat{f}(b)) \; . \tag{E.181}$$

Proof of Corollary E.4a: I first consider the case where $\hat{\lambda}$ is invariant under $(\varphi, \bar{\varphi})$, i.e., $\varphi(\hat{E}_\psi) = \hat{E}_\psi$. Let $b \in B$. I pick a $g \in G$ such that $\psi(b, e_G) = gH$ whence, by (E.152), $(b, g) \in \hat{E}_\psi$. I define $(b', g') \in B \times G$ by

$$(b', g') := \varphi(b, g) = (\bar{\varphi}(b), f(b)g) \; , \tag{E.182}$$

where in the second equality I used (E.178). Since $\varphi(\hat{E}_\psi) = \hat{E}_\psi$ we have $(b', g') \in \hat{E}_\psi$ whence, by (E.152), $\psi(b', e_G) = g'H$ so that, by (E.62),(E.182),

$$\psi(\bar{\varphi}(b), e_G) = \psi(b', e_G) = g'H = (f(b)g)H = L_{G/H}(f(b); gH)\ . \quad \text{(E.183)}$$

Since $\psi(b, e_G) = gH$ I conclude from (E.183) that (E.179) holds.

Conversely, let (E.179) hold for every $b \in B$ and let $(b, g) \in \hat{E}_\psi$. Thus, by (E.152), $\psi(b, e_G) = gH$. My aim is to show that $\varphi(b, g) \in \hat{E}_\psi$. With the notation of (E.182) I compute, by using (E.62),(E.179),

$$\psi(b', e_G) = \psi(\bar{\varphi}(b), e_G) = L_{G/H}(f(b); \psi(b, e_G)) = L_{G/H}(f(b); gH) = (f(b)g)H = g'H\ ,$$

whence $(b', g') \in \hat{E}_\psi$. I thus have shown that $\varphi(\hat{E}_\psi) \subset \hat{E}_\psi$ whence, since φ is a bijection onto $B \times G$, $\varphi(\hat{E}_\psi) = \hat{E}_\psi$ so that $\hat{\lambda}$ is invariant under $(\varphi, \bar{\varphi})$. □

Proof of Corollary E.4b: It follows from (E.180) that $\hat{\gamma}_{\lambda,G/H,L_{G/H}}(\hat{f}) = \psi$ whence, for $b \in B$, by (E.123), $\psi(b, e_G) = \hat{f}(b)$ so that (E.179) is equivalent to (E.181). □

Appendix F

Proofs

F.1 Proof of Proposition 6.4

Proof of Proposition 6.4: Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$ and let n be an integer.

I first consider the case $n = 0$. Since $\Psi_{\omega,A}(0;\phi) = I_{3\times 3}$ we have, by Definition C.14, that $Ind_{3,d}(\Psi_{\omega,A}(0;\cdot)) = (1, ..., 1)^T$ whence $(Ind_{3,d}(A))^0 = (1, ..., 1)^T = Ind_{3,d}(\Psi_{\omega,A}(0;\cdot))$ which proves the claim in the present case.

I now consider the case where n is positive. By (6.4) we have, $\Psi_{\omega,A}(n;\cdot) = A(\cdot + 2\pi(n-1)\omega)\cdots A(\cdot)$, whence, by Theorem C.15a,

$$\begin{aligned} Ind_{3,d}(\Psi_{\omega,A}(n;\cdot)) &= Ind_{3,d}(A(\cdot + 2\pi(n-1)\omega)\cdots A(\cdot)) \\ &= Ind_{3,d}(A(\cdot + 2\pi(n-1)\omega))\cdots Ind_{3,d}(A(\cdot)) \ , \end{aligned}$$

so that, by Proposition C.18f,

$$Ind_{3,d}(\Psi_{\omega,A}(n;\cdot)) = Ind_{3,d}(A(\cdot))\cdots Ind_{3,d}(A(\cdot)) = (Ind_{3,d}(A(\cdot)))^n \ ,$$

which proves the claim in the present case.

I now consider the case where n is negative. By (6.7) we have $\Psi^T_{\omega,A}(n;\cdot) = \Psi_{\omega,A}(-n;\cdot + 2\pi n\omega)$ whence, by Theorem C.15a and Proposition C.18f,

$$\begin{aligned} Ind_{3,d}(\Psi_{\omega,A}(n;\cdot)) &= Ind_{3,d}(\Psi^T_{\omega,A}(n;\cdot)) = Ind_{3,d}(\Psi_{\omega,A}(-n;\cdot + 2\pi n\omega)) \\ &= Ind_{3,d}(\Psi_{\omega,A}(-n;\cdot)) \ . \end{aligned} \tag{F.1}$$

Since I already proved the claim for positive n I have $Ind_{3,d}(\Psi_{\omega,A}(-n;\cdot)) = (Ind_{3,d}(A(\cdot)))^{-n}$ whence, by (F.1),

$$Ind_{3,d}(\Psi_{\omega,A}(n;\cdot)) = (Ind_{3,d}(A(\cdot)))^{-n} \ . \tag{F.2}$$

Since, due to the special structure of the group $\{1,-1\}^d$, $(Ind_{3,d}(A(\cdot)))^{-n} = (Ind_{3,d}(A(\cdot)))^n$, eq. (F.2) gives me (6.25) which proves the claim in the present case. □

F.2 Proof of Proposition 7.1

Proof of Proposition 7.1a: Let $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. Clearly $L_{T^T} \circ L_T = L_T \circ L_{T^T} = id_{\mathbb{R}^{d+3}}$ whence L_{T^T} is the inverse of L_T. Since L_T and L_{T^T} are continuous, it follows that L_T is a homeomorphism onto $\mathbb{R}^{d+3}$. □

Proof of Proposition 7.1b: Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$ and $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$.

I use (6.9) and Proposition 7.1a to get, for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d, S \in \mathbb{R}^3$,

$$\begin{aligned} &\left(L_T \circ L_{\omega,A}(n;\cdot) \circ L_T^{-1}\right)\begin{pmatrix}\phi \\ S\end{pmatrix} = \left(L_T \circ L_{\omega,A}(n;\cdot) \circ L_{T^T}\right)\begin{pmatrix}\phi \\ S\end{pmatrix} \\ &= \left(L_T \circ L_{\omega,A}(n;\cdot)\right)\begin{pmatrix}\phi \\ T(\phi)S\end{pmatrix} = L_T\begin{pmatrix}\phi + 2\pi n\omega \\ \Psi_{\omega,A}(n;\phi)T(\phi)S\end{pmatrix} \\ &= \begin{pmatrix}\phi + 2\pi n\omega \\ T^T(\phi + 2\pi n\omega)\Psi_{\omega,A}(n;\phi)T(\phi)S\end{pmatrix} , \end{aligned} \tag{F.3}$$

which proves (7.3). It is also clear by (7.4) that $A' \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ whence, by Definition 6.1, $(\omega, A') \in \mathcal{SOT}(d,\omega)$. To prove (7.5) I define the function $\Psi' : \mathbb{Z} \times \mathbb{R}^d \to SO(3)$ for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d$ by $\Psi'(n;\phi) := T^T(\phi + 2\pi n\omega)\Psi_{\omega,A}(n;\phi)T(\phi)$ whence my aim is to show that $\Psi' = \Psi_{\omega,A'}$. By (6.4) I have $\Psi'(0;\phi) = I_{3\times 3}$ and, by (7.4) and the remarks on (6.5), I have, for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d$,

$$
\begin{aligned}
&\Psi'(n+1;\phi) = T^T(\phi + 2\pi(n+1)\omega)\Psi_{\omega,A}(n+1;\phi)T(\phi) \\
&= T^T(\phi + 2\pi(n+1)\omega)A(\phi + 2\pi n\omega)\Psi_{\omega,A}(n;\phi)T(\phi) \\
&= T^T(\phi + 2\pi(n+1)\omega)A(\phi + 2\pi n\omega)T(\phi + 2\pi n\omega)T^T(\phi + 2\pi n\omega)\Psi_{\omega,A}(n;\phi)T(\phi) \\
&= A'(\phi + 2\pi n\omega)T^T(\phi + 2\pi n\omega)\Psi_{\omega,A}(n;\phi)T(\phi) = A'(\phi + 2\pi n\omega)\Psi'(n;\phi) \, .
\end{aligned}
$$

Thus Ψ' satisfies the initial value problem

$$
\Psi'(n+1;\phi) = A'(\phi + 2\pi n\omega)\Psi'(n;\phi) \, , \qquad \Psi'(0;\phi) = I_{3\times 3} \, ,
$$

which, by the remarks on (6.5), implies that $\Psi' = \Psi_{\omega,A'}$ whence (7.5) holds. To prove (7.6) I conclude from (7.3), (7.5) that, for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d, S \in \mathbb{R}^3$,

$$
\Big(L_T \circ L_{\omega,A}(n;\cdot) \circ L_T^{-1}\Big)\begin{pmatrix}\phi \\ S\end{pmatrix} = \begin{pmatrix}\phi + 2\pi n\omega \\ \Psi_{\omega,A'}(n;\phi)S\end{pmatrix} . \tag{F.4}
$$

It follows from (6.9),(F.4) that (7.6) holds. Recalling Appendix B, I conclude that L_T is a continuous $\mathbb{Z}$-map from the topological $\mathbb{Z}$-space $(\mathbb{R}^{d+3}, L_{\omega,A})$ to the topological $\mathbb{Z}$-space $(\mathbb{R}^{d+3}, L_{\omega,A'})$ and that both topological $\mathbb{Z}$-spaces are conjugate. □

Proof of Proposition 7.1c: Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$ and $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. Let $\begin{pmatrix}\phi(\cdot) \\ S(\cdot)\end{pmatrix}$ be a spin-orbit trajectory of (ω, A) and let $S'(\cdot)$ be defined by $S'(n) :=$

$T^T(\phi(n))S(n)$. It follows from (6.2),(6.3),(7.4) that

$$\begin{aligned}
S'(n+1) &= T^T(\phi(n+1))S(n+1) = T^T(\phi(0)+2\pi(n+1)\omega)S(n+1) \\
&= T^T(\phi(0)+2\pi(n+1)\omega)A(\phi(0)+2\pi n\omega)S(n) \\
&= T^T(\phi(0)+2\pi(n+1)\omega)A(\phi(0)+2\pi n\omega)T(\phi(n))S'(n) \\
&= T^T(\phi(0)+2\pi(n+1)\omega)A(\phi(0)+2\pi n\omega)T(\phi(0)+2\pi n\omega)S'(n) \\
&= A'(\phi(0)+2\pi n\omega)S'(n) = A'(\phi(n))S'(n)\ .
\end{aligned}$$

Thus, by (6.2), $S'(\cdot)$ is a spin trajectory, over $\phi(0)$, of the spin-orbit torus (ω, A') and $\begin{pmatrix} \phi(\cdot) \\ S'(\cdot) \end{pmatrix}$ is a spin-orbit trajectory of (ω, A'). □

Proof of Proposition 7.1d: Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$ and $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. Let also $\phi_0 \in \mathbb{R}^d$ and let $t : \mathbb{Z} \to SO(3)$ be defined by $t(n) := T(\phi_0 + 2\pi n\omega)$. Let furthermore $S(\cdot)$ be a spin trajectory, over ϕ_0, of (ω, A) and let me define the function $S' : \mathbb{Z} \to \mathbb{R}^3$ by $S'(n) := t^T(n)S(n)$. Defining the orbital trajectory $\phi(\cdot)$ by $\phi(n) := \phi_0 + 2\pi n\omega$, one observes that $\begin{pmatrix} \phi(\cdot) \\ S(\cdot) \end{pmatrix}$ is a spin-orbit trajectory of (ω, A) and that (7.7) holds. It follows from Proposition 7.1c that $\begin{pmatrix} \phi(\cdot) \\ S'(\cdot) \end{pmatrix}$ is a spin-orbit trajectory of (ω, A'). Thus $S'(\cdot)$ is a spin trajectory of (ω, A'). Clearly $S'(\cdot)$ is over ϕ_0. □

F.3 Proof of Theorem 7.3

Proof of Theorem 7.3a: The claim follows from Definition 7.2 and Proposition 7.1b. □

Proof of Theorem 7.3b: Eq. (7.8) follows from Definition 7.2 and Proposition 7.1b.

To prove the second claim I first note that, if $f, g \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$, then the product is defined by $(fg)(\phi) := f(\phi)g(\phi)$. Clearly the constant function in $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ whose constant value is $I_{3\times3}$, is the unit element of the group. If there is no danger of confusion, I denote the unit element by $I_{3\times3}$. Furthermore the inverse of $f \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ is the transpose f^T since $(f^T f)(\phi) = f^T f(\phi) = I_{3\times3}$. Thus $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ is a group under pointwise multiplication of $SO(3)$-valued functions which proves the second claim.

To prove the third claim I first note that, by Definition 7.2, $R_{d,\omega}$ is a function from $\mathcal{C}_{per}(\mathbb{R}^d, SO(3)) \times \mathcal{SOT}(d,\omega)$ into $\mathcal{SOT}(d,\omega)$. Thus I only have to show the two group action properties of $R_{d,\omega}$ (see also Appendix B). First of all, it follows from Definition 7.2 that $R_{d,\omega}(I_{3\times3}; \omega, A) = (\omega, A)$. Secondly, it follows from Definition 7.2 that if $(\omega, A) \in \mathcal{SOT}(d,\omega)$ and $T_1, T_2 \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$, then, by defining

$$(\omega, A_1) := R_{d,\omega}(T_1; \omega, A) \ , \tag{F.5}$$

I get

$$A_1(\phi) = T_1^T(\phi + 2\pi\omega)A(\phi)T_1(\phi) \ . \tag{F.6}$$

Defining

$$(\omega, A') := R_{d,\omega}(T_1 T_2; \omega, A) \ , \qquad (\omega, A'') := R_{d,\omega}(T_2; R_{d;\omega}(T_1; \omega, A)) \ , \tag{F.7}$$

I conclude from Definition 7.2 and (F.5) that

$$\begin{aligned} A'(\phi) &= (T_1 T_2)^T(\phi + 2\pi\omega)A(\phi)(T_1 T_2)(\phi) \ , \\ A''(\phi) &= T_2^T(\phi + 2\pi\omega)A_1(\phi)T_2(\phi) \ . \end{aligned} \tag{F.8}$$

Using (F.6),(F.8) I get $A' = A''$ whence by (F.7)

$$R_{d,\omega}(T_1 T_2; \omega, A) = R_{d,\omega}(T_2; R_{d,\omega}(T_1; \omega, A)) \ ,$$

which completes the proof that $R_{d,\omega}$ is a right $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$-action on $\mathcal{SOT}(d,\omega)$. □

Proof of Theorem 7.3c: The claim follows from Proposition 7.1c and Definition 7.2. □

Proof of Theorem 7.3d: Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$ and $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. Let also $\mathcal{S}_G$ be a polarization field of (ω, A). I abbreviate

$$(\omega, A') := R_{d,\omega}(T; \omega, A) \; . \tag{F.9}$$

Since $G \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$ I have $H := T^T G \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$ and, by (6.16), (7.5),(7.9),

$$\begin{aligned}
\Psi_{\omega,A'}(n; \phi - 2\pi n\omega) H(\phi - 2\pi n\omega) &= \Psi_{\omega,A'}(n; \phi - 2\pi n\omega) T^T(\phi - 2\pi n\omega) G(\phi - 2\pi n\omega) \\
&= T^T(\phi)\Psi_{\omega,A}(n; \phi - 2\pi n\omega) T(\phi - 2\pi n\omega) T^T(\phi - 2\pi n\omega) G(\phi - 2\pi n\omega) \\
&= T^T(\phi)\Psi_{\omega,A}(n; \phi - 2\pi n\omega) G(\phi - 2\pi n\omega) = T^T(\phi)\mathcal{S}_G(n, \phi) = \mathcal{S}'(n, \phi) \; .
\end{aligned}$$

Thus, by Definition 6.2, $\mathcal{S}'$ is a polarization field of the spin-orbit torus (ω, A') with generator H. By (6.20),(7.9) I have for $n \in \mathbb{Z}, G \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$

$$L^{(PF)}_{\omega,A'}(n; T^T G) = L^{(PF)}_{\omega,A'}(n; H) = \mathcal{S}'(n, \cdot) = T^T \mathcal{S}_G(n, \cdot) = T^T L^{(PF)}_{\omega,A}(n; G) \; ,$$

whence (7.10) follows. Clearly the polarization field $\mathcal{S}'$ is invariant if $\mathcal{S}_G$ is and $\mathcal{S}'$ is a spin field if $\mathcal{S}_G$ is. □

Proof of Theorem 7.3e: The claim is an immediate consequence of Definition 6.2 and parts b) and d) of Theorem 7.3. □

Proof of Theorem 7.3f: Let $(\omega, A), (\omega, A') \in \mathcal{SOT}(d, \omega)$ belong to the same $R_{d,\omega}$-orbit. Then, by Definition 7.2, a $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ exists such that $R_{d,\omega}(T; \omega, A) = (\omega, A')$ whence (7.5) holds for arbitrary $n \in \mathbb{Z}, \phi \in \mathbb{R}^d$. It follows from (7.5) and Theorem C.15a that, for $n \in \mathbb{Z}$,

$$\begin{aligned}
& Ind_{3,d}(\Psi_{\omega,A'}(n; \cdot)) = Ind_{3,d}\Big(T^T(\cdot + 2\pi n\omega)\Psi_{\omega,A}(n; \cdot)T(\cdot)\Big) \\
&= Ind_{3,d}(T^T(\cdot + 2\pi n\omega)) Ind_{3,d}(\Psi_{\omega,A}(n; \cdot)) Ind_{3,d}(T) \\
&= Ind_{3,d}(T(\cdot + 2\pi n\omega)) Ind_{3,d}(\Psi_{\omega,A}(n; \cdot)) Ind_{3,d}(T) \\
&= Ind_{3,d}(\Psi_{\omega,A}(n; \cdot)) Ind_{3,d}(T(\cdot + 2\pi n\omega)) Ind_{3,d}(T) \; .
\end{aligned}$$

Thus, by Proposition C.18f, for $n \in \mathbb{Z}$,

$$Ind_{3,d}(\Psi_{\omega,A'}(n;\cdot)) = Ind_{3,d}(\Psi_{\omega,A}(n;\cdot))Ind_{3,d}(T(\cdot + 2\pi n\omega))Ind_{3,d}(T)$$
$$= Ind_{3,d}(\Psi_{\omega,A}(n;\cdot))Ind_{3,d}(T)Ind_{3,d}(T) = Ind_{3,d}(\Psi_{\omega,A}(n;\cdot)) \; ,$$

which proves the first claim. The second claim follows from the first claim and Theorem C.22c. □

F.4 Proof of Proposition 7.5

Proof of Proposition 7.5a: Let $(\omega, A) \in \mathcal{WT}(d,\omega)$ and $N := Ind_2(A), g := PHF(A)$. Thus by Definition C.12, for $\phi \in \mathbb{R}^d$,

$$A(\phi) = \exp\Big(\mathcal{J}[N^T\phi + 2\pi g(\phi)]\Big) \; ,$$

whence, by (6.4), for $\phi \in \mathbb{R}^d$ and positive integer n,

$$\begin{aligned}
&\Psi_{\omega,A}(n;\phi) = A(\phi + 2\pi(n-1)\omega)A(\phi + 2\pi(n-2)\omega)\cdots A(\phi + 2\pi\omega)A(\phi)\\
&= \exp\Big(\mathcal{J}[N^T(\phi + 2\pi(n-1)\omega) + 2\pi g(\phi + 2\pi(n-1)\omega)]\Big)\cdots\\
&\qquad \cdots\exp\Big(\mathcal{J}[N^T(\phi + 2\pi\omega) + 2\pi g(\phi + 2\pi\omega)]\Big)\exp\Big(\mathcal{J}[N^T\phi + 2\pi g(\phi)]\Big)\\
&= \exp\Big(\mathcal{J}[N^T(\phi + 2\pi\omega(n-1)) + \cdots + N^T\phi\\
&\qquad +2\pi g(\phi + 2\pi(n-1)\omega) + \cdots + 2\pi g(\phi)]\Big)\\
&= \exp\Big(\mathcal{J}[nN^T\phi + 2\pi\sum_{j=0}^{n-1}\Big(jN^T\omega + g(\phi + 2\pi j\omega)\Big)]\Big) \; ,
\end{aligned}$$

which implies (7.13). Using Definition C.12, it follows from (7.13) that, for positive integer n,

$$Ind_2(\Psi_{\omega,A}(n;\cdot)) = nN = nInd_2(A) \; . \qquad \text{(F.10)}$$

Using (6.7),(7.13),(C.1) I get, for negative integer n and $\phi \in \mathbb{R}^d$,

$$\Psi_{\omega,A}(n;\phi) = \Psi^T_{\omega,A}(-n;\phi + 2\pi n\omega)$$
$$= \left(\exp\left(\mathcal{J}[-nN^T(\phi + 2\pi n\omega) + \pi n(n+1)N^T\omega + 2\pi \sum_{j=0}^{-n-1} g(\phi + 2\pi(n+j)\omega)]\right)\right)^T$$
$$= \exp\left(-\mathcal{J}[-nN^T(\phi + 2\pi n\omega) + \pi n(n+1)N^T\omega + 2\pi \sum_{j=0}^{-n-1} g(\phi + 2\pi(n+j)\omega)]\right),$$

whence, by Definition C.12, for negative integer n, eq. (F.10) holds. Moreover, since $\Psi_{\omega,A}(0;\phi) = I_{3\times 3}$, it follows from Definition C.12 that $Ind_2(\Psi_{\omega,A}(0;\cdot)) = 0$ whence (F.10) holds. I thus have shown that (F.10) (whence (7.14)) holds for all integers n.

That $\Psi_{\omega,A}(n;\cdot)$ is 2π-nullhomotopic w.r.t. $SO(3)$ iff $Ind_{3,d}(\Psi_{\omega,A}(n;\cdot)) = (1,...,1)^T$ follows from Theorem C.22g. Using (F.10) and Theorem C.15b I conclude that $((-1)^{nN_1},...,(-1)^{nN_d})$ is the $SO(3)$-index of $\Psi_{\omega,A}(n;\cdot)$ which proves the last claim. □

Proof of Proposition 7.5b: Let $(\omega, A) \in \mathcal{AT}(d,\omega)$. Thus, by (C.2), I have $A = \exp(\mathcal{J}2\pi\nu)$ where $\nu := PH(A)$. Applying (6.4),(C.2) I obtain (7.16). It follows from (7.16) and Definition C.12 that, for all integers n, $Ind_2(\Psi_{\omega,A}(n;\cdot)) = 0$ and $PHF(\Psi_{\omega,A}(n;\cdot))$ is the constant function in $\mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R})$ whose value is $\lfloor n\nu \rfloor$. It also follows from (7.16) that (ω, A) is trivial iff $\nu = 0$. □

Proof of Proposition 7.5c: Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$. If $(\omega, A) \in \mathcal{WT}(d,\omega)$ then, by the definition of $\mathcal{WT}(d,\omega)$, A is $SO_3(2)$-valued. If A is $SO_3(2)$-valued then, by (6.4),(C.2), $\Psi_{\omega,A}(n;\cdot)$ is $SO_3(2)$-valued for all integers n whence $(\omega, A) \in \mathcal{WT}(d,\omega)$.

If $(\omega, A) \in \mathcal{AT}(d,\omega)$ then, by the definition of $\mathcal{AT}(d,\omega)$, A is $SO_3(2)$-valued and constant. If A is $SO_3(2)$-valued and constant then, by (6.4),(C.2), $\Psi_{\omega,A}(n;\cdot)$ is $SO_3(2)$-valued and constant for all integers n whence $(\omega, A) \in \mathcal{AT}(d,\omega)$. □

Proof of Proposition 7.5d: Let $(\omega, A), (\omega, A') \in \mathcal{WT}(d,\omega)$.

To prove the first claim let n be an even integer. Then, by Proposition 7.5a, $Ind_{3,d}(\Psi_{\omega,A}(n;\cdot)) = (1,...,1)^T = Ind_{3,d}(\Psi_{\omega,A'}(n;\cdot))$ whence, by Theorem C.22g, $\Psi_{\omega,A}(n;\cdot) \simeq^{2\pi}_{SO(3)} \Psi_{\omega,A'}(n;\cdot)$.

To prove the second claim let n be an odd integer. Then, by Proposition 7.5a, $Ind_{3,d}(\Psi_{\omega,A}(n;\cdot)) = Ind_{3,d}(A)$ and $Ind_{3,d}(\Psi_{\omega,A'}(n;\cdot)) = Ind_{3,d}(A')$. On the other hand, by Theorem C.22g, I have that $\Psi_{\omega,A}(n;\cdot) \simeq^{2\pi}_{SO(3)} \Psi_{\omega,A'}(n;\cdot)$ iff $Ind_{3,d}(\Psi_{\omega,A}(n;\cdot)) = Ind_{3,d}(\Psi_{\omega,A'}(n;\cdot))$. I conclude that $\Psi_{\omega,A}(n;\cdot) \simeq^{2\pi}_{SO(3)} \Psi_{\omega,A'}(n;\cdot)$ iff $Ind_{3,d}(A) = Ind_{3,d}(A')$.

To prove the third claim let $(\omega, A) \sim_{d,\omega} (\omega, A')$ and m be an arbitrary integer. By Theorem 7.3f I have $Ind_{3,d}(A) = Ind_{3,d}(A')$. If m is even then, by the first claim, $\Psi_{\omega,A}(m;\cdot) \simeq^{2\pi}_{SO(3)} \Psi_{\omega,A'}(m;\cdot)$. If m is odd then, since $Ind_{3,d}(A) = Ind_{3,d}(A')$, the second claim gives me $\Psi_{\omega,A}(m;\cdot) \simeq^{2\pi}_{SO(3)} \Psi_{\omega,A'}(m;\cdot)$. □

F.5 Proof of Proposition 7.7

Proof of Proposition 7.7a: Let $(\omega, A), (\omega, A') \in \mathcal{SOT}(d,\omega)$ and $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ with $R_{d,\omega}(T;\omega,A) = (\omega, A') \in \mathcal{WT}(d,\omega)$. I abbreviate $N := Ind_{2,d}(A')$. By Proposition 7.5a, we have $Ind_{3,d}(\Psi_{\omega,A'}(n;\cdot)) = ((-1)^{nN_1}, ..., (-1)^{nN_d})^T$. Applying Theorem 7.3f, the claim follows. □

Proof of Proposition 7.7b: Let $(\omega, A) \in \mathcal{ACB}(d,\omega)$. Then, by Definition 7.6, a $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ exists such that $(\omega, A') := R_{d,\omega}(T;\omega,A) \in \mathcal{AT}(d,\omega)$ whence, by Theorem 7.3a, for $n \in \mathbb{Z}$,

$$\Psi_{\omega,A'}(n;\cdot) = T^T(\cdot + 2\pi n\omega)\Psi_{\omega,A}(n;\cdot)T(\cdot) \ . \tag{F.11}$$

Applying Proposition 7.5b, a $\nu \in [0,1)$ exists such that, for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d$,

$$\Psi_{\omega,A'}(n;\phi) = \exp(\mathcal{J}2\pi n\nu) \ . \tag{F.12}$$

Since $\simeq^{2\pi}_{SO(3)}$ is an equivalence relation on $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$, (F.11) gives me, for $n \in \mathbb{Z}$,

$$\Psi_{\omega,A'}(n;\cdot) \simeq^{2\pi}_{SO(3)} T^T(\cdot + 2\pi n\omega)\Psi_{\omega,A}(n;\cdot)T(\cdot) \,. \tag{F.13}$$

Because, by (F.12), $\Psi_{\omega,A'}(n;\cdot)$ is a constant function, we have, by Proposition C.18c, that $\Psi_{\omega,A'}(n;\cdot) \simeq^{2\pi}_{SO(3)} I_{3\times 3}$ whence, by (F.13), for $n \in \mathbb{Z}$,

$$I_{3\times 3} \simeq^{2\pi}_{SO(3)} T^T(\cdot + 2\pi n\omega)\Psi_{\omega,A}(n;\cdot)T(\cdot) \,, \tag{F.14}$$

where, for brevity, $I_{3\times 3}$ denotes the constant function in $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ whose only value is $I_{3\times 3}$. Applying Proposition C.20b to (F.14) I get, for $n \in \mathbb{Z}$,

$$T(\cdot + 2\pi n\omega) \simeq^{2\pi}_{SO(3)} \Psi_{\omega,A}(n;\cdot)T(\cdot) \,. \tag{F.15}$$

Applying Proposition C.18f to (F.15) I get, for $n \in \mathbb{Z}$, $T(\cdot) \simeq^{2\pi}_{SO(3)} \Psi_{\omega,A}(n;\cdot)T(\cdot)$, whence, by Proposition C.20b, for $n \in \mathbb{Z}$,

$$I_{3\times 3} \simeq^{2\pi}_{SO(3)} \Psi_{\omega,A}(n;\cdot)T(\cdot)T^T(\cdot) = \Psi_{\omega,A}(n;\cdot) \,. \tag{F.16}$$

I conclude from (F.16) and Proposition C.18b that, for every $n \in \mathbb{Z}$, $\Psi_{\omega,A}(n;\cdot)$ is 2π-nullhomotopic w.r.t. $SO(3)$. Applying Proposition C.18e, gives me $Ind_{3,d}(\Psi_{\omega,A}(n;\cdot)) = (1, ..., 1)^T$. □

F.6 Proof of Lemma 7.8

Proof of Lemma 7.8a: Let R be in $SO(3)$ and $Re^3 = e^3$. Thus the third column of R is e^3 and $R^T e^3 = e^3$ whence the third row of R is $(e^3)^T$. I conclude that

$$R = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \,, \tag{F.17}$$

where a, b, c, d are real numbers. Using again that R is in $SO(3)$, it follows from (C.1),(C.2),(F.17) that $R \in SO_3(2)$. □

Proof of Lemma 7.8b: The claim follows from Proposition 7.5c and Lemma 7.8a. □

F.7 Proof of Theorem 7.9

Proof of Theorem 7.9: Let $(\omega, A) \in \mathcal{SOT}(d,\omega), (\omega, A') \in \mathcal{WT}(d,\omega), T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ and $R_{d,\omega}(T;\omega, A) = (\omega, A')$. By Theorem 7.3a, $\Psi_{\omega,A'}$ satisfies (7.5). I define $G \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$ by $G := Te^3$. Of course, G is the generator of a spin field $\mathcal{S}_G$ of (ω, A) and by Definitions 6.2,7.2 and Lemma 7.8b I obtain

$$\begin{aligned}
&\mathcal{S}_G(1,\phi) = A(\phi - 2\pi\omega)G(\phi - 2\pi\omega) = A(\phi - 2\pi\omega)T(\phi - 2\pi\omega)e^3 \\
&= T(\phi)T^T(\phi)A(\phi - 2\pi\omega)T(\phi - 2\pi\omega)e^3 = T(\phi)A'(\phi - 2\pi\omega)e^3 = T(\phi)e^3 \\
&= G(\phi) = \mathcal{S}_G(0,\phi) \ . \qquad \text{(F.18)}
\end{aligned}$$

With (F.18) and Proposition 6.3 I have shown that the spin field $\mathcal{S}_G$ is invariant.

To demonstrate the converse direction, let $(\omega, A) \in \mathcal{SOT}(d,\omega), T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ and let $G := Te^3$ be the generator of an ISF of (ω, A). I write $R_{d,\omega}(T;\omega, A) =: (\omega, A') \in \mathcal{SOT}(d,\omega)$ whence A' satisfies (7.4). I obtain by (7.5) and Definition 6.2 that

$$\begin{aligned}
&A'(\phi)e^3 = T^T(\phi + 2\pi\omega)A(\phi)T(\phi)e^3 = T^T(\phi + 2\pi\omega)A(\phi)G(\phi) \\
&= T^T(\phi + 2\pi\omega)\mathcal{S}_G(1, \phi + 2\pi\omega) = T^T(\phi + 2\pi\omega)G(\phi + 2\pi\omega) = e^3 \ .
\end{aligned}$$

Thus, by Lemma 7.8b, the spin-orbit torus (ω, A') is weakly trivial, i.e., $R_{d,\omega}(T;\omega, A) \in \mathcal{WT}(d,\omega)$.

To prove the second claim, let first of all $(\omega, A) \in \mathcal{WCB}(d,\omega)$. Thus a $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ exists such that $R_{d,\omega}(T;\omega, A) \in \mathcal{WT}(d,\omega)$ whence, by the first claim, Te^3 is the generator of an ISF. Conversely let there be a $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ such that Te^3 is the generator of an ISF. Thus by the first claim, $R_{d,\omega}(T;\omega, A) \in \mathcal{WT}(d,\omega)$ whence $(\omega, A) \in \mathcal{WCB}(d,\omega)$. □

F.8 Proof of Theorem 7.10

Proof of Theorem 7.10a: Let G be the generator of an ISF and let G be 2π-nullhomotopic w.r.t. $\mathbb{S}^2$. Then by Theorem C.24a a $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ exists such that $G = Te^3$. It thus follows from Theorem 7.9 that $R_{d,\omega}(T; \omega, A) \in \mathcal{WT}(d, \omega)$. Clearly, $(\omega, A) \in \mathcal{WCB}(d, \omega)$. □

Proof of Theorem 7.10b: Let G be the generator of an ISF and let $d = 1$. Then by Theorem C.24b a $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ exists such that $G = Te^3$. Thus by Theorem 7.9 $R_{d,\omega}(T; \omega, A) \in \mathcal{WT}(d, \omega)$. Clearly, $(\omega, A) \in \mathcal{WCB}(d, \omega)$. □

Proof of Theorem 7.10c: Let G be the generator of an ISF and let $d = 2$. Then by Theorem C.24b G is 2π-nullhomotopic w.r.t. $\mathbb{S}^2$ iff a $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ exists such that $G = Te^3$. The claim now follows from Theorem 7.9. □

F.9 Proof of Proposition 7.12

Proof of Proposition 7.12a: Let $(\omega, A), (\omega, A') \in \mathcal{SOT}(d, \omega)$. If $(\omega, A) \sim_{d,\omega} (\omega, A')$ then, since $\sim_{d,\omega}$ is an equivalence relation on $\mathcal{SOT}(d, \omega)$, we have

$$\begin{aligned}&\{(\omega, A'') \in \mathcal{AT}(d, \omega) : (\omega, A'') \sim_{d,\omega} (\omega, A)\} \\ &\quad = \{(\omega, A'') \in \mathcal{AT}(d, \omega) : (\omega, A'') \sim_{d,\omega} (\omega, A')\} ,\end{aligned}$$

whence, by Definition 7.11, $\Xi_1(\omega, A) = \Xi_1(\omega, A')$.

To prove the second claim let $(\omega, A'') \in \mathcal{ACB}(d, \omega)$ and $\Xi_1(\omega, A) = \Xi_1(\omega, A'')$. Because of the first claim, the second claim is proven if I show that $(\omega, A) \sim_{d,\omega} (\omega, A'')$. In fact, picking a $\nu \in \Xi_1(\omega, A) = \Xi_1(\omega, A'')$, Definition 7.11 gives me a $(\omega, A''') \in \mathcal{AT}(d, \omega)$ with $(\omega, A) \sim_{d,\omega} (\omega, A''')$, $(\omega, A'') \sim_{d,\omega} (\omega, A''')$ and $PH(A''') = \nu$. By the transitivity of $\sim_{d,\omega}$ I get $(\omega, A) \sim_{d,\omega} (\omega, A'')$. □

Proof of Proposition 7.12b: Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$.

To prove the first claim, let (ω, A) be on spin-orbit resonance of first kind. Thus, by Definition 7.11, a $(\omega, A') \in \mathcal{AT}(d, \omega)$ exists such that $(\omega, A') \sim_{d,\omega} (\omega, A)$ and $PH(A') = 0$. Therefore, by Proposition 7.5b, (ω, A') is trivial whence, by Definition 7.6, $(\omega, A) \in \mathcal{CB}(d, \omega)$. Conversely, let $(\omega, A) \in \mathcal{CB}(d, \omega)$ so that a trivial spin-orbit torus (ω, A') exists such that $(\omega, A') \sim_{d,\omega} (\omega, A)$. Thus, by Proposition 7.5b, $PH(A') = 0$ whence, by Definition 7.11, $0 \in \Xi_1(\omega, A)$ so that (ω, A) is on spin-orbit resonance of first kind.

The second claim follows from the first claim and Definition 7.11. □

Proof of Proposition 7.12c: Let $(\omega, A), (\omega, A') \in \mathcal{SOT}(d, \omega)$ with $(\omega, A) \sim_{d,\omega} (\omega, A')$.

If $(\omega, A) \in \mathcal{CB}(d, \omega)$ then, by Proposition 7.12b and Definition 7.11, $0 \in \Xi_1(\omega, A)$ whence, by Proposition 7.12a, $0 \in \Xi_1(\omega, A')$ so that, by Proposition 7.12b and Definition 7.11, $(\omega, A') \in \mathcal{CB}(d, \omega)$. Reversing the roles of A, A' it follows that either both spin-orbit tori are coboundaries or neither of them.

The two remaining claims follow from the fact that $\sim_{d,\omega}$ is an equivalence relation on $\mathcal{SOT}(d, \omega)$. □

Proof of Proposition 7.12d: Let $(\omega, A) \in \mathcal{ACB}(d, \omega)$. Then there exists $(\omega, A') \in \mathcal{AT}(d, \omega)$ such that $(\omega, A) \sim_{d,\omega} (\omega, A')$. By Definition 7.4, $A'(\phi)$ is independent of ϕ.

To prove the converse direction let $(\omega, A) \sim_{d,\omega} (\omega, A')$ such that $A'(\phi)$ is independent of ϕ. By some simple Linear Algebra, $R \in SO(3), \nu \in [0, 1)$ exist such that $R^T A' R = \exp(\mathcal{J} 2\pi\nu)$ (see, e.g., [BEH04, Lemma 2.1]). Defining $(\omega, A'') := R_{d,\omega}(R; \omega, A')$, we have, by Definition 7.2, that $A'' = \exp(\mathcal{J} 2\pi\nu)$. It follows from Proposition 7.5c that $(\omega, A'') \in \mathcal{AT}(d, \omega)$. Since $(\omega, A) \sim_{d,\omega} (\omega, A')$ and $(\omega, A') \sim_{d,\omega} (\omega, A'')$, the transitivity of $\sim_{d,\omega}$ implies $(\omega, A) \sim_{d,\omega} (\omega, A'')$ whence $(\omega, A) \in \mathcal{ACB}(d, \omega)$.
□

F.10 Proof of Theorem 7.13

Proof of Theorem 7.13: Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$ and let $(1, \omega)$ be nonresonant. Let (ω, A) have ISF's $\mathcal{S}_{G^1}, \mathcal{S}_{G^2}$ such that $\mathcal{S}_{G^2}$ is different from $\mathcal{S}_{G^1}$ and $-\mathcal{S}_{G^1}$. Thus a $\phi_0 \in \mathbb{R}^d$ exists such that

$$G^1(\phi_0) \times G^2(\phi_0) \neq 0 \ .$$

I define the function $f \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$ by $f(\phi) := |G^1(\phi) \times G^2(\phi)|$. Since $\mathcal{S}_{G^1}, \mathcal{S}_{G^2}$ are invariant polarization fields we have by (6.23) that, for $\phi \in \mathbb{R}^d$,

$$\begin{aligned}
&f(\phi) = |G^1(\phi) \times G^2(\phi)| \\
&= |\Big(A(\phi - 2\pi\omega)G^1(\phi - 2\pi\omega) \Big) \times \Big(A(\phi - 2\pi\omega)G^2(\phi - 2\pi\omega) \Big)| \\
&= |A(\phi - 2\pi\omega)\Big(G^1(\phi - 2\pi\omega) \times G^2(\phi - 2\pi\omega) \Big)| \\
&= |G^1(\phi - 2\pi\omega) \times G^2(\phi - 2\pi\omega)| = f(\phi - 2\pi\omega) \ .
\end{aligned}$$

Thus, by Corollary D.3a, f is constant with constant value, say λ. Clearly $f(\phi_0) \neq 0$ whence $\lambda \neq 0$. Hence I can define a function $G^3 : \mathbb{R}^d \to \mathbb{S}^2$ by $G^3(\phi) := (G^1(\phi) \times G^2(\phi))/\lambda$. Of course, $G^3 \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$ whence G^3 generates a spin field $\mathcal{S}_{G^3}$. Since $\mathcal{S}_{G^1}, \mathcal{S}_{G^2}$ are invariant polarization fields I compute by (6.23) for $\phi \in \mathbb{R}^d$

$$\begin{aligned}
&A(\phi - 2\pi\omega)G^3(\phi - 2\pi\omega) = \frac{1}{\lambda} A(\phi - 2\pi\omega)\Big(G^1(\phi - 2\pi\omega) \times G^2(\phi - 2\pi\omega) \Big) \\
&= \frac{1}{\lambda}\Big(A(\phi - 2\pi\omega)G^1(\phi - 2\pi\omega)) \times A(\phi - 2\pi\omega)G^2(\phi - 2\pi\omega) \Big) \\
&= \frac{1}{\lambda}(G^1(\phi) \times G^2(\phi)) = G^3(\phi) \ .
\end{aligned}$$

Thus, by Proposition 6.3, the polarization field $\mathcal{S}_{G^3}$ is invariant whence $\mathcal{S}_{G^3}$ is an ISF. I define the function $T \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^{3\times3})$ by

$$T(\phi)e^1 := G^3(\phi) \times G^2(\phi) \ , \qquad T(\phi)e^2 := G^3(\phi) \ , \qquad T(\phi)e^3 := G^2(\phi) \ .$$

Clearly the columns of $T(\phi)$ are orthonormal and $\det(T(\phi)) = (G^3(\phi)\times G^2(\phi))^T(G^3(\phi)\times G^2(\phi)) = 1$ whence $T\in\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. Since $\mathcal{S}_{G^2},\mathcal{S}_{G^3}$ are invariant polarization fields I obtain from (6.23) for $\phi\in\mathbb{R}^d$

$$A(\phi-2\pi\omega)\Big(G^2(\phi-2\pi\omega)\times G^3(\phi-2\pi\omega)\Big)$$
$$=\Big(A(\phi-2\pi\omega)G^2(\phi-2\pi\omega))\times A(\phi-2\pi\omega)G^3(\phi-2\pi\omega)\Big) = G^2(\phi)\times G^3(\phi)\ ,$$

so that, by Proposition 6.3, the polarization field $\mathcal{S}_{G^2\times G^3}$ is invariant whence $\mathcal{S}_{G^2\times G^3}$ is an ISF. I can summarize that all three columns of T are generators of invariant spin fields, whence, for $i=1,2,3,\phi\in\mathbb{R}^d$, by (6.23), $A(\phi-2\pi\omega)T(\phi-2\pi\omega)e^i = T(\phi)e^i$, so that $T^T(\phi+2\pi\omega)A(\phi-2\pi\omega)T(\phi)e^i = e^i$, i.e., for $\phi\in\mathbb{R}^d$,

$$T^T(\phi+2\pi\omega)A(\phi)T(\phi) = I_{3\times 3}\ .$$

This implies by Definition 7.2 that $R_{d,\omega}(T;\omega,A) = (\omega, I_{3\times3})$ whence $(\omega,A)\in\mathcal{CB}(d,\omega)$. Applying Proposition 7.12b, I obtain that (ω,A) is on spin-orbit resonance of first kind. □

F.11 Proof of Theorem 7.14

Proof of Theorem 7.14a: I first consider the case when a $T\in\mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$ exists such that $R_{d,\omega}(T;\omega,A_1) = (\omega,A_2)$ and so I abbreviate $N := Ind_2(T)$, $g := PHF(T)$. Thus, by Definition 7.2 and (7.22), we have, for $\phi\in\mathbb{R}^d$,

$$\exp\Big(-\mathcal{J}[N^T(\phi+2\pi\omega)+2\pi g(\phi+2\pi\omega)]\Big)\exp\Big(\mathcal{J}[M_1^T\phi+2\pi f_1(\phi)]\Big)$$
$$\cdot\exp\Big(\mathcal{J}[N^T\phi+2\pi g(\phi)]\Big) = T^T(\phi+2\pi\omega)A_1(\phi)T(\phi) = A_2(\phi)$$
$$=\exp\Big(\mathcal{J}[M_2^T\phi+2\pi f_2(\phi)]\Big)\ ,$$

i.e.,

$$\exp\Big(\mathcal{J}[2\pi g(\phi) - 2\pi g(\phi + 2\pi\omega) - 2\pi N^T\omega + 2\pi f_1(\phi) - 2\pi f_2(\phi) + (M_1 - M_2)^T\phi]\Big) = I_{3\times 3}\ . \tag{F.19}$$

It follows from (F.19) and Theorem C.11a that an integer n exists such that, for $\phi \in \mathbb{R}^d$,

$$g(\phi) - g(\phi + 2\pi\omega) - N^T\omega + f_1(\phi) - f_2(\phi) = \frac{(M_2 - M_1)^T\phi}{2\pi} + n\ . \tag{F.20}$$

Since f_1, f_2 and g are 2π-periodic it follows from (F.20) that (7.23) holds and that, for $\phi \in \mathbb{R}^d$,

$$g(\phi) - g(\phi + 2\pi\omega) - N^T\omega + f_1(\phi) - f_2(\phi) = n\ . \tag{F.21}$$

Taking the zeroth Fourier coefficient on both sides of (F.21) I get $-N^T\omega + f_{1,0} - f_{2,0} = n$, which implies (7.24) and, by (F.21), that (7.25) holds.

I finally consider the case when a $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$ exists such that $R_{d,\omega}(T\mathcal{J}'; \omega, A_1) = (\omega, A_2)$ and I again abbreviate $N := Ind_2(T)$, $g := PHF(T)$. By Definition 7.2 and (7.20),(7.21), (7.22) I get, for $\phi \in \mathbb{R}^d$,

$$\begin{aligned}
&\exp\Big(-\mathcal{J}[-N^T(\phi + 2\pi\omega) - 2\pi g(\phi + 2\pi\omega) + M_1^T\phi + 2\pi f_1(\phi) + N^T\phi + 2\pi g(\phi)]\Big)\\
&= \exp\Big(\mathcal{J}'\mathcal{J}\mathcal{J}'[-N^T(\phi + 2\pi\omega) - 2\pi g(\phi + 2\pi\omega) + M_1^T\phi + 2\pi f_1(\phi) + N^T\phi + 2\pi g(\phi)]\Big)\\
&= \mathcal{J}'\exp\Big(\mathcal{J}[-N^T(\phi + 2\pi\omega) - 2\pi g(\phi + 2\pi\omega) + M_1^T\phi + 2\pi f_1(\phi) + N^T\phi + 2\pi g(\phi)]\Big)\mathcal{J}'\\
&= \mathcal{J}'\exp\Big(\mathcal{J}[-N^T(\phi + 2\pi\omega) - 2\pi g(\phi + 2\pi\omega)]\Big)\exp\Big(\mathcal{J}[M_1^T\phi + 2\pi f_1(\phi)]\Big)\\
&\quad\cdot\exp\Big(\mathcal{J}[N^T\phi + 2\pi g(\phi)]\Big)\mathcal{J}'\\
&= \mathcal{J}'T^T(\phi + 2\pi\omega)A_1(\phi)T(\phi)\mathcal{J}' = (T(\phi + 2\pi\omega)\mathcal{J}')^T A_1(\phi)T(\phi)\mathcal{J}' = A_2(\phi)\\
&= \exp\Big(\mathcal{J}[M_2^T\phi + 2\pi f_2(\phi)]\Big)\ ,
\end{aligned}$$

i.e.,

$$\exp\Big(\mathcal{J}[-2\pi g(\phi) + 2\pi g(\phi + 2\pi\omega) + 2\pi N^T\omega - 2\pi f_1(\phi) - 2\pi f_2(\phi) - (M_1 + M_2)^T\phi]\Big) = I_{3\times 3} \ . \tag{F.22}$$

It follows from (F.22) and Theorem C.11a that an integer n exists such that, for $\phi \in \mathbb{R}^d$,

$$-g(\phi) + g(\phi + 2\pi\omega) + N^T\omega - f_1(\phi) - f_2(\phi) = \frac{(M_1 + M_2)^T\phi}{2\pi} + n \ . \tag{F.23}$$

Since f_1, f_2 and g are 2π-periodic it follows from (F.23) that (7.26) holds and that, for $\phi \in \mathbb{R}^d$,

$$-g(\phi) + g(\phi + 2\pi\omega) + N^T\omega - f_1(\phi) - f_2(\phi) = n \ . \tag{F.24}$$

Taking the zeroth Fourier coefficient on both sides of (F.24) I get

$$f_{1,0} + f_{2,0} - N^T\omega = -n \ ,$$

which implies (7.27) and, by (F.24), that (7.28) holds. □

Proof of Theorem 7.14b: Let $(\omega, A_1) \sim_{d,\omega} (\omega, A_2)$, i.e., let a $T' \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ exist such that $R_{d,\omega}(T'; \omega, A_1) = (\omega, A_2)$. Thus, by Definition 7.2 and for $\phi \in \mathbb{R}^d$,

$$A_1(\phi)T'(\phi) = T'(\phi + 2\pi\omega)A_2(\phi) \ . \tag{F.25}$$

Defining $t := T'e^3 \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$, I conclude from (F.25) and, for $\phi \in \mathbb{R}^d$,

$$A_1(\phi)t(\phi) = t(\phi + 2\pi\omega) \ . \tag{F.26}$$

Clearly the third component t_3 of t is an element of $\mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$ which by (F.26) satisfies

$$t_3(\phi) = t_3(\phi + 2\pi\omega) \ . \tag{F.27}$$

Because $(1, \omega)$ is nonresonant I conclude from (F.27) and Corollary D.3a that t_3 is constant so that, since $|t_3| \leq |t| = 1$, only the following three cases can occur: Case (i) where $t_3 = 1$, Case (ii) where $t_3 = -1$, Case (iii) where $|t_3| < 1$.

I first consider Case (i). Since $|t| = 1$ we have in the present case that $t = t_3 e^3 = e^3$, i.e., $T' e^3 = e^3$. Due to Lemma 7.8a, I thus obtain that T' is $SO_3(2)$-valued whence $T' \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$. Therefore, $T := T'$ satisfies the claim.

I now consider Case (ii). Since $|t| = 1$ we have in the present case that $t = t_3 e^3 = -e^3$, i.e., $T' e^3 = -e^3$. Due to Lemma 7.8a, I obtain that $T' \mathcal{J}'$ is $SO_3(2)$-valued whence $T := T' \mathcal{J}' \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$. Thus $R_{d,\omega}(T\mathcal{J}'; \omega, A_1) = (\omega, A_2)$ which proves the claim.

I now consider Case (iii). Because the constant $t_0 := \sqrt{1 - t_3^2}$ is positive, we have that $g_1 \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$, defined by

$$g_1(\phi) := \begin{pmatrix} \frac{t_1(\phi)}{t_0} & -\frac{t_2(\phi)}{t_0} & 0 \\ \frac{t_2(\phi)}{t_0} & \frac{t_1(\phi)}{t_0} & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{F.28}$$

satisfies, for all $\phi \in \mathbb{R}^d$,

$$t(\phi) = g_1(\phi)(t_0 e^1 + t_3 e^3) . \tag{F.29}$$

Combining (F.26) with (F.29) results, for all $\phi \in \mathbb{R}^d$, in

$$A_1(\phi) g_1(\phi)[t_0 e^1 + t_3 e^3] = A_1(\phi) t(\phi) = t(\phi + 2\pi\omega) = g_1(\phi + 2\pi\omega)[t_0 e^1 + t_3 e^3] ,$$

i.e.,

$$A_1(\phi) g_1(\phi) g_1^T(\phi + 2\pi\omega)[t_0 e^1 + t_3 e^3] = t_0 e^1 + t_3 e^3 . \tag{F.30}$$

Since $A_1(\phi) g_1(\phi) g_1^T(\phi + 2\pi\omega) \in SO_3(2)$, I conclude from (F.30) that

$$A_1(\phi) g_1(\phi) g_1^T(\phi + 2\pi\omega) e^1 = e^1 , \tag{F.31}$$

where I also used the fact that t_0 is nonzero. Using again that $A_1(\phi)g_1(\phi)g_1^T(\phi+2\pi\omega)$ is in $SO_3(2)$, eq. (F.31) implies that

$$A_1(\phi)g_1(\phi)g_1^T(\phi + 2\pi\omega) = I_{3\times3} \; . \tag{F.32}$$

By (F.32) and Definition 7.2 I obtain

$$R_{d,\omega}(g_1; \omega, A_1) = (\omega, I_{3\times3}) \; . \tag{F.33}$$

I conclude that $(\omega, A_1) \in \mathcal{CB}(d, \omega)$ (and therefore $(\omega, A_2) \in \mathcal{CB}(d, \omega)$). Thus the present case is highly exceptional. Since $t_0 e^1 + t_3 e^3$ is a constant unit vector, a constant matrix $\tilde{t}$ exists in $SO(3)$ such that $\tilde{t}e^3 = t_0 e^1 + t_3 e^3$, whence (F.29) and the definition of t imply $T' e^3 = t = g_1 \tilde{t} e^3$, i.e.,

$$T'^T g_1 \tilde{t} e^3 = e^3 \; . \tag{F.34}$$

Thus and due to Lemma 7.8a I obtain that $T'^T g_1 \tilde{t}$ is $SO_3(2)$-valued whence $g_2 := T'^T g_1 \tilde{t} \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$. Therefore $T' = g_1 \tilde{t} g_2^T$ whence (F.25) yields, for $\phi \in \mathbb{R}^d$,

$$A_1(\phi)g_1(\phi)\tilde{t}g_2^T(\phi) = g_1(\phi + 2\pi\omega)\tilde{t}g_2^T(\phi + 2\pi\omega)A_2(\phi) \; ,$$

i.e.,

$$A_1(\phi)g_1(\phi)g_1^T(\phi + 2\pi\omega)\tilde{t}g_2^T(\phi) = \tilde{t}g_2^T(\phi + 2\pi\omega)A_2(\phi) \; ,$$

so that, due to (F.32),

$$\tilde{t}g_2^T(\phi) = \tilde{t}g_2^T(\phi + 2\pi\omega)A_2(\phi) \; ,$$

which implies

$$A_2(\phi)g_2(\phi)g_2^T(\phi + 2\pi\omega) = I_{3\times3} \; . \tag{F.35}$$

It follows from (F.32),(F.35) that, for $\phi \in \mathbb{R}^d$,

$$A_2(\phi) = [g_1(\phi + 2\pi\omega)g_2^T(\phi + 2\pi\omega)]^T A_1(\phi)g_1(\phi)g_2^T(\phi) \; . \tag{F.36}$$

Thus, by Definition 7.2, $R_{d,\omega}(T;\omega,A_1)=(\omega,A_2)$, where $T:=g_1g_2^T\in\mathcal{C}_{per}(\mathbb{R}^d,SO_3(2))$, which proves the claim. □

Proof of Theorem 7.14c: ⇒: Let $(\omega,A_1)\sim_{d,\omega}(\omega,A_2)$. Then, by Theorem 7.14b, a $T\in\mathcal{C}_{per}(\mathbb{R}^d,SO_3(2))$ exists such that either
$R_{d,\omega}(T;\omega,A_1)=(\omega,A_2)$ or $R_{d,\omega}(T\mathcal{J}';\omega,A_1)=(\omega,A_2)$. In the former case we have, by Theorem 7.14a, that (7.23),(7.24), (7.25) hold where $N:=Ind_2(T)$ and $g:=PHF(T)$. In the latter case we have, by Theorem 7.14a, that (7.26),(7.27), (7.28) hold where $N:=Ind_2(T)$ and $g:=PHF(T)$.

⇐: Let (7.23) hold and let $g\in\mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R})$, $N\in\mathbb{Z}^d$ exist such that (7.24), (7.25) hold. I define $T\in\mathcal{C}_{per}(\mathbb{R}^d,SO_3(2))$ by (7.29). Clearly by (7.22),(7.23), (7.24),(7.25),(7.29) we have, for $\phi\in\mathbb{R}^d$,

$$\begin{aligned}
T^T(\phi+2\pi\omega)A_1(\phi)T(\phi) &= \exp\Big(\mathcal{J}[-N^T(\phi+2\pi\omega)-2\pi g(\phi+2\pi\omega)]\Big)\\
&\quad\cdot\exp\Big(\mathcal{J}[M_1^T\phi+2\pi f_1(\phi)]\Big)\exp\Big(\mathcal{J}[N^T\phi+2\pi g(\phi)]\Big)\\
&= \exp\Big(\mathcal{J}[2\pi g(\phi)-2\pi g(\phi+2\pi\omega)-2\pi N^T\omega+2\pi f_1(\phi)+M_1^T\phi]\Big)\\
&= \exp\Big(\mathcal{J}[2\pi g(\phi)-2\pi g(\phi+2\pi\omega)-2\pi N^T\omega+2\pi\tilde{f}_1(\phi)+2\pi f_{1,0}+M_1^T\phi]\Big)\\
&= \exp\Big(\mathcal{J}[-2\pi N^T\omega+2\pi\tilde{f}_2(\phi)+2\pi f_{1,0}+M_1^T\phi]\Big)\\
&= \exp\Big(\mathcal{J}[2\pi\tilde{f}_2(\phi)+2\pi f_{2,0}+M_1^T\phi]\Big)=\exp\Big(\mathcal{J}[2\pi f_2(\phi)+M_1^T\phi]\Big)\\
&= \exp\Big(\mathcal{J}[M_2^T\phi+2\pi f_2(\phi)]\Big)=A_2(\phi)\ ,
\end{aligned}$$

whence, by Definition 7.2, $R_{d,\omega}(T;\omega,A_1)=(\omega,A_2)$.

Let (7.26) hold and let $g\in\mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R})$, $N\in\mathbb{Z}^d$ exist such that (7.27),(7.28) hold. I define $T\in\mathcal{C}_{per}(\mathbb{R}^d,SO_3(2))$ by (7.29). Clearly by (7.20),(7.21), (7.22),(7.26), (7.27),(7.28),(7.29) we have, for $\phi \in \mathbb{R}^d$,

$$(T(\phi + 2\pi\omega)\mathcal{J}')^T A_1(\phi)T(\phi)\mathcal{J}' = \mathcal{J}'T^T(\phi + 2\pi\omega)A_1(\phi)T(\phi)\mathcal{J}'$$
$$= \mathcal{J}' \exp\Big(\mathcal{J}[-N^T(\phi + 2\pi\omega) - 2\pi g(\phi + 2\pi\omega)]\Big) \exp\Big(\mathcal{J}[M_1^T\phi + 2\pi f_1(\phi)]\Big)$$
$$\cdot \exp\Big(\mathcal{J}[N^T\phi + 2\pi g(\phi)]\Big)\mathcal{J}'$$
$$= \mathcal{J}' \exp\Big(\mathcal{J}[-N^T(\phi + 2\pi\omega) - 2\pi g(\phi + 2\pi\omega) + M_1^T\phi + 2\pi f_1(\phi) + N^T\phi + 2\pi g(\phi)]\Big)\mathcal{J}'$$
$$= \mathcal{J}' \exp\Big(\mathcal{J}[2\pi g(\phi) - 2\pi g(\phi + 2\pi\omega) - 2\pi N^T\omega + 2\pi\tilde{f}_1(\phi) + 2\pi f_{1,0} + M_1^T\phi]\Big)\mathcal{J}'$$
$$= \mathcal{J}' \exp\Big(\mathcal{J}[-2\pi N^T\omega - 2\pi\tilde{f}_2(\phi) + 2\pi f_{1,0} + M_1^T\phi]\Big)\mathcal{J}'$$
$$= \mathcal{J}' \exp\Big(\mathcal{J}[-2\pi\tilde{f}_2(\phi) - 2\pi f_{2,0} + M_1^T\phi]\Big)\mathcal{J}'$$
$$= \mathcal{J}' \exp\Big(\mathcal{J}[-2\pi f_2(\phi) + M_1^T\phi]\Big)\mathcal{J}' = \mathcal{J}' \exp\Big(\mathcal{J}[-2\pi f_2(\phi) - M_2^T\phi]\Big)\mathcal{J}'$$
$$= \exp\Big(\mathcal{J}'\mathcal{J}\mathcal{J}'[-2\pi f_2(\phi) - M_2^T\phi]\Big) = \exp\Big(-\mathcal{J}[-2\pi f_2(\phi) - M_2^T\phi]\Big)$$
$$= \exp\Big(\mathcal{J}[M_2^T\phi + 2\pi f_2(\phi)]\Big) = A_2(\phi) ,$$

whence, by Definition 7.2, $R_{d,\omega}(T\mathcal{J}'; \omega, A_1) = (\omega, A_2)$. □

F.12 Proof of Corollary 7.15

Proof of Corollary 7.15a: I first note that $M_2 := Ind_2(A_2) = 0$ and that $f_2 := PHF(A_2)$ is the constant function whose value is $\nu \in [0, 1)$. Thus the fractional part of the zeroth Fourier coefficient $f_{2,0}$ of f_2 equals ν and I have $\tilde{f}_2 := f_2 - f_{2,0} = 0$.

I can now apply Theorem 7.14a. First let $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$ such that $R_{d,\omega}(T; \omega, A_1) = (\omega, A_2)$ and let me abbreviate $N := Ind_2(T)$, $g := PHF(T)$. Thus, by (7.23) I obtain $M_1 = M_2 = 0$ whence (7.32) holds. By (7.24) I obtain $f_{1,0} - \nu - N^T\omega = f_{1,0} - f_{2,0} - N^T\omega \in \mathbb{Z}$ whence (7.33) holds. Furthermore, for

$\phi \in \mathbb{R}^d$, I get from (7.25) $g(\phi + 2\pi\omega) - g(\phi) = \tilde{f}_1(\phi) - \tilde{f}_2(\phi) = \tilde{f}_1(\phi)$ whence (7.34) holds.

Now let $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$ such that $R_{d,\omega}(T\mathcal{J}'; \omega, A_1) = (\omega, A_2)$ and let again $N := Ind_2(T)$, $g := PHF(T)$. Thus, by (7.26) I obtain $M_1 = -M_2 = 0$ whence (7.32) holds. By (7.27) I obtain $f_{1,0} + \nu - N^T\omega = f_{1,0} + f_{2,0} - N^T\omega \in \mathbb{Z}$ whence (7.35) holds. Furthermore, for $\phi \in \mathbb{R}^d$, I get from (7.28) that $g(\phi + 2\pi\omega) - g(\phi) = \tilde{f}_1(\phi) + \tilde{f}_2(\phi) = \tilde{f}_1(\phi)$ whence (7.34) holds. □

Proof of Corollary 7.15b: As in the proof of Corollary 7.15a I first note that $M_2 := Ind_2(A_2) = 0$ and that $f_2 := PHF(A_2)$ is the constant function whose value is ν. Thus the zeroth Fourier coefficient $f_{2,0}$ of f_2 equals ν and $\tilde{f}_2 := f_2 - f_{2,0} = 0$. The claims now follows from Theorem 7.14c. □

F.13 Proof of Theorem 8.1

Proof of Theorem 8.1a: Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$ have a polarization field $\mathcal{S}_G$. Let $\phi_0 \in \mathbb{R}^d$ and the function $S : \mathbb{Z} \to \mathbb{R}^3$ be defined by $S(n) := \mathcal{S}_G(n, \phi_0 + 2\pi n\omega)$. By Definition 6.2 we have $S(n) = \mathcal{S}_G(n, \phi_0 + 2\pi n\omega) = \Psi_{\omega,A}(n; \phi_0)G(\phi_0) = \Psi_{\omega,A}(n; \phi_0)\mathcal{S}_G(0, \phi_0) = \Psi_{\omega,A}(n; \phi_0)S(0)$. Then, by (6.3), S is a spin trajectory over ϕ_0.

If the polarization field $\mathcal{S}_G$ is invariant, then by Definition 6.2 we have $S(n) = \mathcal{S}_G(n, \phi_0 + 2\pi n\omega) = \mathcal{S}_G(0, \phi_0 + 2\pi n\omega) = G(\phi_0 + 2\pi n\omega)$ so that, by Definition D.1, $u \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$, defined by $u(\phi) := G(\phi_0 + \phi)$, is an ω-generator of S whence S is ω–quasiperiodic. □

Proof of Theorem 8.1b: Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$ and let $(1, \omega)$ be nonresonant. Let (ω, A) have, for some $\phi_0 \in \mathbb{R}^d$, an ω–quasiperiodic spin trajectory S over ϕ_0. By Corollary D.3b, the ω–quasiperiodic function S has a unique ω-generator u and this ω-generator is $\mathbb{R}^3$-valued, i.e., $u \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$. Of course, for every integer n, $S(n) = u(2\pi n\omega)$. The function $G \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$, defined by $G(\phi) := u(\phi - \phi_0)$, generates a polarization field $\mathcal{S}_G$ of (ω, A). I will first show that the polarization field $\mathcal{S}_G$ is invariant and satisfies (8.1).

Since S is a spin trajectory over ϕ_0 we have by (6.8) that $S(n) = A(\phi_0 + 2\pi(n-1)\omega)S(n-1)$ whence, for $n \in \mathbb{Z}$,

$$\begin{aligned} G(\phi_0 + 2\pi n\omega) &= u(2\pi n\omega) = S(n) = A(\phi_0 + 2\pi(n-1)\omega)S(n-1) \\ &= A(\phi_0 + 2\pi(n-1)\omega)u(2\pi(n-1)\omega) \\ &= A(\phi_0 + 2\pi(n-1)\omega)G(\phi_0 + 2\pi(n-1)\omega) \ . \end{aligned} \tag{F.37}$$

Since G and A are 2π-periodic we thus have for $m \in \mathbb{Z}^d, n \in \mathbb{Z}$ that $G(\phi_0 + 2\pi n\omega + 2\pi m) = A(\phi_0 + 2\pi(n-1)\omega + 2\pi m)G(\phi_0 + 2\pi(n-1)\omega + 2\pi m)$. Thus, defining the set $\tilde{A} := \{\phi_0 + 2\pi n\omega + 2\pi m : m \in \mathbb{Z}^d, n \in \mathbb{Z}\}$, we see that (6.23) holds for all $\phi \in \tilde{A}$. Since $(1, \omega)$ is nonresonant, I conclude from Theorem D.2 that the set $\tilde{A}$ is dense in $\mathbb{R}^d$. Since $\tilde{A}$ is dense in $\mathbb{R}^d$ and since G and A are continuous, it thus follows that (6.23) holds for all $\phi \in \mathbb{R}^d$. By Proposition 6.3 I conclude that the polarization field $\mathcal{S}_G$ is invariant. Of course, (F.37) implies (8.1).

To show the uniqueness of $\mathcal{S}_G$ let $\mathcal{S}_H$ be an arbitrary invariant polarization field such that, for all integers n, $S(n) = H(\phi_0 + 2\pi n\omega)$. Thus $v \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$, defined by $v(\phi) := H(\phi_0 + \phi)$, is an ω-generator of S. However, since u is the unique ω-generator of S, I conclude that $v = u$ whence $H = G$.

Let in addition S be normalized to 1, i.e., $|S(n)| = 1$. To show that $\mathcal{S}_G$ is a spin field, I note that if $m \in \mathbb{Z}^d, n \in \mathbb{Z}$ then $G(\phi_0 + 2\pi n\omega + 2\pi m) = u(2\pi n\omega + 2\pi m) = u(2\pi n\omega) = S(n)$ whence $|G(\phi_0 + 2\pi n\omega + 2\pi m)| = |S(n)| = 1$. Thus, for $\phi \in \tilde{A}$, we have $|G(\phi)| = 1$. Since $|G(\phi)| = 1$ on a dense set of points ϕ I conclude, by the continuity of $|G|$, that $|G(\phi)| = 1$ for all ϕ in $\mathbb{R}^d$ whence the polarization field $\mathcal{S}_G$ is a spin field. □

F.14 Proof of Theorem 8.3

Proof of Theorem 8.3a: Let $(\omega, A) \in \mathcal{WCB}(d, \omega)$ and $(\omega, A') := R_{d,\omega}(T; \omega, A) \in \mathcal{WT}(d, \omega)$ with $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. Since $(\omega, A') \in \mathcal{WT}(d, \omega)$ the $SO_3(2)$-index and phase function of $\Psi_{\omega,A'}(n; \cdot)$ are well defined so that I can abbreviate $N_n := Ind_2(\Psi_{\omega,A'}(n; \cdot)), f(n, \cdot) := PHF(\Psi_{\omega,A'}(n; \cdot))$. Let also $\phi_0 \in \mathbb{R}^d$.

Defining the function $t : \mathbb{Z} \to SO(3)$ by $t(n) := T(\phi_0 + 2\pi n\omega)$, it follows from the lines after Definition 8.2 that t is an SPF over ϕ_0. Furthermore, $T(\phi_0 + \cdot)$ is an $\mathbb{R}^{3\times 3}$-valued ω-generator of t whence t is ω–quasiperiodic. I obtain from Definition 7.2 and (7.14), (7.15),(8.4) that the differential phase function λ of t satisfies, for $n \in \mathbb{Z}$,

$$\begin{aligned}
&\exp(2\pi\lambda(n)\mathcal{J}) = t^T(n+1)A(\phi_0 + 2\pi n\omega)t(n) \\
&= T^T(\phi_0 + 2\pi(n+1)\omega)A(\phi_0 + 2\pi n\omega)T(\phi_0 + 2\pi n\omega) = A'(\phi_0 + 2\pi n\omega) \\
&= \exp(\mathcal{J}[N_1^T(\phi_0 + 2\pi n\omega) + 2\pi f(1, \phi_0 + 2\pi n\omega)]) \\
&= \exp(\mathcal{J}[N_1^T\phi_0 + 2\pi N_n^T\omega + 2\pi f(1, \phi_0 + 2\pi n\omega)]) \ .
\end{aligned} \tag{F.38}$$

Since $\lambda(n) \in [0, 1)$, it follows from (C.2),(F.38) that (8.7) holds. Also I obtain from Theorem 7.3a and (7.14), (7.15),(8.6) that the integral phase function μ of t satisfies, for $n \in \mathbb{Z}$,

$$\begin{aligned}
&\exp(2\pi\mu(n)\mathcal{J}) = t^T(n)\Psi_{\omega,A}(n; \phi_0)t(0) = T^T(\phi_0 + 2\pi n\omega)\Psi_{\omega,A}(n; \phi_0)T(\phi_0) \\
&= \Psi_{\omega,A'}(n; \phi_0) = \exp(\mathcal{J}[nN_1^T\phi_0 + 2\pi f(n, \phi_0)]) = \exp(\mathcal{J}[N_n^T\phi_0 + 2\pi f(n, \phi_0)]) \ .
\end{aligned} \tag{F.39}$$

Since $\mu(n) \in [0, 1)$, it follows from (C.2),(F.39) that (8.8) holds. □

Proof of Theorem 8.3b: Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$ and let $(1, \omega)$ be nonresonant. Let also (ω, A) have an ω–quasiperiodic SPF t over some $\phi_0 \in \mathbb{R}^d$.

By Corollary D.3b, the ω–quasiperiodic function t has a unique ω-generator $\tilde{t}$ and this ω-generator is $\mathbb{R}^{3\times3}$-valued, i.e., $\tilde{t} \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^{3\times3})$. Of course, for every integer n, $t(n) = \tilde{t}(2\pi n\omega)$. I define the function $T \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^{3\times3})$ by $T(\phi) := \tilde{t}(\phi - \phi_0)$. Clearly, for every integer n, $t(n) = \tilde{t}(2\pi n\omega) = T(\phi_0 + 2\pi n\omega)$.

To show the uniqueness of T, let T' be an arbitrary function in $\mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^{3\times3})$ such that, for all integers n, $t(n) = T'(\phi_0 + 2\pi n\omega)$. Thus $\tilde{t}' \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^{3\times3})$, defined by $\tilde{t}'(\phi) := T'(\phi_0 + \phi)$, satisfies, for every integer n, $\tilde{t}'(2\pi n\omega) = T'(\phi_0 + 2\pi n\omega) = t(n)$ whence $\tilde{t}'$ is an ω-generator of t. However, since $\tilde{t}$ is the unique ω-generator of t, I conclude that $\tilde{t}' = \tilde{t}$ whence $T = T'$.

I now show that $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. If $m \in \mathbb{Z}^d, n \in \mathbb{Z}$ then $T(\phi_0 + 2\pi n\omega + 2\pi m) = \tilde{t}(2\pi n\omega + 2\pi m) = \tilde{t}(2\pi n\omega) = t(n)$ whence $T^T(\phi_0 + 2\pi n\omega + 2\pi m)T(\phi_0 + 2\pi n\omega + 2\pi m) = t^T(n)t(n) = I_{3\times3}$ and $\det(T(\phi_0 + 2\pi n\omega + 2\pi m)) = \det(t(n)) = 1$. Thus, defining the set $\tilde{A} := \{\phi_0 + 2\pi n\omega + 2\pi m : m \in \mathbb{Z}^d, n \in \mathbb{Z}\}$, we have for $\phi \in \tilde{A}$ that $T^T(\phi)T(\phi) = I_{3\times3}$ and $\det(T(\phi)) = 1$. Since $(1, \omega)$ is nonresonant, I conclude from Theorem D.2 that the set $\tilde{A}$ is dense in $\mathbb{R}^d$. Thus $T^T(\phi)T(\phi) = I_{3\times3}$ and $\det(T(\phi)) = 1$ on a dense set of points ϕ so that, by the continuity of T, I conclude that $T^T(\phi)T(\phi) = I_{3\times3}$ and $\det(T(\phi)) = 1$ for *all* $\phi \in \mathbb{R}^d$ whence T is $SO(3)$-valued. Since $T \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^{3\times3})$ I conclude that $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$.

I now show that $R_{d,\omega}(T; \omega, A) \in \mathcal{WT}(d, \omega)$. By Definition 8.2, the function $S : \mathbb{Z} \to \mathbb{R}^3$, defined by $S(n) := t(n)e^3$, is an ω–quasiperiodic spin trajectory over ϕ_0 such that $|S(n)| = 1$. Thus by Theorem 8.1b an ISF $\mathcal{S}_G$ exists such that (8.1) holds for all integers n. It follows, for every integer n, that $T(\phi_0 + 2\pi n\omega)e^3 = t(n)e^3 = S(n) = G(\phi_0 + 2\pi n\omega)$ whence, for $m \in \mathbb{Z}^d, n \in \mathbb{Z}$, we have $T(\phi_0 + 2\pi n\omega + 2\pi m)e^3 = G(\phi_0 + 2\pi n\omega + 2\pi m)$, i.e., for $\phi \in \tilde{A}$,

$$T(\phi)e^3 = G(\phi) . \tag{F.40}$$

Since the set $\tilde{A}$ is dense in $\mathbb{R}^d$ and since T and G are continuous I conclude that

(F.40) holds for *all* $\phi \in \mathbb{R}^d$. Thus the third column of T is the generator of an ISF whence, by Theorem 7.9, $R_{d,\omega}(T;\omega,A) \in \mathcal{WT}(d,\omega)$. This implies that (ω,A) is a weak coboundary which completes the proof. □

F.15 Proof of Theorem 8.5

Proof of Theorem 8.5a: Let $(\omega,A) \in \mathcal{SOT}(d,\omega)$ and let $\nu \in \Xi_2(\omega,A,\phi_0)$ for some $\phi_0 \in \mathbb{R}^d$. Then, by Definition 8.4, there exists an ω–quasiperiodic UPF t over ϕ_0 with UPR ν. Furthermore, for every integer n, eq. (8.10) holds for $\lambda = \nu$ and $t(n) = u(2\pi n\omega)$ where u is an $\mathbb{R}^{3\times 3}$-valued ω-generator of t. Thus $v \in \mathcal{C}_{per}(\mathbb{R}^{d+1},\mathbb{R}^{3\times 3})$, defined by $v(\phi,\psi) := u(\phi)\exp(\mathcal{J}\psi)u^T(0)$, is an (ω,ν)-generator of $\Psi_{\omega,A}(\cdot;\phi_0)$ since $\Psi_{\omega,A}(n;\phi_0) = v(2\pi n\omega, 2\pi n\nu)$. Therefore every spin trajectory over ϕ_0 is (ω,ν)–quasiperiodic. □

Proof of Theorem 8.5b: Let $(\omega,A) \in \mathcal{ACB}(d,\omega)$ and $(\omega,A') := R_{d,\omega}(T;\omega,A) \in \mathcal{AT}(d,\omega)$ with $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. Let $\phi_0 \in \mathbb{R}^d$ and let the function $t : \mathbb{Z} \to SO(3)$ be defined by $t(n) := T(\phi_0 + 2\pi n\omega)$.

Due to the inclusions (7.12) we have $(\omega,A') \in \mathcal{WT}(d,\omega)$ so that I can apply Theorem 8.3a leading me to the result that t is an ω–quasiperiodic SPF over ϕ_0. Thus to show that t is a UPF I have to compute its differential phase function. In fact using Proposition 7.5b and Theorem 8.3a I obtain, for $n \in \mathbb{Z}$, that $\lambda(n) = \lfloor \nu \rfloor = \nu$ where $\nu := PH(A')$. □

Proof of Theorem 8.5c: Let $(\omega,A) \in \mathcal{SOT}(d,\omega)$ and let $(1,\omega)$ be nonresonant. Let (ω,A) have an ω–quasiperiodic UPF t over some $\phi_0 \in \mathbb{R}^d$ with UPR ν.

Since t is an ω–quasiperiodic SPF I can apply Theorem 8.3b by which a unique $T \in \mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^{3\times 3})$ exists such that, for all integers n, $t(n) = T(\phi_0 + 2\pi n\omega)$. Moreover $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$.

To prove the remaining claims I compute, by using (8.9),

$$\begin{aligned}\exp(2\pi\nu\mathcal{J}) &= t^T(n+1)A(\phi_0+2\pi n\omega)t(n)\\ &= T^T(\phi_0+2\pi(n+1)\omega)A(\phi_0+2\pi n\omega)T(\phi_0+2\pi n\omega)\\ &= T^T(\phi_0+2\pi\omega+2\pi n\omega+2\pi m)A(\phi_0+2\pi n\omega+2\pi m)T(\phi_0+2\pi n\omega+2\pi m)\ ,\end{aligned}$$

where $n\in\mathbb{Z}, m\in\mathbb{Z}^d$ and where in the third equality I used the 2π-periodicity of A and T. I conclude that for $\phi\in\tilde{A}:=\{\phi_0+2\pi n\omega+2\pi m : m\in\mathbb{Z}^d, n\in\mathbb{Z}\}$, we have

$$\exp(2\pi\nu\mathcal{J}) = T^T(\phi+2\pi\omega)A(\phi)T(\phi)\ . \tag{F.41}$$

Since, by Theorem D.2, $\tilde{A}$ is dense in $\mathbb{R}^d$ and since A and T are continuous functions I conclude that (F.41) holds for all ϕ in $\mathbb{R}^d$. Defining $(\omega, A')\in\mathcal{AT}(d,\omega)$ by $\Psi_{\omega,A'}(n;\phi):=\exp(2\pi\nu\mathcal{J})$ I get $PH(A')=\nu$ and, by (F.41), for $\phi\in\mathbb{R}^d$,

$$A'(\phi) = T^T(\phi+2\pi\omega)A(\phi)T(\phi)\ . \tag{F.42}$$

Applying Definition 7.2 to (F.42), yields that $(\omega, A') = R_{d,\omega}(T;\omega,A)$ which completes the proof. Clearly, since $PH(A')=\nu$ we also have $\nu\in\Xi_1(\omega,A)$. □

F.16 Proof of Theorem 8.6

Proof of Theorem 8.6: Let $(\omega,A)\in\mathcal{SOT}(d,\omega)$ and let $\phi_0\in\mathbb{R}^d$. Let also $\nu\in\Xi_2(\omega,A,\phi_0)$. In the lines before Theorem 8.6, I already showed that $[\nu]_\omega\subset\Xi_2(\omega,A,\phi_0)$ so my task is to prove the converse inclusion $[\nu]_\omega\supset\Xi_2(\omega,A,\phi_0)$.

Let $\tilde{\nu}\in\Xi_2(\omega,A,\phi_0)$ so I am done when I show that $\tilde{\nu}\sim_\omega\nu$. Let $t,\tilde{t}$ be ω–quasiperiodic UPF's over ϕ_0 and let ν be the UPR of t and $\tilde{\nu}$ be the UPR of $\tilde{t}$. I define the two functions $g_\pm:\mathbb{Z}\to\mathbb{C}$ by

$$g_\pm(n) := \Big(t(n)(e^1\pm ie^2)\Big)^T\Big(\tilde{t}(n)(e^1+ie^2)\Big)\ . \tag{F.43}$$

Because t and $\tilde{t}$ are ω-quasiperiodic, $g_\pm$ is ω-quasiperiodic. By (8.9) we have, for $n \in \mathbb{Z}$,

$$
\begin{aligned}
t(n+1)(e^1 \pm ie^2) &= A(\phi_0 + 2\pi n\omega)t(n)\exp(-2\pi\nu\mathcal{J})(e^1 \pm ie^2)\ , \\
\tilde{t}(n+1)(e^1 + ie^2) &= A(\phi_0 + 2\pi n\omega)\tilde{t}(n)\exp(-2\pi\tilde{\nu}\mathcal{J})(e^1 + ie^2)\ ,
\end{aligned}
\tag{F.44}
$$

and by (C.2)

$$
\begin{aligned}
\exp(-2\pi\nu\mathcal{J})(e^1 \pm ie^2) &= \exp(\pm i2\pi\nu)(e^1 \pm ie^2)\ , \\
\exp(-2\pi\tilde{\nu}\mathcal{J})(e^1 + ie^2) &= \exp(i2\pi\tilde{\nu})(e^1 + ie^2)\ .
\end{aligned}
\tag{F.45}
$$

It follows from (F.44),(F.45) that, for $n \in \mathbb{Z}$,

$$
\begin{aligned}
t(n+1)(e^1 \pm ie^2) &= \exp(\pm i2\pi\nu)A(\phi_0 + 2\pi n\omega)t(n)(e^1 \pm ie^2)\ , \\
\tilde{t}(n+1)(e^1 + ie^2) &= \exp(i2\pi\tilde{\nu})A(\phi_0 + 2\pi n\omega)\tilde{t}(n)(e^1 + ie^2)\ ,
\end{aligned}
$$

whence (F.43) yields

$$
\begin{aligned}
g_\pm(n+1) &= \Big(t(n+1)(e^1 \pm ie^2)\Big)^T\Big(\tilde{t}(n+1)(e^1 + ie^2)\Big) = \exp(i2\pi(\pm\nu + \tilde{\nu})) \\
&\quad \cdot\Big(A(\phi_0 + 2\pi n\omega)t(n)(e^1 \pm ie^2)\Big)^T\Big(A(\phi_0 + 2\pi n\omega)\tilde{t}(n)(e^1 + ie^2)\Big) \\
&= \exp(i2\pi(\pm\nu + \tilde{\nu}))\Big(t(n)(e^1 \pm ie^2)\Big)^T\Big(\tilde{t}(n)(e^1 + ie^2)\Big) \\
&= \exp(i2\pi(\pm\nu + \tilde{\nu}))g_\pm(n)\ .
\end{aligned}
\tag{F.46}
$$

By induction in n I obtain from (F.46) that

$$
g_\pm(n) = \exp(i2\pi n(\pm\nu + \tilde{\nu}))g_\pm(0)\ . \tag{F.47}
$$

To exploit (F.47), I show that either $g_+(0) \neq 0$ or $g_-(0) \neq 0$. In fact, if $g_+(0) = g_-(0) = 0$ then by (F.43) the $11, 12, 21, 22$ matrix elements of $t^T(0)\tilde{t}(0)$ vanish whence

$t^T(0)\tilde{t}(0)$ has zero determinant which is a contradiction to the fact that $t^T(0)\tilde{t}(0) \in SO(3)$. I thus have shown that either $g_+(0) \neq 0$ or $g_-(0) \neq 0$.

I first consider the case when $g_+(0) \neq 0$. Then by (F.47) $g_+(n)/g_+(0) = \exp(i2\pi n(\nu + \tilde{\nu}))$ is an ω-quasiperiodic function of n. Since this function is exponential I can apply Theorem D.5 giving me that $\nu+\tilde{\nu} \in Y_\omega$ whence $\tilde{\nu} \sim_\omega \nu$. In the case when $g_-(0) \neq 0$ I obtain by (F.47) that $g_-(n)/g_-(0) = \exp(i2\pi n(-\nu+\tilde{\nu}))$ is a ω-quasiperiodic function of n. Applying again Theorem D.5, gives me that $-\nu + \tilde{\nu} \in Y_\omega$ whence $\tilde{\nu} \sim_\omega \nu$. Thus I have shown that in any case $\tilde{\nu} \sim_\omega \nu$, which completes the proof. □

F.17 Proof of Theorem 8.7

Proof of Theorem 8.7a: Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$.

Let $\nu \in \Xi_1(\omega, A)$. To prove the first claim, I have to show that $[\nu]_\omega \subset \Xi_1(\omega, A)$. By Definition 7.11, a $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ exists such that $(\omega, \tilde{A}) := R_{d,\omega}(T; \omega, A) \in \mathcal{AT}(d, \omega)$ and $PH(\tilde{A}) = \nu$. For $j \in \mathbb{Z}^d$ I define $T_{\pm,j} \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ by

$$T_{+,j}(\phi) := T(\phi)\exp(-\mathcal{J} j^T \phi) \ , \qquad T_{-,j}(\phi) := T(\phi)\exp(\mathcal{J} j^T \phi)\mathcal{J}' \ ,$$

and abbreviate $(\omega, A_{\pm,j}) := R_{d,\omega}(T_{\pm,j}; \omega, A)$. I obtain by Definition 7.2 that, for $\phi \in \mathbb{R}^d$,

$$\begin{aligned} A_{+,j}(1;\phi) &= T^T_{+,j}(\phi + 2\pi\omega)A(\phi)T_{+,j}(\phi) \\ &= \exp(\mathcal{J} j^T(\phi + 2\pi\omega))T^T(\phi + 2\pi\omega)A(\phi)T(\phi)\exp(-\mathcal{J} j^T\phi) \\ &= \exp(\mathcal{J} j^T(\phi + 2\pi\omega))\tilde{A}(\phi)\exp(-\mathcal{J} j^T\phi) \\ &= \exp(\mathcal{J} j^T(\phi + 2\pi\omega))\exp(\mathcal{J}2\pi\nu)\exp(-\mathcal{J} j^T\phi) = \exp(\mathcal{J}2\pi(\nu + j^T\omega)) \ . \end{aligned} \tag{F.48}$$

It follows from (F.48) and Proposition 7.5c that $(\omega, A_{+,j}) \in \mathcal{AT}(d, \omega)$ and $PH(A_{+,j}) =$

$\lfloor \nu + j^T\omega \rfloor$ whence, by Definition 7.11, $\lfloor \nu + j^T\omega \rfloor \in \Xi_1(\omega, A)$. I also obtain by Definition 7.2 and (7.20),(7.21) that, for $\phi \in \mathbb{R}^d$,

$$\begin{aligned}
A_{-,j}(\phi) &= T_{-,j}^T(\phi + 2\pi\omega)A(\phi)T_{-,j}(\phi) \\
&= \mathcal{J}' \exp(-\mathcal{J} j^T(\phi + 2\pi\omega))T^T(\phi + 2\pi\omega)A(\phi)T(\phi) \exp(\mathcal{J} j^T\phi)\mathcal{J}' \\
&= \mathcal{J}' \exp(-\mathcal{J} j^T(\phi + 2\pi\omega))\tilde{A}(\phi) \exp(\mathcal{J} j^T\phi)\mathcal{J}' \\
&= \mathcal{J}' \exp(-\mathcal{J} j^T(\phi + 2\pi\omega)) \exp(\mathcal{J}2\pi\nu) \exp(\mathcal{J} j^T\phi)\mathcal{J}' \\
&= \mathcal{J}' \exp(\mathcal{J}2\pi(\nu - j^T\omega))\mathcal{J}' = \exp(\mathcal{J}'\mathcal{J}\mathcal{J}'2\pi(\nu - j^T\omega)) = \exp(-\mathcal{J}2\pi(\nu - j^T\omega)) \\
&= \exp(\mathcal{J}2\pi(-\nu + j^T\omega)) \; . \qquad \text{(F.49)}
\end{aligned}$$

It follows from (F.49) and Proposition 7.5c that $(\omega, A_{-,j}) \in \mathcal{AT}(d, \omega)$ and $PH(A_{-,j}) = \lfloor -\nu + j^T\omega \rfloor$ whence, by Definition 7.11, $\lfloor -\nu + j^T\omega \rfloor \in \Xi_1(\omega, A)$.

I thus can summarize that for $\varepsilon \in \{1, -1\}, j \in \mathbb{Z}^d$ I have $\lfloor \varepsilon\nu + j^T\omega \rfloor \in \Xi_1(\omega, A)$. Therefore, using (8.11), I conclude, for $\nu' \in [\nu]_\omega$, that $\nu' \in \Xi_1(\omega, A)$ which proves the first claim.

To prove the second claim, let $y \in ([0, 1) \cap Y_\omega)$. If $y' \sim_\omega y$ then $y' = \varepsilon y + y''$ with $\varepsilon \in \{1, -1\}, y'' \in Y_\omega$. Clearly $y' \in ([0, 1) \cap Y_\omega)$ whence $[y]_\omega \subset ([0, 1) \cap Y_\omega)$. If conversely $y' \in ([0, 1) \cap Y_\omega)$ then $y' = y + (y' - y)$. Since $(y' - y) \in Y_\omega$ I conclude that $y' \in [y]_\omega$ whence $[y]_\omega \supset ([0, 1) \cap Y_\omega)$. This completes the proof of the second claim.

Let $\mu \in \Xi_1(\omega, A) \cap Y_\omega$. Thus $\mu \in ([0, 1) \cap Y_\omega)$ whence, by the second claim, $[\mu]_\omega = ([0, 1) \cap Y_\omega)$. Since $\mu \in \Xi_1(\omega, A)$ we thus get by the first claim that $([0, 1) \cap Y_\omega) = [\mu]_\omega \subset \Xi_1(\omega, A)$. Thus if $\Xi_1(\omega, A) \cap Y_\omega \neq \emptyset$ then $([0, 1) \cap Y_\omega) \subset \Xi_1(\omega, A)$ which proves the third claim. □

Proof of Theorem 8.7b: Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$, let $\phi_0 \in \mathbb{R}^d$ and $\nu \in \Xi_1(\omega, A)$.

By Definition 7.11, a $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ exists such that $(\omega, A') := R_{d,\omega}(T; \omega, A) \in \mathcal{AT}(d, \omega)$ and $PH(A') = \nu$. Thus by Theorem 8.5b an ω–quasiperiodic UPF t exists over ϕ_0 and which has the UPR ν whence $\nu \in \Xi_2(\omega, A, \phi_0)$.

I conclude that the inclusion (8.14) holds.

Since $\nu \in \Xi_2(\omega, A, \phi_0)$ we have by Theorem 8.6 that $[\nu]_\omega = \Xi_2(\omega, A, \phi_0)$. I thus conclude from Theorem 8.7a that

$$\Xi_2(\omega, A, \phi_0) = [\nu]_\omega \subset \Xi_1(\omega, A) \subset \Xi_2(\omega, A, \phi_0) \ ,$$

whence $\Xi_1(\omega, A) = \Xi_2(\omega, A, \phi_0)$. I thus have shown that if $\Xi_1(\omega, A)$ is nonempty, then $\Xi_1(\omega, A) = \Xi_2(\omega, A, \phi_0)$. □

Proof of Theorem 8.7c: Let $(\omega, A) \in \mathcal{SOT}(d, \omega), \phi_0 \in \mathbb{R}^d$ and let $(1, \omega)$ be nonresonant. By the inclusion (8.14) I only have to show that $\Xi_1(\omega, A) \supset \Xi_2(\omega, A, \phi_0)$ so let $\nu \in \Xi_2(\omega, A, \phi_0)$. Thus an ω–quasiperiodic UPF exists over ϕ_0 whose UPR is ν. Applying Theorem 8.5c now gives $\nu \in \Xi_1(\omega, A)$. □

Proof of Theorem 8.7d: Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$ and let $R_{d,\omega}(T; \omega, A) = (\omega, A')$ where $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. Let also $\phi_0 \in \mathbb{R}^d$ and $\nu \in \Xi_2(\omega, A, \phi_0)$, i.e., let there be an ω-quasiperiodic UPF t of (ω, A) over ϕ_0 with UPR ν. I define the function $t' : \mathbb{Z} \to SO(3)$ by $t'(n) := T^T(\phi_0 + 2\pi n\omega)t(n)$. Clearly t' is ω-quasiperiodic. Using Definitions 7.2 and 8.4 I obtain for all integers n

$$\begin{aligned}
& t'^T(n+1)A'(\phi_0 + 2\pi n\omega)t'(n) = \\
& t^T(n+1)T(\phi_0 + 2\pi(n+1)\omega)A'(\phi_0 + 2\pi n\omega)T^T(\phi_0 + 2\pi n\omega)t(n) \\
& = t^T(n+1)A(\phi_0 + 2\pi n\omega)t(n) = \exp(2\pi\nu\mathcal{J}) \ .
\end{aligned} \tag{F.50}$$

Using again Definition 8.4, I obtain from (F.50) that t' is a UPF of (ω, A') over ϕ_0 with UPR ν. Since t' is ω-quasiperiodic I conclude that $\Xi_2(\omega, A, \phi_0) \subset \Xi_2(\omega, A', \phi_0)$. Reversing the roles of A and A' I also obtain that $\Xi_2(\omega, A, \phi_0) \supset \Xi_2(\omega, A', \phi_0)$ whence $\Xi_2(\omega, A, \phi_0) = \Xi_2(\omega, A', \phi_0)$. □

F.18 Proof of Proposition 8.9

Proof of Proposition 8.9a: Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$.

If $(\omega, A) \in \mathcal{ACB}(d, \omega)$ then $\Xi_1(\omega, A)$ is nonempty whence, by Theorem 8.7b, $\Xi_1(\omega, A) = \Xi_2(\omega, A, \phi_0)$ for all ϕ_0 in $\mathbb{R}^d$ so that (ω, A) is well–tuned and the spin tunes of first and second kind are the same.

If $\nu \in \Xi_1(\omega, A)$ then, by Theorem 8.7b, $\nu \in \Xi_2(\omega, A, \phi_0)$ for arbitrary $\phi_0 \in \mathbb{R}^d$ whence, by Theorem 8.6, $\Xi_2(\omega, A, \phi_0) = [\nu]_\omega$. Also, if $\nu \in \Xi_1(\omega, A)$ then $\Xi_1(\omega, A)$ is nonempty whence, by Theorem 8.7b, $\Xi_1(\omega, A) = \Xi_2(\omega, A, \phi_0)$. Thus if $\nu \in \Xi_1(\omega, A)$ then $\Xi_1(\omega, A) = [\nu]_\omega$. The third claim follows from Theorem 8.6. □

Proof of Proposition 8.9b: Let $(\omega, A), (\omega, A') \in \mathcal{SOT}(d, \omega)$ and $(\omega, A) \in \mathcal{ACB}(d, \omega)$.

If $\nu \in \Xi_1(\omega, A) \cap \Xi_1(\omega, A')$ then, by Proposition 8.9a, $\Xi_1(\omega, A) = [\nu]_\omega = \Xi_1(\omega, A')$. Thus either $\Xi_1(\omega, A) \cap \Xi_1(\omega, A') = \emptyset$ or $\Xi_1(\omega, A) = \Xi_1(\omega, A')$. Clearly, in the former case, we have $(\omega, A) \not\sim_{d,\omega} (\omega, A')$ since otherwise, by Proposition 7.12a, I would have $\Xi_1(\omega, A) = \Xi_1(\omega, A')$. In the latter case we have, by Proposition 7.12a, $(\omega, A) \sim_{d,\omega} (\omega, A')$ whence $(\omega, A') \in \mathcal{ACB}(d, \omega)$. □

Proof of Proposition 8.9c: Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$ and let $(1, \omega)$ be nonresonant.

If (ω, A) is well–tuned then, by Theorem 8.7c, $\Xi_1(\omega, A)$ is nonempty whence $(\omega, A) \in \mathcal{ACB}(d, \omega)$. If $(\omega, A) \in \mathcal{ACB}(d, \omega)$ then, by Proposition 8.9a, (ω, A) is well–tuned. I thus have shown that (ω, A) is well–tuned iff $(\omega, A) \in \mathcal{ACB}(d, \omega)$.

If (ω, A) is well–tuned then, by the first claim, $\Xi_1(\omega, A)$ is nonempty whence, by Theorem 8.7b, all $\Xi_2(\omega, A, \phi_0)$ are equal to $\Xi_1(\omega, A)$ where ϕ_0 varies over $\mathbb{R}^d$. Thus $\Xi_1(\omega, A) = \Xi_2(\omega, A)$. □

Proof of Proposition 8.9d: Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$ and $\phi_0 \in \mathbb{R}^d$. If ν is a spin tune of second kind then $\nu \in \Xi_2(\omega, A, \phi_0)$ whence by Theorem 8.5a every spin trajectory over ϕ_0 is (ω, ν)–quasiperiodic. The second claim follows from the first claim and (8.14). □

Proof of Proposition 8.9e: Clearly if (ω, A) is well–tuned then the $\Xi_2(\omega, A, \phi_0)$ have a common element. I now consider the case that the $\Xi_2(\omega, A, \phi_0)$ have a common element ν. Then by Theorem 8.6 for every ϕ_0, $\Xi_2(\omega, A, \phi_0) = [\nu]_\omega$ whence all $\Xi_2(\omega, A, \phi_0)$ are nonempty an equal, i.e., (ω, A) is well–tuned. □

Proof of Proposition 8.9f: By Theorem 8.6 either $\Xi_2(\omega, A, \phi_0)$ is empty or $\Xi_2(\omega, A, \phi_0) = [\nu]_\omega$ for some ν whence $\Xi_2(\omega, A, \phi_0)$ has countably many elements. It follows by Theorem 8.7b that $\Xi_1(\omega, A)$ has countably many elements.

Since each $\Xi_2(\omega, A, \phi_0)$ has countably many elements, it follows for a well-tuned (ω, A) that $\Xi_2(\omega, A)$ has countably many elements. Thus if $\Xi_2(\omega, A)$ has uncountably many elements, then (ω, A) is ill–tuned. □

Proof of Proposition 8.9g: Let $(\omega, A), (\omega, A') \in \mathcal{SOT}(d, \omega)$ with $(\omega, A) \sim_{d,\omega} (\omega, A')$.

If (ω, A) is well-tuned then all $\Xi_2(\omega, A, \phi_0)$ are nonempty and equal whence, by Theorem 8.7d, all $\Xi_2(\omega, A', \phi_0)$ are nonempty and equal. Reversing the roles of A and A' I conclude that either both spin-orbit tori $(\omega, A), (\omega, A')$ are well-tuned or both of them are ill-tuned.

To prove the last claim let $(\omega, A), (\omega, A')$ be well-tuned. Then, by Theorem 8.7d, $\Xi_2(\omega, A) = \Xi_2(\omega, A, \phi_0) = \Xi_2(\omega, A', \phi_0) = \Xi_2(\omega, A')$ where ϕ_0 is any element of $\mathbb{R}^d$. □

F.19 Proof of Proposition 8.10

Proof of Proposition 8.10a: Let (ω, A) be on spin-orbit resonance of first kind. Then $0 \in \Xi_1(\omega, A)$ and, by Proposition 8.9a, (ω, A) is well-tuned and 0 is a spin tune of second kind. Thus (ω, A) is on spin-orbit resonance of second kind.

Let (ω, A) be off spin-orbit resonance of first kind. Then $0 \notin \Xi_1(\omega, A)$ and, by Proposition 8.9a, (ω, A) is well-tuned and 0 is not a spin tune of second kind. Thus (ω, A) is off spin-orbit resonance of second kind. $\square$

Proof of Proposition 8.10b: Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$.

If (ω, A) is on spin-orbit resonance of second kind, then 0 is a spin tune of second kind so that, by Proposition 8.9d, every spin trajectory is $(\omega, 0)$–quasiperiodic whence ω–quasiperiodic.

I now consider the case that every spin trajectory is ω–quasiperiodic. Let $\phi_0 \in \mathbb{R}^d$. Since $\Psi_{\omega,A}(\cdot; \phi_0)$ is ω–quasiperiodic, we have by Remark 2 of Section 8.3 that $\Psi_{\omega,A}(\cdot; \phi_0)$ is an ω–quasiperiodic UPF over ϕ_0 with zero UPR. Thus $0 \in \Xi_2(\omega, A, \phi_0)$ whence, by Proposition 8.9e, (ω, A) is well-tuned and 0 is a spin tune of second kind. Therefore (ω, A) is on spin-orbit resonance of second kind. $\square$

Proof of Proposition 8.10c: Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$. I first consider the case when (ω, A) is on spin-orbit resonance of first kind. Thus $0 \in \Xi_1(\omega, A)$ whence, by Theorem 8.7a and Proposition 8.9a, $\Xi_1(\omega, A) = [0]_\omega = [0, 1) \cap Y_\omega$. On the other hand if $\Xi_1(\omega, A) = [0, 1) \cap Y_\omega$ then $0 \in \Xi_1(\omega, A)$ so that (ω, A) is on spin-orbit resonance of first kind. I thus have shown that (ω, A) is on spin-orbit resonance of first kind iff $\Xi_1(\omega, A) = [0, 1) \cap Y_\omega$.

To prove the second claim let first of all (ω, A) be on spin-orbit resonance of first kind. Then $0 \in \Xi_1(\omega, A)$ whence $m \in \mathbb{Z}^d, n \in \mathbb{Z}$ exist such that (8.15) holds for $\nu = 0$. If conversely $\nu \in \Xi_1(\omega, A)$ and $m \in \mathbb{Z}^d, n \in \mathbb{Z}$ exist such that (8.15) holds then $\nu \in ([0, 1) \cap Y_\omega)$ and, by Theorem 8.7a and Proposition 8.9a, $\Xi_1(\omega, A) = [\nu]_\omega = [0, 1) \cap Y_\omega$ whence, by the first claim, (ω, A) is on spin-orbit resonance of first kind. $\square$

Proof of Proposition 8.10d: Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$. I first consider the case that (ω, A) is on spin-orbit resonance of second kind. Thus 0 is a spin tune of second kind whence, by Theorem 8.7a and Proposition 8.9a, $\Xi_2(\omega, A, \phi_0) = [0]_\omega = [0, 1) \cap Y_\omega$ for all $\phi_0 \in \mathbb{R}^d$. I now consider the case that $\Xi_2(\omega, A, \phi_0) = [0, 1) \cap Y_\omega$ for all $\phi_0 \in \mathbb{R}^d$. Clearly, (ω, A) is well-tuned and $0 \in \Xi_2(\omega, A)$ whence (ω, A) is on spin-orbit resonance of second kind.

To prove the second claim let first of all (ω, A) be on spin-orbit resonance of second kind. Then 0 is a spin tune of second kind whence $m \in \mathbb{Z}^d, n \in \mathbb{Z}$ exist such that (8.15) holds for $\nu = 0$. If conversely ν is a spin tune of second kind and $m \in \mathbb{Z}^d, n \in \mathbb{Z}$ exist such that (8.15) holds then $\nu \in ([0, 1) \cap Y_\omega)$ whence, by Theorem 8.7a and Proposition 8.9a, $\Xi_2(\omega, A) = [\nu]_\omega = [0, 1) \cap Y_\omega$ so that 0 is a spin tune of second kind which implies that (ω, A) is on spin-orbit resonance of second kind. □

Proof of Proposition 8.10e: Let $(\omega, A), (\omega, A') \in \mathcal{SOT}(d, \omega)$ be on spin-orbit resonance of first kind. Thus, by Definition 7.11, $0 \in \Xi_1(\omega, A), \Xi_1(\omega, A')$ whence, by Proposition 8.9a, $\Xi_1(\omega, A) = [0]_\omega = \Xi_1(\omega, A')$ so that, by Proposition 7.12a, $(\omega, A) \sim_{d,\omega} (\omega, A')$. □

Proof of Proposition 8.10f: Let $(\omega, A), (\omega, A') \in \mathcal{SOT}(d, \omega)$ with $(\omega, A) \sim_{d,\omega} (\omega, A')$.

If (ω, A) is on spin-orbit resonance of second kind then 0 is a spin tune of second kind of (ω, A) whence, by Proposition 8.9g, (ω, A') is well-tuned and 0 is a spin tune of second kind of (ω, A') so that (ω, A') is on spin-orbit resonance of second kind. Reversing the roles of A and A' one sees that either both of $(\omega, A), (\omega, A')$ are on spin-orbit resonance of second kind or neither of them.

If (ω, A) is off spin-orbit resonance of second kind then (ω, A) is well-tuned and 0 is not a spin tune of second kind of (ω, A). Thus, by Proposition 8.9g, (ω, A') is well-tuned and 0 is not a spin tune of second kind of (ω, A') so that (ω, A') is off spin-orbit resonance of second kind. Reversing the roles of A and A' we see that either both of $(\omega, A), (\omega, A')$ are off spin-orbit resonance of second kind or neither of them. □

Proof of Proposition 8.10g: Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$ and let $(1, \omega)$ be nonresonant. Let (ω, A) have an ISF $\mathcal{S}_G$ and an ISF which is different from $\mathcal{S}_G$ and $-\mathcal{S}_G$. Then, by Theorem 7.13, (ω, A) is on spin-orbit resonance of first kind. Applying now Proposition 8.10a, one concludes that (ω, A) is on spin-orbit resonance of second kind. □

F.20 Proof of Theorem 8.11

Proof of Theorem 8.11a: ⇒: Let $(\omega, A_1) \in \mathcal{ACB}(d, \omega)$. Thus, by Definition 7.6, a $T' \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ exists such that $(\omega, A_2) := R_{d,\omega}(T'; \omega, A_1) \in \mathcal{AT}(d, \omega)$. Then, by Theorem 7.14b, a $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$ exists such that either $R_{d,\omega}(T; \omega, A_1) = (\omega, A_2)$ or $R_{d,\omega}(T\mathcal{J}'; \omega, A_1) = (\omega, A_2)$. In both cases we have, by Corollary 7.15a, that (7.32),(7.34) hold where $g := PHF(T) \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$.

⇐: Let $M_1 = 0$ and let $g \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$ such that (7.34) holds. Defining $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$ by

$$T(\phi) := \exp(\mathcal{J} 2\pi g(\phi)) \; ,$$

I get from (7.30),(7.32), (7.34), for $\phi \in \mathbb{R}^d$,

$$\begin{aligned} T^T(\phi + 2\pi\omega) A_1(\phi) T(\phi) &= \exp\Big(\mathcal{J}[-2\pi g(\phi + 2\pi\omega)]\Big) \\ &\quad \cdot \exp\Big(\mathcal{J}[2\pi f_1(\phi)]\Big) \exp\Big(\mathcal{J}[2\pi g(\phi)]\Big) \\ &= \exp\Big(\mathcal{J}[2\pi g(\phi) - 2\pi g(\phi + 2\pi\omega) + 2\pi f_1(\phi)]\Big) \\ &= \exp\Big(\mathcal{J}[2\pi g(\phi) - 2\pi g(\phi + 2\pi\omega) + 2\pi \tilde{f}_1(\phi) + 2\pi f_{1,0}]\Big) = \exp(\mathcal{J}[2\pi f_{1,0}]) \; . \end{aligned} \tag{F.51}$$

Thus Definition 7.2 and Proposition 7.5c give $R_{d,\omega}(T;\omega,A_1) \in \mathcal{AT}(d,\omega)$ whence (ω, A_1) is an almost coboundary. □

Proof of Theorem 8.11b: Let $M_1 = 0$ and let $g \in \mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R})$ exist such that (7.34) holds. I pick a $N \in \mathbb{Z}^d$ and define $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$ by (7.29). Defining $(\omega, A_2) := R_{d,\omega}(T;\omega,A_1)$, I obtain from Definition 7.2 and (7.29),(7.30),(7.32), (7.34) that, for $\phi \in \mathbb{R}^d$,

$$\begin{aligned}
A_2(\phi) = T^T(\phi+2\pi\omega)A_1(\phi)T(\phi) &= \exp\Big(\mathcal{J}[-N^T(\phi+2\pi\omega) - 2\pi g(\phi+2\pi\omega)]\Big) \\
&\quad \cdot \exp\Big(\mathcal{J}[2\pi f_1(\phi)]\Big)\exp\Big(\mathcal{J}[N^T\phi + 2\pi g(\phi)]\Big) \\
&= \exp\Big(\mathcal{J}[2\pi g(\phi) - 2\pi g(\phi+2\pi\omega) - 2\pi N^T\omega + 2\pi f_1(\phi)]\Big) \\
&= \exp\Big(\mathcal{J}[2\pi g(\phi) - 2\pi g(\phi+2\pi\omega) - 2\pi N^T\omega + 2\pi \tilde{f}_1(\phi) + 2\pi f_{1,0}]\Big) \\
&= \exp\Big(\mathcal{J}[-2\pi N^T\omega + 2\pi f_{1,0}]\Big) \,,
\end{aligned}$$

whence (8.16) holds and Proposition 7.5c, Definition 7.11 give $(\omega, A_2) \in \mathcal{AT}(d,\omega)$ and $\lfloor -N^T\omega + f_{1,0}\rfloor \in \Xi_1(\omega, A_1)$. Defining $(\omega, A_3) := R_{d,\omega}(T\mathcal{J}';\omega,A_1)$, I obtain from

Definition 7.2 and (7.20),(7.21), (7.29),(7.30),(7.32), (7.34) that, for $\phi \in \mathbb{R}^d$,

$$\begin{aligned}
A_3(\phi) &= (T(\phi+2\pi\omega)\mathcal{J}')^T A_1(\phi)T(\phi)\mathcal{J}' = \mathcal{J}'T^T(\phi+2\pi\omega)A_1(\phi)T(\phi)\mathcal{J}' \\
&= \mathcal{J}' \exp\Big(\mathcal{J}[-N^T(\phi+2\pi\omega) - 2\pi g(\phi+2\pi\omega)]\Big)\exp\Big(\mathcal{J}[2\pi f_1(\phi)]\Big) \\
&\qquad \cdot \exp\Big(\mathcal{J}[N^T\phi + 2\pi g(\phi)]\Big)\mathcal{J}' \\
&= \mathcal{J}' \exp\Big(\mathcal{J}[-N^T(\phi+2\pi\omega) - 2\pi g(\phi+2\pi\omega) + 2\pi f_1(\phi) + N^T\phi + 2\pi g(\phi)]\Big)\mathcal{J}' \\
&= \mathcal{J}' \exp\Big(\mathcal{J}[2\pi g(\phi) - 2\pi g(\phi+2\pi\omega) - 2\pi N^T\omega + 2\pi \tilde{f}_1(\phi) + 2\pi f_{1,0}]\Big)\mathcal{J}' \\
&= \mathcal{J}' \exp\Big(\mathcal{J}[-2\pi N^T\omega + 2\pi f_{1,0}]\Big)\mathcal{J}' \\
&= \exp\Big(\mathcal{J}'\mathcal{J}\mathcal{J}'[-2\pi N^T\omega + 2\pi f_{1,0}]\Big) = \exp\Big(-\mathcal{J}[-2\pi N^T\omega + 2\pi f_{1,0}]\Big) \\
&= \exp\Big(\mathcal{J}[2\pi N^T\omega - 2\pi f_{1,0}]\Big) ,
\end{aligned}$$

whence (8.17) holds and Proposition 7.5c, Definition 7.11 give $(\omega, A_3) \in \mathcal{AT}(d,\omega)$ and $\lfloor N^T\omega - f_{1,0}\rfloor \in \Xi_1(\omega, A_1)$. □

Proof of Theorem 8.11c: Let $(\omega, A_1) \in \mathcal{ACB}(d,\omega)$. Thus, by Proposition 8.9a, (ω, A_1) is well-tuned. Moreover, by Theorem 8.11a, $M_1 = 0$ and a $g \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$ exists such that (7.34) is true for all $\phi \in \mathbb{R}^d$. Thus I can apply Theorem 8.11b so that, by choosing $N := 0 \in \mathbb{Z}^d$, I find $\lfloor f_{1,0}\rfloor \in \Xi_1(\omega, A_1)$. This implies, by Proposition 8.9a, that (8.18) holds. □

F.21 Proof of Corollary 8.12

Proof of Corollary 8.12a: By the transitivity of $\sim_{d,\omega}$ we have $(\omega, A), (\omega, A_1) \in \mathcal{ACB}(d,\omega)$ whence, by Proposition 8.9a, (ω, A) and (ω, A_1) are well-tuned. Since $(\omega, A_1) \in \mathcal{ACB}(d,\omega)$ I obtain from Theorem 8.11c that (8.18) holds. On the other hand, since $(\omega, A) \sim_{d,\omega} (\omega, A_1)$, I obtain from Proposition 7.12a and Theorem 8.7d that

$$\Xi_1(\omega, A) = \Xi_1(\omega, A_1) \;, \qquad \Xi_2(\omega, A) = \Xi_2(\omega, A_1) \;,$$

whence, by (8.18), I conclude that (8.19) holds. □

Proof of Corollary 8.12b: Recalling from the proof of Corollary 8.12a that $(\omega, A_1) \in \mathcal{ACB}(d, \omega)$ I obtain from Theorem 8.11a that $M_1 = 0$ whence Proposition 7.5a gives (8.20). Defining the function $F : \mathbb{Z} \to \mathbb{R}$ by $F(n) := f_1(2\pi n\omega)$ I note that f_1 is an ω-generator of the ω-quasiperiodic function F. We recall from the definition of $a_k(F, 0)$ (with $k = 0, 1, ...$) in Section D.3 that, for $n = 1, 2, ...,$

$$a_{n-1}(F, 0) = \frac{1}{n} \sum_{j=0}^{n-1} F(j) = \frac{1}{n} \sum_{j=0}^{n-1} f_1(2\pi j\omega) \;. \tag{F.52}$$

Since $(1, \omega)$ is nonresonant, Lemma D.4c gives

$$f_{1,0} = \lim_{n \to \infty} a_n(F, 0) =: a(F, 0) \;, \tag{F.53}$$

where in the second equality I used the definition of $a(F, 0)$ from Section D.3. Collecting (F.52), (F.53), I obtain (8.21). □

Proof of Corollary 8.12c: I define the function $t : \mathbb{Z} \to SO(3)$ by $t(n) := T(2\pi n\omega)$. Since $R_{d,\omega}(T; \omega, A) \in \mathcal{WT}(d, \omega)$ I can apply Theorem 8.3a by which t is an ω–quasiperiodic SPF of (ω, A) over $0 \in \mathbb{R}^d$. Since t is an SPF of (ω, A) over $0 \in \mathbb{R}^d$ I can apply (8.6) so that, for $n \in \mathbb{Z}$,

$$\Psi_{\omega,A}(n; 0) = t(n) \exp(\mathcal{J} 2\pi\mu(n)) t^T(0) \;, \tag{F.54}$$

where μ is the integral phase function of t. Using again Theorem 8.3a and noting that, by Corollary 8.12b, $Ind_2(A_1) = M_1 = 0$, I obtain, for $n \in \mathbb{Z}$,

$$\mu(n) = \lfloor f(n, 0) \rfloor \;, \tag{F.55}$$

where $f(n,\cdot) := PHF(\Psi_{\omega,A_1}(n;\cdot))$. Since, by Corollary 8.12b, we have, for $\phi \in \mathbb{R}^d, n = 1,2,...$, that

$$\lfloor f(n,\phi)\rfloor = \lfloor\sum_{j=0}^{n-1} f_1(\phi + 2\pi j\omega)\rfloor \ ,$$

I get from (F.55) that, for $n = 1,2,...$,

$$\mu(n) = \lfloor\sum_{j=0}^{n-1} f_1(2\pi j\omega)\rfloor \ . \tag{F.56}$$

I conclude from (F.54),(F.56) that, for $n = 1,2,...$,

$$\Psi_{\omega,A}(n;0) = t(n)\exp\Big(\mathcal{J}2\pi\lfloor\sum_{j=0}^{n-1} f_1(2\pi j\omega)\rfloor\Big)t^T(0) \ ,$$

whence (8.22) holds for $n = 1,2,...$ which proves the first claim. Note incidentally that by the definition of PHF I have $\lfloor f(n,0)\rfloor = f(n,0)$, but this fact is not needed here since it does not simplify the above argumentation.

To prove the second claim, I define the function $S : \mathbb{Z} \to \mathbb{S}^2$, by $S(n) := \Psi_{\omega,A}(n;0)t(0)e^1$. It is clear by (6.3) that S is a spin trajectory of (ω, A) over $0 \in \mathbb{R}^d$. It follows from (F.54),(C.2) that, for $n \in \mathbb{Z}$,

$$t^T(n)S(n) = t^T(n)\Psi_{\omega,A}(n;0)t(0)e^1 = \exp(\mathcal{J}2\pi\mu(n))e^1 = \begin{pmatrix} \cos(2\pi\mu(n)) \\ \sin(2\pi\mu(n)) \\ 0 \end{pmatrix} ,$$

whence, for $n \in \mathbb{Z}$,

$$(e^1 + ie^2)^T t^T(n)S(n) = (e^1 + ie^2)^T \begin{pmatrix} \cos(2\pi\mu(n)) \\ \sin(2\pi\mu(n)) \\ 0 \end{pmatrix}$$
$$= \cos(2\pi\mu(n)) + i\sin(2\pi\mu(n)) = \exp(i2\pi\mu(n)) \ ,$$

so that, by (F.56), we have, for $n = 1,2,...$,

$$(e^1 + ie^2)^T t^T(n)S(n) = \exp\Big(i2\pi\lfloor\sum_{j=0}^{n-1} f_1(2\pi j\omega)\rfloor\Big) \ ,$$

which implies (8.23). □

F.22 Proof of Proposition 8.14

Proof of Proposition 8.14: Let $(\omega, A) \in \mathcal{WCB}(d,\omega)$ and let me pick a $T' \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ such that $R_{d,\omega}(T';\omega,A) =: (\omega, A') \in \mathcal{WT}(d,\omega)$. I pick any $s' \in \{1,-1\}^d$ and define $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ by

$$T := T' g_d^{(s')} \ . \tag{F.57}$$

Defining $(\omega, A'') := R_{d,\omega}(T;\omega,A)$ I get, by Definition C.14, Theorem 7.3a and (F.57), that, for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d$,

$$\begin{aligned}
&\Psi_{\omega,A''}(n;\phi) = T^T(\phi + 2\pi n\omega)\Psi_{\omega,A}(n;\phi)T(\phi) \\
&= (g_d^{(s')})^T(\phi + 2\pi n\omega)(T')^T(\phi + 2\pi n\omega)\Psi_{\omega,A}(n;\phi)T'(\phi)g_d^{(s')}(\phi) \\
&= (g_d^{(s')})^T(\phi + 2\pi n\omega)\Psi_{\omega,A'}(n;\phi)g_d^{(s')}(\phi) \ .
\end{aligned} \tag{F.58}$$

Since $\Psi_{\omega,A'}(n;\cdot)$ is $SO_3(2)$-valued I conclude from (F.58) that $\Psi_{\omega,A''}(n;\cdot)$ is $SO_3(2)$-valued whence $(\omega, A'') \in \mathcal{WT}(d,\omega)$. Eq. (F.57) and Theorem C.15a give me $Ind_{3,d}(T) = Ind_{3,d}(T'g_d^{(s')}) = Ind_{3,d}(T')Ind_{3,d}(g_d^{(s')})$ whence, by Theorem C.15c, $Ind_{3,d}(T) = Ind_{3,d}(T')s'$. Thus choosing s' appropriately, $Ind_{3,d}(T)$ can assume any value s in $\{1,-1\}^d$ which proves the first claim.

To prove the second claim let $(\omega, A) \in \mathcal{ACB}(d,\omega)$ and let me pick a $T' \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ such that $R_{d,\omega}(T';\omega,A) =: (\omega, A') \in \mathcal{AT}(d,\omega)$. I pick any $s' \in \{1,-1\}^d$ and define $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ by (F.57). Defining $(\omega, A'') := R_{d,\omega}(T;\omega,A)$ I get, by Definition C.14 and (F.58), that, for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d$,

$$\begin{aligned}
&\Psi_{\omega,A''}(n;\phi) = (g_d^{(s')})^T(\phi + 2\pi n\omega)\Psi_{\omega,A'}(n;\phi)g_d^{(s')}(\phi) \\
&= (g_d^{(s')})^T(\phi + 2\pi n\omega)g_d^{(s')}(\phi)\Psi_{\omega,A'}(n;\phi) \\
&= \exp(-\mathcal{J}\pi n\sum_{i=1}^d(1 - s_i')\omega_i)\Psi_{\omega,A'}(n;\phi) \ .
\end{aligned} \tag{F.59}$$

Since $\Psi_{\omega,A'}(n;\phi)$ is in $SO_3(2)$ and independent of ϕ I conclude from (F.59) that $\Psi_{\omega,A''}(n;\phi)$ is in $SO_3(2)$ and independent of ϕ whence $(\omega, A'') \in \mathcal{AT}(d,\omega)$. From the proof of the first claim I know that $Ind_{3,d}(T) = Ind_{3,d}(T')s'$. Therefore choosing s' appropriately, $Ind_{3,d}(T)$ can assume any value t in $\{1,-1\}^d$. Thus, by Definition 8.13, every $\Xi_1^t(\omega, A)$ is nonempty which proves the second claim. □

F.23 Proof of Theorem 8.15

Proof of Theorem 8.15a: Let $(\omega, A) \in \mathcal{ACB}(d,\omega)$ and let $(1,\omega)$ be nonresonant. Let $T_i \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ such that $(\omega, A_i) := R_{d,\omega}(T_i;\omega, A) \in \mathcal{AT}(d,\omega)$ and $\nu_i := PH(A_i)$ where $i = 1,2$. I also abbreviate $s := Ind_{3,d}(T)$ where $T := T_1^T T_2 \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. The proof goes along the lines of the proof of Theorem 7.14b. By Definition 7.2 and Proposition 7.5b, I have, for $\phi \in \mathbb{R}^d$,

$$T_1(\phi + 2\pi\omega)\exp(\mathcal{J}2\pi\nu_1)T_1^T(\phi) = A(\phi) = T_2(\phi + 2\pi\omega)\exp(\mathcal{J}2\pi\nu_2)T_2^T(\phi) \ ,$$

whence, for $\phi \in \mathbb{R}^d$,

$$\exp(\mathcal{J}2\pi\nu_1)T(\phi) = T(\phi + 2\pi\omega)\exp(\mathcal{J}2\pi\nu_2) \ . \tag{F.60}$$

Abbreviating $t := Te^3 \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$, I conclude from (F.60) that, for $\phi \in \mathbb{R}^d$,

$$\exp(\mathcal{J}2\pi\nu_1)t(\phi) = t(\phi + 2\pi\omega) \ . \tag{F.61}$$

Defining, for $j = 1,2,3$, $t_j := t^T e^j \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$ I have, by (F.61), for $\phi \in \mathbb{R}^d$,

$$t_3(\phi) = t_3(\phi + 2\pi\omega) \ . \tag{F.62}$$

Because $(1,\omega)$ is nonresonant I conclude from (F.62) and Corollary D.3a that t_3 is constant so that only the following three cases can occur: Case (i) where $|t_3| < 1$, Case (ii) where $t_3 = 1$, Case (iii) where $t_3 = -1$.

I first consider Case (i). Because the constant $t_0 := \sqrt{1 - t_3^2}$ is nonzero, the function $g_1 : \mathbb{R}^d \to \mathbb{R}^{3\times 3}$, defined by

$$g_1(\phi) := \begin{pmatrix} \frac{t_1(\phi)}{t_0} & -\frac{t_2(\phi)}{t_0} & 0 \\ \frac{t_2(\phi)}{t_0} & \frac{t_1(\phi)}{t_0} & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{F.63}$$

belongs to $\mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$ and satisfies, for $\phi \in \mathbb{R}^d$,

$$t(\phi) = t_1(\phi)e^1 + t_2(\phi)e^2 + t_3 e^3 = g_1(\phi)(t_0 e^1 + t_3 e^3) . \tag{F.64}$$

Combining (F.61) with (F.64) results, for $\phi \in \mathbb{R}^d$, in

$$\exp(\mathcal{J} 2\pi\nu_1) g_1(\phi)(t_0 e^1 + t_3 e^3) = \exp(\mathcal{J} 2\pi\nu_1) t(\phi) = t(\phi + 2\pi\omega)$$
$$= g_1(\phi + 2\pi\omega)(t_0 e^1 + t_3 e^3) ,$$

whence, for $\phi \in \mathbb{R}^d$,

$$\exp(\mathcal{J} 2\pi\nu_1) g_1(\phi) g_1^T(\phi + 2\pi\omega)(t_0 e^1 + t_3 e^3)$$
$$= g_1^T(\phi + 2\pi\omega) \exp(\mathcal{J} 2\pi\nu_1) g_1(\phi)(t_0 e^1 + t_3 e^3) = (t_0 e^1 + t_3 e^3) . \tag{F.65}$$

Since $\exp(\mathcal{J} 2\pi\nu_1) g_1(\phi) g_1^T(\phi + 2\pi\omega)$ is in $SO_3(2)$, I conclude from (F.65) that, for $\phi \in \mathbb{R}^d$,

$$\exp(\mathcal{J} 2\pi\nu_1) g_1(\phi) g_1^T(\phi + 2\pi\omega) t_0 e^1 = t_0 e^1 . \tag{F.66}$$

Using again the fact that $\exp(\mathcal{J} 2\pi\nu_1) g_1(\phi) g_1^T(\phi + 2\pi\omega)$ is in $SO_3(2)$ and since t_0 is nonzero, (F.66) implies that, for $\phi \in \mathbb{R}^d$,

$$\exp(\mathcal{J} 2\pi\nu_1) g_1(\phi) g_1^T(\phi + 2\pi\omega) = I_{3\times 3} . \tag{F.67}$$

Since $t_0 e^1 + t_3 e^3$ is a constant unit vector, a constant matrix $\tilde{t}$ exists in $SO(3)$ such that $\tilde{t} e^3 = t_0 e^1 + t_3 e^3$, whence (F.64) and the definition of t imply

$$g_1 \tilde{t} e^3 = T e^3 . \tag{F.68}$$

Thus and due to Lemma 7.8a I obtain that $T^T g_1 \tilde{t}$ is $SO_3(2)$-valued. I thus can define $g_2 \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$ by

$$g_2 := T^T g_1 \tilde{t} \ . \tag{F.69}$$

Since $T = g_1 \tilde{t} g_2^T$, (F.60) gives me, for $\phi \in \mathbb{R}^d$,

$$\begin{aligned} &\exp(\mathcal{J} 2\pi\nu_1) g_1(\phi) \tilde{t} g_2^T(\phi) = \exp(\mathcal{J} 2\pi\nu_1) T(\phi) = T(\phi + 2\pi\omega) \exp(\mathcal{J} 2\pi\nu_2) \\ &= g_1(\phi + 2\pi\omega) \tilde{t} g_2^T(\phi + 2\pi\omega) \exp(\mathcal{J} 2\pi\nu_2) \ , \end{aligned}$$

whence, for $\phi \in \mathbb{R}^d$,

$$\left(\exp(\mathcal{J} 2\pi\nu_1) g_1(\phi) g_1^T(\phi + 2\pi\omega) \right) \tilde{t} \left(\exp(-\mathcal{J} 2\pi\nu_2) g_2^T(\phi) g_2(\phi + 2\pi\omega) \right) = \tilde{t} \ ,$$

so that, due to (F.67), for $\phi \in \mathbb{R}^d$,

$$\exp(-\mathcal{J} 2\pi\nu_2) g_2^T(\phi) g_2(\phi + 2\pi\omega) = I_{3\times 3} \ . \tag{F.70}$$

Since $g_1 \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$, Definition C.12 gives me, for $\phi \in \mathbb{R}^d$, $g_1(\phi) = \exp(\mathcal{J}[N^T\phi + 2\pi f(\phi)])$ where $N := Ind_{2,d}(g_1)$, $f := PHF(g_1)$. Thus (F.67) gives me, for $\phi \in \mathbb{R}^d$,

$$\exp(\mathcal{J} 2\pi[\nu_1 + f(\phi) - f(\phi + 2\pi\omega) - N^T\omega]) = I_{3\times 3} \ ,$$

whence, by Theorem C.11a,

$$\nu_1 + f(\phi) - f(\phi + \omega) - N^T\omega = M \ , \tag{F.71}$$

where M is a constant integer. Using the 2π-periodicity of f and taking the integral $\int_0^{2\pi} d\phi_1 \cdots \int_0^{2\pi} d\phi_d$ of (F.71) it follows that

$$\nu_1 = N^T\omega + M \ . \tag{F.72}$$

Since (F.67) implies (F.72), analogously (F.70) implies

$$\nu_2 = N'^T\omega + M' \ , \tag{F.73}$$

where M' is a constant integer and $N' := Ind_{2,d}(g_2)$. Eq. (F.72),(F.73) give me, for $\phi \in \mathbb{R}^d$,

$$\nu_1 - \nu_2 = (N - N')^T \omega + M - M' \ , \tag{F.74}$$

whence, by Definition D.1, $(\nu_1 - \nu_2) \in Y_\omega$. To show that $(\nu_1 - \nu_2) \in Y_\omega^s$ I abbreviate $s^i := Ind_{3,d}(g_i)$ where $i = 1, 2$. Then, by (F.69) and Theorem C.15a, we have

$$\begin{aligned} s = Ind_{3,d}(T) &= Ind_{3,d}(g_1 \tilde{t} g_2^T) = Ind_{3,d}(g_1) Ind_{3,d}(\tilde{t}) Ind_{3,d}(g_2^T) \\ &= Ind_{3,d}(g_1) Ind_{3,d}(\tilde{t}) Ind_{3,d}(g_2) = s^1 Ind_{3,d}(\tilde{t}) s^2 \ , \end{aligned}$$

whence, by Definition C.14,

$$s = s^1 Ind_{3,d}(\tilde{t}) s^2 = s^1 (1, ..., 1)^T s^2 = s^1 s^2 \ . \tag{F.75}$$

Since $N = Ind_{2,d}(g_1)$ and $N' = Ind_{2,d}(g_2)$, Theorem C.15b gives me
$s^1 = ((-1)^{N_1}, ..., (-1)^{N_d})^T$ and $s^2 = ((-1)^{N'_1}, ..., (-1)^{N'_d})^T$ whence, by (F.75),

$$s = \left((-1)^{N_1 + N'_1}, ..., (-1)^{N_d + N'_d} \right)^T = \left((-1)^{N_1 - N'_1}, ..., (-1)^{N_d - N'_d} \right)^T \ . \tag{F.76}$$

I conclude from (F.74),(F.76) and Definition 8.13 that $(\nu_1 - \nu_2) \in Y_\omega^s$.

I now consider Case (ii). Because $Te^3 = e^3$ I obtain, due to Lemma 7.8a, that T is $SO_3(2)$-valued. Since $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$, we have, by Definition C.12, that, for $\phi \in \mathbb{R}^d$, $T(\phi) = \exp(\mathcal{J}[\hat{N}^T \phi + 2\pi \hat{f}(\phi)])$ where $\hat{N} := Ind_{2,d}(T)$, $\hat{f} := PHF(T)$. Thus (F.60) gives me, for $\phi \in \mathbb{R}^d$,

$$\exp(\mathcal{J} 2\pi [\nu_1 - \nu_2 + \hat{f}(\phi) - \hat{f}(\phi + 2\pi\omega) - \hat{N}^T \omega]) = I_{3\times 3} \ ,$$

whence, by Theorem C.11a,

$$\nu_1 - \nu_2 + \hat{f}(\phi) - \hat{f}(\phi + 2\pi\omega) - \hat{N}^T \omega = \hat{M} \ , \tag{F.77}$$

where $\hat{M}$ is a constant integer. Using the 2π-periodicity of $\hat{f}$ and taking the integral $\int_0^{2\pi} d\phi_1 \cdots \int_0^{2\pi} d\phi_d$ of (F.77) it follows that

$$\nu_1 - \nu_2 = \hat{N}^T \omega + \hat{M} \ . \tag{F.78}$$

Thus, by Definition D.1, $(\nu_1 - \nu_2) \in Y_\omega$. To show that $(\nu_1 - \nu_2) \in Y_\omega^s$ I recall that $\hat{N} = Ind_{2,d}(T)$ whence, by Theorem C.15b, $s = Ind_{3,d}(T) = ((-1)^{\hat{N}_1}, ..., (-1)^{\hat{N}_d})$. I thus conclude, by (F.78) and Definition 8.13, that $(\nu_1 - \nu_2) \in Y_\omega^s$.

I now consider Case (iii). Because $Te^3 = -e^3$, due to Lemma 7.8a, I obtain that $T\mathcal{J}'$ is $SO_3(2)$-valued, where $\mathcal{J}'$ is given by (7.20). Since $T\mathcal{J}' \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$, we have, by Definition C.12, that, for $\phi \in \mathbb{R}^d$, $T(\phi)\mathcal{J}' = \exp(\mathcal{J}[\check{N}^T\phi + 2\pi\check{f}(\phi)])$ where $\check{N} := Ind_{2,d}(T\mathcal{J}')$, $\check{f} := PHF(T\mathcal{J}')$. Thus (7.21),(F.60) give me, for $\phi \in \mathbb{R}^d$,

$$\begin{aligned}
&\exp(\mathcal{J}[2\pi\nu_1 + 2\pi\check{f}(\phi) + \check{N}^T\phi])\mathcal{J}' = \exp(\mathcal{J}2\pi\nu_1)T(\phi)\mathcal{J}'\mathcal{J}' = \exp(\mathcal{J}2\pi\nu_1)T(\phi) \\
&= T(\phi + 2\pi\omega)\exp(\mathcal{J}2\pi\nu_2) = T(\phi + 2\pi\omega)\mathcal{J}'\mathcal{J}'\exp(\mathcal{J}2\pi\nu_2) \\
&= \exp(\mathcal{J}[2\pi\check{f}(\phi + 2\pi\omega) + \check{N}^T\phi + 2\pi\check{N}^T\omega])\mathcal{J}'\exp(\mathcal{J}2\pi\nu_2) \\
&= \exp(\mathcal{J}[2\pi\check{f}(\phi + 2\pi\omega) + \check{N}^T\phi + 2\pi\check{N}^T\omega])\mathcal{J}'\exp(\mathcal{J}2\pi\nu_2)\mathcal{J}'\mathcal{J}' \\
&= \exp(\mathcal{J}[2\pi\check{f}(\phi + 2\pi\omega) + \check{N}^T\phi + 2\pi\check{N}^T\omega])\exp(\mathcal{J}'\mathcal{J}\mathcal{J}'2\pi\nu_2)\mathcal{J}' \\
&= \exp(\mathcal{J}[2\pi\check{f}(\phi + 2\pi\omega) + \check{N}^T\phi + 2\pi\check{N}^T\omega])\exp(-\mathcal{J}2\pi\nu_2)\mathcal{J}' ,
\end{aligned}$$

whence, for $\phi \in \mathbb{R}^d$,

$$\exp(\mathcal{J}2\pi[\nu_1 + \nu_2 + \check{f}(\phi) - \check{f}(\phi + 2\pi\omega) - \check{N}^T\omega]) = I_{3\times 3} ,$$

so that, by Theorem C.11a, for $\phi \in \mathbb{R}^d$,

$$\nu_1 + \nu_2 + \check{f}(\phi) - \check{f}(\phi + 2\pi\omega) - \check{N}^T\omega = \check{M} , \tag{F.79}$$

where $\check{M}$ is a constant integer. Using the 2π-periodicity of $\check{f}$ and taking the integral $\int_0^{2\pi} d\phi_1 \cdots \int_0^{2\pi} d\phi_d$ of (F.79) it follows that

$$\nu_1 + \nu_2 = \check{N}^T\omega + \check{M} . \tag{F.80}$$

Thus, by Definition D.1, $(\nu_1 + \nu_2) \in Y_\omega$. To show that $(\nu_1 + \nu_2) \in Y_\omega^s$ we recall that $\check{N} = Ind_{2,d}(T\mathcal{J}')$ whence, by Theorem C.15b,

$$Ind_{3,d}(T\mathcal{J}') = ((-1)^{\check{N}_1}, ..., (-1)^{\check{N}_d})^T . \tag{F.81}$$

By Theorem C.15a I have $Ind_{3,d}(T\mathcal{J}') = Ind_{3,d}(T)Ind_{3,d}(\mathcal{J}') = sInd_{3,d}(\mathcal{J}')$, whence, by Definition C.14, $Ind_{3,d}(T\mathcal{J}') = sInd_{3,d}(\mathcal{J}') = s(1,...,1)^T = s$, so that, by (F.81), $s = ((-1)^{\check{N}_1}, ..., (-1)^{\check{N}_d})^T$. I thus conclude by (F.80) and Definition 8.13 that $(\nu_1 + \nu_2) \in Y_\omega^s$. □

Proof of Theorem 8.15b: Let $(\omega, A) \in \mathcal{ACB}(d, \omega)$ and let $\nu \in \Xi_1^{(1,...,1)}(\omega, A)$ (such a ν exists by Proposition 8.14). Thus a $T_2 \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ exists such that $(\omega, A_2) := R_{d,\omega}(T_2; \omega, A) \in \mathcal{AT}(d, \omega)$ with $\nu = PH(A_2)$ and $Ind_{3,d}(T_2) = (1, ..., 1)^T$. Let $s \in \{1, -1\}^d$ and $\nu' \in \Xi_1^s(\omega, A)$. Thus a $T_1 \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ exists such that $(\omega, A_1) := R_{d,\omega}(T_1; \omega, A) \in \mathcal{AT}(d, \omega)$ with $\nu' = PH(A_1)$ and $Ind_{3,d}(T_1) = s$. By Theorem C.15a I have

$$Ind_{3,d}(T_1^T T_2) = Ind_{3,d}(T_1^T)Ind_{3,d}(T_2) = Ind_{3,d}(T_1)Ind_{3,d}(T_2) = s(1, ..., 1)^T = s \ ,$$

whence, by Theorem 8.15a, I conclude that either $(\nu' - \nu) \in Y_\omega^s$ or $(\nu' + \nu) \in Y_\omega^s$. Thus a $y \in Y_\omega^s$ exists such that either $\nu' = \nu + y$ or $\nu' = -\nu + y$ which proves the claim. □

Proof of Theorem 8.15c: Let $\Xi_1(\omega, A) \cap Y_\omega^{half} = \emptyset$ and $s, s' \in \{1, -1\}^d$ with $s \neq s'$. If $\Xi_1(\omega, A) = \emptyset$ then the claim is trivial so let me assume that $(\omega, A) \in \mathcal{ACB}(d, \omega)$. Thus, by Proposition 8.14, I can pick a ν in $\Xi_1^{(1,...,1)}(\omega, A)$. It follows from Theorem 8.15b that if the set Y, defined by

$$Y := \{\varepsilon\nu + y : y \in Y_\omega^s, \varepsilon \in \{1, -1\}\} \cap \{\varepsilon'\nu + y' : y' \in Y_\omega^{s'}, \varepsilon' \in \{1, -1\}\} \ , \tag{F.82}$$

is empty, then $\Xi_1^s(\omega, A) \cap \Xi_1^{s'}(\omega, A) = \emptyset$ which proves the claim.

Thus I am done if I show that Y is empty. I show this by contraposition, so let's assume that $Y \neq \emptyset$. Then, due to (F.82), $\varepsilon, \varepsilon' \in \{1, -1\}$ and $x \in Y_\omega^s, x' \in Y_\omega^{s'}$ exist such that $\varepsilon\nu + x = \varepsilon'\nu + x'$ whence, by Definition 8.13, $j, j' \in \mathbb{Z}, m, m' \in \mathbb{Z}^d$ exist such that

$$\varepsilon\nu + j + m^T\omega = \varepsilon'\nu + j' + m'^T\omega \ , \tag{F.83}$$

$$s = ((-1)^{m_1}, ..., (-1)^{m_d})^T \ , \qquad s' = ((-1)^{m'_1}, ..., (-1)^{m'_d})^T \ . \tag{F.84}$$

Note that, due to (F.84), $(1, ..., 1)^T \neq s'/s = ((-1)^{m'_1 - m_1}, ..., (-1)^{m'_d - m_d})^T$ whence, by Definition 8.13, we have, for every integer n,

$$\frac{(m'-m)^T\omega + n}{2}, -\frac{(m'-m)^T\omega + n}{2} \in Y_\omega^{half} . \tag{F.85}$$

In the case $\varepsilon = \varepsilon'$, (F.83) gives me $(m'-m)^T\omega + j' - j = 0$ so that, since $(1, \omega)$ is nonresonant, $m = m', j = j'$ which, by (F.84), leads me to the contradiction that $s = s'$. In the case $\varepsilon = -\varepsilon'$, eq. (F.83) gives me $2\varepsilon\nu = (m'-m)^T\omega + j' - j$ whence $2\nu = \varepsilon[(m'-m)^T\omega + j' - j]$, so that, by (F.85), $\nu \in Y_\omega^{half}$ which contradicts that $\nu \in \Xi_1(\omega, A)$ and $\Xi_1(\omega, A) \cap Y_\omega^{half} = \emptyset$. This completes the proof that the assumption $Y \neq \emptyset$ is wrong. □

Proof of Theorem 8.15d: Let (ω, A) have an ISF $\mathcal{S}_G$ and let it also have an ISF which is different from $\mathcal{S}_G$ and $-\mathcal{S}_G$. It follows from Theorem 7.13 that (ω, A) is on spin-orbit resonance of first kind. Thus $\Xi_1(\omega, A) \neq \emptyset$ and, by Proposition 8.10c, $\Xi_1(\omega, A) \subset Y_\omega$. Since $(1, \omega)$ is nonresonant, Definition 8.13 gives me $Y_\omega^{half} \cap Y_\omega = \emptyset$, whence $\Xi_1(\omega, A) \cap Y_\omega^{half} = \emptyset$. Theorem 8.15c now implies that $\Xi_1^s(\omega, A) \cap \Xi_1^t(\omega, A) = \emptyset$ if $s \neq t$. □

Proof of Theorem 8.15e: The claim is trivial if $\Xi_1(\omega, A) \cap Y_\omega^{half} = \emptyset$ so let $\Xi_1(\omega, A) \cap Y_\omega^{half} \neq \emptyset$ and let me pick a $\nu_1 \in \Xi_1(\omega, A) \cap Y_\omega^{half}$. Let $\nu_2 \in \Xi_1(\omega, A)$. Thus I am done if the show that $\nu_2 \in Y_\omega^{half}$. Since $\nu_1 \in Y_\omega^{half}$, Definition 8.13 gives me $2\nu_1 = j + m^T\omega$, where $j \in \mathbb{Z}, m \in \mathbb{Z}^d$ and $((-1)^{m_1}, ..., (-1)^{m_d}) \neq (1, ..., 1)$. Because, by Theorem 8.15a, either $\nu_1 - \nu_2$ or $\nu_1 + \nu_2$ is in Y_ω, it follows that $k \in \mathbb{Z}, n \in \mathbb{Z}^d, \varepsilon \in \{1, -1\}$ exist such that $\nu_2 = \varepsilon\nu_1 + k + n^T\omega$ whence

$$\nu_2 = \varepsilon\frac{j + m^T\omega}{2} + k + n^T\omega = \frac{\varepsilon j + 2k + (\varepsilon m + 2n)^T\omega}{2} . \tag{F.86}$$

Clearly $((-1)^{\varepsilon m_1 + 2n_1}, ..., (-1)^{\varepsilon m_d + 2n_d}) = ((-1)^{m_1}, ..., (-1)^{m_d}) \neq (1, ..., 1)$ whence, by (F.86) and Definition 8.13, $\nu_2 \in Y_\omega^{half}$. □

F.24 Proof of Lemma 8.16

Proof of Lemma 8.16: Let $G \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$ be of class C^1 and let $\omega \in \mathbb{R}^d$. The following proof is a simple suspension argument of the subgroup $\mathbb{Z}$ of $\mathbb{R}$. Defining the function $\Omega \in \mathcal{C}(\mathbb{R}^{d+1}, \mathbb{R}^3)$, for $\theta \in \mathbb{R}, \phi \in \mathbb{R}^d$, by

$$\Omega(\theta, \phi) := G(\phi + \theta\omega) \times (\omega^T \nabla) G(\phi + \theta\omega) \ , \tag{F.87}$$

where ∇ is the gradient on $\mathbb{R}^d$, I consider the following family of initial value problems:

$$\dot{S}(\theta) = \Omega(\theta, \phi) \times S(\theta) \ , \tag{F.88}$$

$$S(0) \in \mathbb{R}^3, \phi \in \mathbb{R}^d \ . \tag{F.89}$$

Since the ODE (F.88) is linear in S and since Ω is continuous and $\Omega(\theta, \cdot)$ is 2π-periodic, there exists [Am] a function $\Phi \in \mathcal{C}(\mathbb{R}^{d+1}, \mathbb{R}^{3\times 3})$ such that $\Phi(\theta; \cdot)$ is 2π-periodic and such that (F.88),(F.89) are solved by

$$S(\theta) = \Phi(\theta; \phi) S(0) \ . \tag{F.90}$$

Defining, for $\phi \in \mathbb{R}^d$,

$$A(\phi) := \Phi(2\pi; \phi) \ , \tag{F.91}$$

one sees that $(\omega, A) \in \mathcal{SOT}(d, \omega)$ if $\Psi_{\omega,A}(n; \phi)$ is defined in terms of $A(\phi)$ by (6.4). Clearly G is the generator of a spin field $\mathcal{S}_G$ of (ω, A).

I am thus done if I show that this spin field is invariant. I now consider for $\phi \in \mathbb{R}^d$ the function $S_\phi : \mathbb{R} \to \mathbb{R}^3$ by $S_\phi(\theta) := G(\phi + \theta\omega)$. Clearly S_ϕ is of class C^1 and from (F.87) I obtain, for $\theta \in \mathbb{R}$,

$$\begin{aligned} \dot{S}_\phi(\theta) &= (\omega^T \nabla) G(\phi + \theta\omega) = \Big(G(\phi + \theta\omega) \times (\omega^T \nabla) G(\phi + \theta\omega) \Big) \times G(\phi + \theta\omega) \\ &= \Omega(\theta, \phi) \times G(\phi + \theta\omega) = \Omega(\theta, \phi) \times S_\phi(\theta) \ , \end{aligned} \tag{F.92}$$

where in the second equality I used the fact that

$$G^T(\omega^T \nabla)G = 0 \; . \tag{F.93}$$

Note that (F.93) holds because G is of class C^1 and $\mathbb{S}^2$ valued. It follows from (F.92) that S_ϕ solves the initial value problem (F.88),(F.89) for $S(0) = S_\phi(0) = G(\phi)$. It thus follows from (F.90) that, for $\theta \in \mathbb{R}, \phi \in \mathbb{R}^d$,

$$G(\phi + \theta\omega) = S_\phi(\theta) = \Phi(\theta;\phi)S_\phi(0) = \Phi(\theta;\phi)G(\phi) \; , \tag{F.94}$$

whence, by (F.91), for $\phi \in \mathbb{R}^d$,

$$G(\phi + 2\pi\omega) = \Phi(2\pi;\phi)G(\phi) = A(\phi)G(\phi) \; . \tag{F.95}$$

I conclude from (F.95) and Proposition 6.3 that the spin field $\mathcal{S}_G$ is invariant. □

F.25 Proof of Theorem 8.17

Proof of Theorem 8.17: Let ω be in $\mathbb{R}^d$ such that $(1,\omega)$ is nonresonant and $d \geq 2$. Then, by Theorem C.24c, a function $G \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$ exists which is of class C^∞ but which has no 2π-periodic lifting w.r.t. $(SO(3), p_3, \mathbb{S}^2)$, i.e., no $T' \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ exists whose third column is G. On the other hand it follows from Lemma 8.16 that a $(\omega, A) \in \mathcal{SOT}(d,\omega)$ exists for which G is the generator of an ISF $\mathcal{S}_G$.

I now prove, by contraposition, that $(\omega, A) \notin \mathcal{WCB}(d,\omega)$. Thus let's assume that there is a $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ such that $R_{d,\omega}(T;\omega,A) \in \mathcal{WT}(d,\omega)$. Clearly $R_{d,\omega}(T\mathcal{J}';\omega,A) \in \mathcal{WT}(d,\omega)$, too, where $\mathcal{J}'$ is defined by (7.20). Note that the third column of $T\mathcal{J}'$ is $-g$ where g denotes the third column of T. By Theorem 7.9, g and $-g$ are generators of ISF's of (ω, A). Clearly $G \neq g, G \neq -g$ since otherwise G would be the third column of T or $T\mathcal{J}'$. Since $(1,\omega)$ is nonresonant, it follows from the proof of Theorem 7.13 that a $T'' \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ exists such that G is the third column of T''. This is a contradiction whence $(\omega, A) \notin \mathcal{WCB}(d,\omega)$.

I now prove, by contraposition, that $\mathcal{S}_G$ and $-\mathcal{S}_G$ are the only ISF's of (ω, A). Thus let's assume that (ω, A) has an ISF $\mathcal{S}_H$ such that $H \neq G, H \neq -G$. Then, by Theorem 7.13, (ω, A) is on spin-orbit resonance of first kind whence I arrive at the contradiction that $(\omega, A) \in \mathcal{ACB}(d, \omega) \subset \mathcal{WCB}(d, \omega)$. □

F.26 Proof of Proposition 9.1

Proof of Proposition 9.1: Let (ω, A) be a d-dimensional spin-orbit torus. By (6.9),(6.14), I have, for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d, S \in \mathbb{R}^3$,

$$L_{\omega,A}(n; \phi, S) = \begin{pmatrix} L_\omega(n; \phi) \\ \Psi_{\omega,A}(n; \phi) S \end{pmatrix} . \tag{F.96}$$

It follows from (6.14),(F.96) and by the definition of h that

$$h(L_{\omega,A}(n; \phi, S)) = L_\omega(n; \phi) = L_\omega(n; h(\phi_1, ..., \phi_d, S)) , \tag{F.97}$$

where $n \in \mathbb{Z}, \phi \in \mathbb{R}^d, S \in \mathbb{R}^3$. Since h is continuous and recalling from Section 6.2 that $(\mathbb{R}^{d+3}, L_{\omega,A})$ is a topological $\mathbb{Z}$-space, I conclude from (F.97) that h is a $\mathbb{Z}$-map from the topological $\mathbb{Z}$-space $(\mathbb{R}^{d+3}, L_{\omega,A})$ to the topological $\mathbb{Z}$-space $(\mathbb{R}^d, L_\omega)$. Since h is also a projection onto the first d-components of $\mathbb{R}^{d+3}$ I thus conclude that the topological $\mathbb{Z}$-space $(\mathbb{R}^{d+3}, L_{\omega,A})$ is a skew product of the topological $\mathbb{Z}$-space $(\mathbb{R}^d, L_\omega)$. □

F.27 Proof of Proposition 9.2

Proof of Proposition 9.2a: It follows from (9.7) that, for $z \in \mathbb{T}^d$,

$$L_\omega^{(T)}(0; z) = z , \tag{F.98}$$

and, for $m, n \in \mathbb{Z}, z \in \mathbb{T}^d$,

$$L_\omega^{(T)}(n; L_\omega^{(T)}(m; z)) = L_\omega^{(T)}(n; \exp(i2\pi m\omega_1)z_1, ..., \exp(i2\pi m\omega_d)z_d)$$
$$= \Big(\exp(i2\pi(m+n)\omega_1)z_1, ..., \exp(i2\pi(m+n)\omega_d)z_d\Big)^T = L_\omega^{(T)}(m+n; z) \ . \tag{F.99}$$

It follows from (F.98),(F.99) that $L_\omega^{(T)}$ is a $\mathbb{Z}$-action on $\mathbb{T}^d$. Since $L_\omega^{(T)}(n; \cdot)$ is continuous, $(\mathbb{T}^d, L_\omega^{(T)})$ is a topological $\mathbb{Z}$-space. I also recall that $(\mathbb{R}^d, L_\omega)$ is a topological $\mathbb{Z}$-space. Using (6.14),(9.7) and the definition of $p_{4,d}$ I get

$$p_{4,d}(L_\omega(n; \phi)) = p_{4,d}(\phi + 2\pi n\omega)$$
$$= \Big(\exp(i[\phi_1 + 2\pi n\omega_1]), ..., \exp(i[\phi_d + 2\pi n\omega_d])\Big)^T$$
$$= \Big(\exp(i2\pi n\omega_1)\exp(i\phi_1), ..., \exp(i2\pi n\omega_d)\exp(i\phi_d)\Big)$$
$$= L_\omega^{(T)}(n; \exp(i\phi_1), ..., \exp(i\phi_d)) = L_\omega^{(T)}(n; p_{4,d}(\phi)) \ , \tag{F.100}$$

where $n \in \mathbb{Z}, \phi \in \mathbb{R}^d$. It follows from (F.100) that $p_{4,d}$ is a $\mathbb{Z}$-map from the topological $\mathbb{Z}$-space $(\mathbb{R}^d, L_\omega)$ to the topological $\mathbb{Z}$-space $(\mathbb{T}^d, L_\omega^{(T)})$. Clearly $p_{4,d}$ is continuous. Since $p_{4,d}$ is also onto $\mathbb{T}^d$ I thus conclude that the topological $\mathbb{Z}$-space $(\mathbb{R}^d, L_\omega)$ is an extension of the topological $\mathbb{Z}$-space $(\mathbb{T}^d, L_\omega^{(T)})$. □

Proof of Proposition 9.2b: By (6.11),(9.6) we have, for $z \in \mathbb{T}^d$,

$$\Psi'_{\omega,A}(0; z) = I_{3\times 3} \ . \tag{F.101}$$

By (9.8),(F.98),(F.101) we have, for $z \in \mathbb{T}^d, S \in \mathbb{R}^3$,

$$L_{\omega,A}^{(T)}(0; z, S) = \begin{pmatrix} z \\ S \end{pmatrix} \ . \tag{F.102}$$

By (6.6),(6.14),(9.6), (F.100) we have, for $m, n \in \mathbb{Z}, \phi \in \mathbb{R}^d$,

$$\begin{aligned}
&\Psi'_{\omega,A}(n+m; p_{4,d}(\phi)) = \Psi_{\omega,A}(n+m;\phi) = \Psi_{\omega,A}(n;\phi+2\pi m\omega)\Psi_{\omega,A}(m;\phi) \\
&= \Psi'_{\omega,A}(n; p_{4,d}(\phi+2\pi m\omega))\Psi'_{\omega,A}(m; p_{4,d}(\phi)) \\
&= \Psi'_{\omega,A}(n; p_{4,d}(L_\omega(m;\phi)))\Psi'_{\omega,A}(m; p_{4,d}(\phi)) \\
&= \Psi'_{\omega,A}(n; L^{(T)}_\omega(m; p_{4,d}(\phi)))\Psi'_{\omega,A}(m; p_{4,d}(\phi)) \ .
\end{aligned} \tag{F.103}$$

Since $p_{4,d}$ is onto $\mathbb{T}^d$ we have by (F.103), and for $m, n \in \mathbb{Z}, z \in \mathbb{T}^d$,

$$\Psi'_{\omega,A}(n+m; z) = \Psi'_{\omega,A}(n; L^{(T)}_\omega(m; z))\Psi'_{\omega,A}(m; z) \ . \tag{F.104}$$

By (9.8),(F.99),(F.104) we have, for $z \in \mathbb{T}^d, S \in \mathbb{R}^3, m, n \in \mathbb{Z}$

$$\begin{aligned}
&L^{(T)}_{\omega,A}\Big(n; L^{(T)}_{\omega,A}(m; z, S)\Big) = L^{(T)}_{\omega,A}\Big(n; L^{(T)}_\omega(m; z), \Psi'_{\omega,A}(m; z)S\Big) \\
&= \begin{pmatrix} L^{(T)}_\omega(n; L^{(T)}_\omega(m; z)) \\ \Psi'_{\omega,A}(n; L^{(T)}_\omega(m; z))\Psi'_{\omega,A}(m; z)S \end{pmatrix} = \begin{pmatrix} L^{(T)}_\omega(n+m; z) \\ \Psi'_{\omega,A}(n+m; z)S \end{pmatrix} \\
&= L^{(T)}_{\omega,A}(m+n; z, S) \ .
\end{aligned} \tag{F.105}$$

It follows from (F.102),(F.105) that $L^{(T)}_{\omega,A}$ is a $\mathbb{Z}$-action on $\mathbb{T}^d \times \mathbb{R}^3$. Choosing the product topology on $\mathbb{T}^d \times \mathbb{R}^3$ and using the fact that $\Psi'_{\omega,A}(n; \cdot), L^{(T)}_\omega(n; \cdot)$ are continuous functions I find by (9.8) that $L^{(T)}_{\omega,A}(n; \cdot)$ is continuous whence $(\mathbb{T}^d \times \mathbb{R}^3, L^{(T)}_{\omega,A})$ is a topological $\mathbb{Z}$-space. It follows from (F.96),(9.6),(9.8), (F.100) that for $\phi \in \mathbb{R}^d, S \in \mathbb{R}^3, n \in \mathbb{Z}$

$$\begin{aligned}
&p_{5,d}(L_{\omega,A}(n;\phi,S)) = p_{5,d}\Big(L_\omega(n;\phi), \Psi_{\omega,A}(n;\phi)S\Big) = \begin{pmatrix} p_{4,d}(L_\omega(n;\phi)) \\ \Psi_{\omega,A}(n;\phi)S \end{pmatrix} \\
&= \begin{pmatrix} L^{(T)}_\omega(n; p_{4,d}(\phi)) \\ \Psi_{\omega,A}(n;\phi)S \end{pmatrix} = \begin{pmatrix} L^{(T)}_\omega(n; p_{4,d}(\phi)) \\ \Psi'_{\omega,A}(n; p_{4,d}(\phi))S \end{pmatrix} = L^{(T)}_{\omega,A}(n; p_{4,d}(\phi), S) \\
&= L^{(T)}_{\omega,A}(n; p_{5,d}(\phi, S)) \ .
\end{aligned} \tag{F.106}$$

It follows from (F.106) that $p_{5,d}$ is a $\mathbb{Z}$-map from the topological $\mathbb{Z}$-space $(\mathbb{R}^{d+3}, L_{\omega,A})$ to the topological $\mathbb{Z}$-space $(\mathbb{T}^d \times \mathbb{R}^3, L^{(T)}_{\omega,A})$. Clearly $p_{5,d}$ is continuous. Since $p_{5,d}$ is is also onto $\mathbb{T}^d \times \mathbb{R}^3$ I thus conclude that the topological $\mathbb{Z}$-space $(\mathbb{R}^{d+3}, L_{\omega,A})$ is an extension of the topological $\mathbb{Z}$-space $(\mathbb{T}^d \times \mathbb{R}^3, L^{(T)}_{\omega,A})$. $\square$

Proof of Proposition 9.2c: Let (ω, A) be a d-dimensional spin-orbit torus and let $(\mathbb{T}^d \times \mathbb{R}^3, L)$ be a topological $\mathbb{Z}$-space. Let also the function $p_{5,d}$ be a $\mathbb{Z}$-map from the topological $\mathbb{Z}$-space $(\mathbb{R}^{d+3}, L_{\omega,A})$ to the topological $\mathbb{Z}$-space $(\mathbb{T}^d \times \mathbb{R}^3, L)$. Thus by (9.5),(F.106) we have, for $\phi \in \mathbb{R}^d, S \in \mathbb{R}^3, n \in \mathbb{Z}$,

$$L^{(T)}_{\omega,A}(n; p_{4,d}(\phi), S) = L^{(T)}_{\omega,A}(n; p_{5,d}(\phi, S))) = p_{5,d}(L_{\omega,A}(n; \phi, S)) = L(n; p_{5,d}(\phi, S)))$$
$$= L(n; p_{4,d}(\phi), S) \ . \tag{F.107}$$

Since $p_{4,d}$ is onto $\mathbb{T}^d$ we have, by (F.107), that $L = L^{(T)}_{\omega,A}$. $\square$

Proof of Proposition 9.2d: Let $(\mathbb{R}^{d+3}, L)$ be a topological $\mathbb{Z}$-space, let (ω, A) be a d-dimensional spin-orbit torus, and let the function $p_{5,d}$ be a $\mathbb{Z}$-map from the topological $\mathbb{Z}$-space $(\mathbb{R}^{d+3}, L)$ to the topological $\mathbb{Z}$-space $(\mathbb{T}^d \times \mathbb{R}^3, L^{(T)}_{\omega,A})$. Thus, for $\phi \in \mathbb{R}^d, S \in \mathbb{R}^3, n \in \mathbb{Z}$, we have, by (9.5),

$$p_{5,d}(L(n; \phi, S)) = L^{(T)}_{\omega,A}(n; p_{5,d}(\phi, S)) = L^{(T)}_{\omega,A}(n; p_{4,d}(\phi), S) \ . \tag{F.108}$$

Abbreviating, for $\phi \in \mathbb{R}^d, S \in \mathbb{R}^3, n \in \mathbb{Z}$,

$$L(n; \phi, S) =: \begin{pmatrix} L_{orb}(n; \phi, S) \\ L_{spin}(n; \phi, S) \end{pmatrix} \ , \tag{F.109}$$

I get from (9.5),(F.106),(F.108), for $\phi \in \mathbb{R}^d, S \in \mathbb{R}^3, n \in \mathbb{Z}$,

$$\begin{pmatrix} p_{4,d}(L_{orb}(n; \phi, S)) \\ L_{spin}(n; \phi, S) \end{pmatrix} = p_{5,d}(L(n; \phi, S)) = L^{(T)}_{\omega,A}(n; p_{4,d}(\phi), S)$$
$$= \begin{pmatrix} L^{(T)}_{\omega}(n; p_{4,d}(\phi)) \\ \Psi_{\omega,A}(n; \phi) S \end{pmatrix}$$

whence, for $\phi \in \mathbb{R}^d, S \in \mathbb{R}^3, n \in \mathbb{Z}$,

$$p_{4,d}(L_{orb}(n; \phi, S)) = L^{(T)}_{\omega}(n; p_{4,d}(\phi)) \ , \tag{F.110}$$

$$L_{spin}(n; \phi, S) = \Psi_{\omega,A}(n; \phi) S \ . \tag{F.111}$$

Note that (F.111) determines L_{spin}. To investigate L_{orb} I use (F.100),(F.110) to get, for $\phi \in \mathbb{R}^d, S \in \mathbb{R}^3, n \in \mathbb{Z}$,

$$p_{4,d}(L_{orb}(n; \phi, S)) = L_\omega^{(T)}(n; p_{4,d}(\phi)) = p_{4,d}(L_\omega(n; \phi)) \ . \qquad \text{(F.112)}$$

Since, for every $n \in \mathbb{Z}$, the functions $L_{orb}(n; \cdot)$ and $L_\omega(n; \cdot)$ are continuous I conclude from (6.14), (F.112) and Theorem C.11d that a function $\tilde{N} : \mathbb{Z} \to \mathbb{Z}^d$ exists such that, for $\phi \in \mathbb{R}^d, S \in \mathbb{R}^3, n \in \mathbb{Z}$,

$$L_{orb}(n; \phi, S) = L_\omega(n; \phi) + 2\pi\tilde{N}(n) = \phi + 2\pi n\omega + 2\pi\tilde{N}(n) \ . \qquad \text{(F.113)}$$

Note that, by (F.113), $L_{orb}(n; \phi, S)$ is independent of S, i.e., for $\phi \in \mathbb{R}^d, S \in \mathbb{R}^3, n \in \mathbb{Z}$,

$$L_{orb}(n; \phi, S) = L_{orb}(n; \phi) \ . \qquad \text{(F.114)}$$

Since L is a $\mathbb{Z}$-action on $\mathbb{R}^{d+3}$ we have, by (F.109),(F.114), that, for $\phi \in \mathbb{R}^d, m, n \in \mathbb{Z}$,

$$\begin{pmatrix} L_{orb}(0; \phi) \\ L_{spin}(0; \phi, S) \end{pmatrix} = L(0; \phi, S) = \begin{pmatrix} \phi \\ S \end{pmatrix} ,$$

$$\begin{pmatrix} L_{orb}(n+m; \phi) \\ L_{spin}(n+m; \phi, S) \end{pmatrix} = L(n+m; \phi, S) = L(n; L(m; \phi, S))$$

$$= L(n; \begin{pmatrix} L_{orb}(m; \phi) \\ L_{spin}(m; \phi, S) \end{pmatrix}) = \begin{pmatrix} L_{orb}(n; L_{orb}(m; \phi)) \\ L_{spin}(n; L_{orb}(m; \phi), L_{spin}(m; \phi, S)) \end{pmatrix} ,$$

whence, for $\phi \in \mathbb{R}^d, m, n \in \mathbb{Z}$,

$$L_{orb}(0; \phi) = \phi \ , \qquad L_{orb}(n+m; \phi) = L_{orb}(n; L_{orb}(m; \phi)) \ . \qquad \text{(F.115)}$$

It follows from (6.14),(F.113),(F.115), that, for $\phi \in \mathbb{R}^d, m, n \in \mathbb{Z}$,

$$\phi + 2\pi\tilde{N}(0) = L_\omega(0; \phi) + 2\pi\tilde{N}(0) = L_{orb}(0; \phi) = \phi \ ,$$

$$\phi + 2\pi(n+m)\omega + 2\pi\tilde{N}(n+m) = L_\omega(n+m; \phi) + 2\pi\tilde{N}(n+m)$$

$$= L_{orb}(n+m; \phi, S) = L_{orb}(n; L_{orb}(m; \phi)) = L_{orb}(n; \phi + 2\pi m\omega + 2\pi\tilde{N}(m))$$

$$= \phi + 2\pi(n+m)\omega + 2\pi\tilde{N}(m) + 2\pi\tilde{N}(n) \ ,$$

whence, for $m, n \in \mathbb{Z}$,

$$\tilde{N}(0) = 0 \;, \qquad \tilde{N}(n+m) = \tilde{N}(m) + \tilde{N}(n) \;. \tag{F.116}$$

It follows from (F.116) that, for $n \in \mathbb{Z}$,

$$\tilde{N}(n) = n\tilde{N}(1) \;. \tag{F.117}$$

I conclude from (F.109),(F.111),(F.113), (F.117) that, for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d, S \in \mathbb{R}^3$,

$$\begin{aligned} L(n; \phi, S) &= \begin{pmatrix} L_{orb}(n; \phi, S) \\ L_{spin}(n; \phi, S) \end{pmatrix} = \begin{pmatrix} L_{orb}(n; \phi, S) \\ \Psi_{\omega,A}(n; \phi)S \end{pmatrix} \\ &= \begin{pmatrix} \phi + 2\pi n\omega + 2\pi\tilde{N}(n) \\ \Psi_{\omega,A}(n; \phi)S \end{pmatrix} = \begin{pmatrix} \phi + 2\pi n\omega + 2\pi n\tilde{N}(1) \\ \Psi_{\omega,A}(n; \phi)S \end{pmatrix} \;. \end{aligned} \tag{F.118}$$

Since $\tilde{N}(1) \in \mathbb{Z}^d$, eq. (9.9) follows from (F.118).

To prove the remaining claim let (ω, A) be a d-dimensional spin-orbit torus and let $L : \mathbb{Z} \times \mathbb{R}^{d+3} \to \mathbb{R}^{d+3}$ be the function defined by (9.9) where $n \in \mathbb{Z}, N \in \mathbb{Z}^d, \phi \in \mathbb{R}^d, S \in \mathbb{R}^3$. Due to (6.11),(9.9) we have, for $\phi \in \mathbb{R}^d, S \in \mathbb{R}^3$,

$$L(0; \phi, S) = \begin{pmatrix} \phi \\ S \end{pmatrix} \;. \tag{F.119}$$

Furthermore we have, by (6.6),(9.9), that, for $\phi \in \mathbb{R}^d, S \in \mathbb{R}^3, m, n \in \mathbb{Z}$,

$$\begin{aligned} L(m+n; \phi, S) &= \begin{pmatrix} \phi + 2\pi(n+m)\omega + 2\pi(n+m)N \\ \Psi_{\omega,A}(n+m; \phi)S \end{pmatrix} \\ &= \begin{pmatrix} \phi + 2\pi(n+m)\omega + 2\pi(n+m)N \\ \Psi_{\omega,A}(n; \phi + 2\pi m\omega)\Psi_{\omega,A}(m; \phi)S \end{pmatrix} \\ &= \begin{pmatrix} \phi + 2\pi(n+m)\omega + 2\pi(n+m)N \\ \Psi_{\omega,A}(n; \phi + 2\pi m\omega + 2\pi mN)\Psi_{\omega,A}(m; \phi)S \end{pmatrix} \\ &= L(n; \phi + 2\pi m\omega + 2\pi mN, \Psi_{\omega,A}(m; \phi)S) = L(n; L(m; \phi, S)) \;, \end{aligned} \tag{F.120}$$

where in the third equality I used the fact that $\Psi_{\omega,A}(n;\cdot)$ is 2π-periodic. It follows from (F.119),(F.120) that L is a $\mathbb{Z}$-action on $\mathbb{R}^{d+3}$. Since $L(n;\cdot)$ is continuous it thus follows that $(\mathbb{R}^{d+3}, L)$ is a topological $\mathbb{Z}$-space.

Finally, by using (6.14),(9.5),(9.9), (F.106), I compute, for $\phi \in \mathbb{R}^d, S \in \mathbb{R}^3, n \in \mathbb{Z}$,

$$
\begin{aligned}
&p_{5,d}(L(n;\phi,S)) = p_{5,d}\Big(\phi + 2\pi n\omega + 2\pi nN, \Psi_{\omega,A}(n;\phi)S\Big) \\
&= \begin{pmatrix} p_{4,d}(\phi + 2\pi n\omega + 2\pi nN) \\ \Psi_{\omega,A}(n;\phi)S \end{pmatrix} = \begin{pmatrix} p_{4,d}(\phi + 2\pi n\omega) \\ \Psi_{\omega,A}(n;\phi)S \end{pmatrix} = \begin{pmatrix} p_{4,d}(L_\omega(n;\phi)) \\ \Psi_{\omega,A}(n;\phi)S \end{pmatrix} \\
&= L^{(T)}_{\omega,A}(n;p_{5,d}(\phi,S)) \ ,
\end{aligned}
\tag{F.121}
$$

where in the third equality I used the fact that the function $p_{4,d}$ is 2π-periodic. With (F.121) I have shown that $p_{5,d}$ is a $\mathbb{Z}$-map from the topological $\mathbb{Z}$-space $(\mathbb{R}^{d+3}, L)$ to the topological $\mathbb{Z}$-space $(\mathbb{T}^d \times \mathbb{R}^3, L^{(T)}_{\omega,A})$. Since $p_{5,d}$ is also onto $\mathbb{T}^d \times \mathbb{R}^3$ I thus conclude that the topological $\mathbb{Z}$-space $(\mathbb{R}^{d+3}, L)$ is an extension of the topological $\mathbb{Z}$-space $(\mathbb{T}^d \times \mathbb{R}^3, L^{(T)}_{\omega,A})$. □

F.28 Proof of Proposition 9.3

Proof of Proposition 9.3: Let $f \in \mathcal{C}(\mathbb{R}^d, SO(3)/H)$. Clearly $SO(3)$ is compact whence I can apply the results of Section E.6.6. It follows from (9.62),(E.164) that

$$
\check{E}_{f,H} = \hat{E}_{\hat{\gamma}(f)} \ , \tag{F.122}
$$

where $\hat{\gamma}$ is defined by (E.123) and $\hat{E}_{\hat{\gamma}(f)}$ is defined by (E.151). I conclude from (F.122),(E.161),(E.163) that $\widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)},H}(f)$, defined by (9.63), is identical with $\widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)},H}(f)$, defined by (E.163). Thus, by Theorem E.3c in Section E.6.6, $\widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)},H}$ is a bijection onto $RED_H(\lambda_{\mathcal{SOT}(d)})$. In particular the rhs of (9.63) is a H-reduction of $\lambda_{\mathcal{SOT}(d)}$. □

F.29 Proof of Proposition 9.4

Proof of Proposition 9.4a: By (E.61) I have for $R \in SO(3)$

$$RSO_3(2) = p_{R_{SO(3)/SO_3(2)}}(R) , \tag{F.123}$$

whence, since $p_{R_{SO(3)/SO_3(2)}}$ is onto $SO(3)/SO_3(2)$, F is defined by (9.64) on the whole set $SO(3)/SO_3(2)$. To show that F is single valued let $R, R' \in SO(3)$ such that $R'SO_3(2) = RSO_3(2)$, i.e., $p_{R_{SO(3)/SO_3(2)}}(R') = p_{R_{SO(3)/SO_3(2)}}(R)$ whence, by (E.58), a $R'' \in SO_3(2)$ exists such that $R' = R_{SO(3)/SO_3(2)}(R''; R) = RR''$ so that I conclude from (9.64)

$$F(R'SO_3(2)) = R'e^3 = RR''e^3 = Re^3 = F(RSO_3(2)) , \tag{F.124}$$

where in the third equality I used Definition C.2. Thus indeed F is a function: $SO(3)/SO_3(2) \to \mathbb{S}^2$. To show that F is continuous I observe by (9.64), (F.123) that, for $R \in SO(3)$,

$$F(p_{R_{SO(3)/SO_3(2)}}(R)) = Re^3 . \tag{F.125}$$

Thus $F \circ p_{SO(3)/SO_3(2)}$ is continuous whence, since $p_{SO(3)/SO_3(2)}$ is onto $SO(3)/SO_3(2)$ and identifying, I conclude from [Hu, Section II.6] that F is continuous. It is clear by (F.125) that the continuous function $F \circ p_{SO(3)/SO_3(2)}$ is onto $\mathbb{S}^2$ whence, since its domain $SO(3)$ is compact and $\mathbb{S}^2$ is Hausdorff, I conclude from [Bro, Section 4.2] that $F \circ p_{SO(3)/SO_3(2)}$ is identifying. Since $F \circ p_{SO(3)/SO_3(2)}$ and $p_{SO(3)/SO_3(2)}$ are identifying and $p_{SO(3)/SO_3(2)}$ is onto $SO(3)/SO_3(2)$ it follows from [Du, Section VI.3] that F is identifying. Of course since $F \circ p_{SO(3)/SO_3(2)}$ is onto $\mathbb{S}^2$ so is F. Furthermore if $R, R' \in SO(3)$ and $F(RSO_3(2)) = F(R'SO_3(2))$ then, by (9.64), $R'e^3 = Re^3$ so that, by Lemma 7.8a, a $R'' \in SO_3(2)$ exists such that $R' = RR''$ whence $R'SO_3(2) = RSO_3(2)$ so that F is one-one. Since F is one-one, onto $\mathbb{S}^2$ and identifying I conclude that $F \in HOMEO(SO(3)/SO_3(2), \mathbb{S}^2)$. I also conclude from

(9.64),(E.62) and the fact that $L^{(3D)}$ is a left $SO(3)$-action that for $R, R' \in SO(3)$

$$F(L_{SO(3)/SO_3(2)}(R'; RSO_3(2))) = F((R'R)SO_3(2)) = L^{(3D)}(R'R; e^3)$$
$$= L^{(3D)}(R'; L^{(3D)}(R; e^3)) = L^{(3D)}(R'; F(RSO_3(2))) ,$$

whence (9.65) holds. Let $S \in \mathbb{S}^2, R' \in SO(3)$. Since F is onto $\mathbb{S}^2$ I can pick a $R \in SO(3)$ such that $S = F(RSO_3(2))$ whence by (9.65) and the fact F is a bijection onto $\mathbb{S}^2$, I obtain

$$F^{-1}(L^{(3D)}(R'; S)) = F^{-1}(L^{(3D)}(R'; F(RSO_3(2))))$$
$$= F^{-1}(F(L_{SO(3)/SO_3(2)}(R'; RSO_3(2)))) = L_{SO(3)/SO_3(2)}(R'; RSO_3(2))$$
$$= L_{SO(3)/SO_3(2)}(R'; F^{-1}(S)) ,$$

whence (9.66) holds. □

Proof of Proposition 9.4b: Let $f \in \mathcal{C}(\mathbb{R}^d, SO(3)/SO_3(2))$. Since F is a bijection onto $\mathbb{S}^2$ and due to (9.64), the relation: $f(\phi) = RSO_3(2)$ is equivalent to: $(F \circ f)(\phi) = Re^3$. Thus $\{(\phi, R) \in \mathbb{R}^d \times SO(3) : f(\phi) = RH\} = \{(\phi, R) \in \mathbb{R}^d \times SO(3) : (F \circ f)(\phi) = Re^3\}$, whence (9.62) implies (9.67). To prove the second claim, let $G \in \mathcal{C}(\mathbb{R}^d, \mathbb{S}^2)$. Since, by Proposition 9.4a, $F \in HOMEO(SO(3)/SO_3(2), \mathbb{S}^2)$ it follows that $F^{-1} \circ G \in \mathcal{C}(\mathbb{R}^d, SO(3)/SO_3(2)$. Moreover we know from Proposition 9.3 that $\widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)},SO_3(2)}$ is a function: $\mathcal{C}(\mathbb{R}^d, SO(3)/SO_3(2)) \to RED_{SO_3(2)}(\lambda_{\mathcal{SOT}(d)})$ which is defined by (9.63). Thus $\overline{MAIN}_{\lambda_{\mathcal{SOT}(d)},SO_3(2)}$, as defined by (9.68), is a function: $\mathcal{C}(\mathbb{R}^d, \mathbb{S}^2) \to RED_{SO_3(2)}(\lambda_{\mathcal{SOT}(d)})$. To show that $\overline{MAIN}_{\lambda_{\mathcal{SOT}(d)},SO_3(2)}$ is one-one let $G, G' \in \mathcal{C}(\mathbb{R}^d, \mathbb{S}^2)$ such that $\overline{MAIN}_{\lambda_{\mathcal{SOT}(d)},SO_3(2)}(G) = \overline{MAIN}_{\lambda_{\mathcal{SOT}(d)},SO_3(2)}(G')$, i.e., $\widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)},SO_3(2)}(F^{-1} \circ G) = \widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)},SO_3(2)}(F^{-1} \circ G')$. Since, by Proposition 9.3, $\widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)},SO_3(2)}$ is one-one, I conclude that $F^{-1} \circ G = F^{-1} \circ G'$ whence, because F is a bijection onto $\mathbb{S}^2$, I obtain that $G = G'$ which entails that $\overline{MAIN}_{\lambda_{\mathcal{SOT}(d)},SO_3(2)}$ is one-one. To show that $\overline{MAIN}_{\lambda_{\mathcal{SOT}(d)},SO_3(2)}$ is onto $RED_{SO_3(2)}(\lambda_{\mathcal{SOT}(d)})$, let $\hat{\lambda}$ be in $RED_{SO_3(2)}(\lambda_{\mathcal{SOT}(d)})$. Thus by Proposition 9.3, a $f \in \mathcal{C}(\mathbb{R}^d, SO(3)/SO_3(2))$ exists such that $\hat{\lambda} = \widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)},SO_3(2)}(f)$. Since,

by Proposition 9.4a, $F \in HOMEO(SO(3)/SO_3(2), \mathbb{S}^2)$ I have $G'' \in \mathcal{C}(\mathbb{R}^d, \mathbb{S}^2)$ which I define by $G'' := F \circ f$. Of course, by (9.68), $\overline{MAIN}_{\lambda_{\mathcal{SOT}(d)}, SO_3(2)}(G'') = \widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)}, SO_3(2)}(F^{-1} \circ G'') = \widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)}, SO_3(2)}(F^{-1} \circ F \circ f)$
$= \widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)}, SO_3(2)}(f) = \hat{\lambda}$ which proves that $\overline{MAIN}_{\lambda_{\mathcal{SOT}(d)}, SO_3(2)}$ is onto $RED_{SO_3(2)}(\lambda_{\mathcal{SOT}(d)})$. This completes the proof that $\overline{MAIN}_{\lambda_{\mathcal{SOT}(d)}, SO_3(2)}$ is a bijection onto $RED_{SO_3(2)}(\lambda_{\mathcal{SOT}(d)})$. □

F.30 Proof of Theorem 9.5

Proof of Theorem 9.5a: Let $f \in \mathcal{C}(\mathbb{R}^d, SO(3)/H)$. I first consider the case when the H-reduction $\widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)}, H}(f)$ is invariant under $\Phi_{\omega,A}(\mathbb{Z})$. Then $\widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)}, H}(f)$ is invariant under $\Phi_{\omega,A}(1)$. Since, by (9.21),(9.22), for $\phi \in \mathbb{R}^d$, $R \in SO(3)$,

$$\Phi_{\omega,A}(1) = (\varphi_{\omega,A}(1;\cdot), L_\omega(1;\cdot)) \,, \qquad \varphi_{\omega,A}(1;\phi,R) = \begin{pmatrix} L_\omega(1;\phi) \\ A(\phi)R \end{pmatrix} \,, \tag{F.126}$$

I thus conclude from Corollary E.4b that (9.70) holds for every $\phi \in \mathbb{R}^d$. Let, conversely, (9.70) hold for every $\phi \in \mathbb{R}^d$. Then, by Corollary E.4b and (F.126), $\widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)}, H}(f)$ is invariant under $\Phi_{\omega,A}(1)$. By the remarks after (9.69) I conclude that $\widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)}, H}(f)$ is invariant under $\Phi_{\omega,A}(\mathbb{Z})$. □

Proof of Theorem 9.5b: Let $G \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$. I first consider the case where (ω, A) has the ISF $\mathcal{S}_G$. Thus, by Proposition 6.3, for $\phi \in \mathbb{R}^d$,

$$G(L_\omega(1;\phi)) = A(\phi)G(\phi) = L^{(3D)}(A(\phi); G(\phi)) \,, \tag{F.127}$$

where in the second equality I used (9.31). I define $f \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3)/SO_3(2))$ by

$$f := F^{-1} \circ G \,, \tag{F.128}$$

where F is defined by (9.64). Note that f is continuous since G is continuous and since, by Proposition 9.4a, $F \in HOMEO(SO(3)/SO_3(2), \mathbb{S}^2)$. Note also that f is

2π-periodic since G is 2π-periodic. I conclude from (9.66),(F.127), (F.128) that, for $\phi \in \mathbb{R}^d$,

$$\begin{aligned} & L_{SO(3)/SO_3(2)}(A(\phi); f(\phi)) = L_{SO(3)/SO_3(2)}(A(\phi); F^{-1}(G(\phi))) \\ & = F^{-1}(L^{(3D)}(A(\phi); G(\phi))) = F^{-1}(G(L_\omega(1;\phi))) = f(L_\omega(1;\phi)) \,. \quad \text{(F.129)} \end{aligned}$$

It follows from Theorem 9.5a and (F.129) that the $SO_3(2)$-reduction $\widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)},SO_3(2)}(f)$ of $\lambda_{\mathcal{SOT}(d)}$ is invariant under the group $\Phi_{\omega,A}(\mathbb{Z})$. Since $f \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3)/SO_3(2))$ I thus conclude from (9.69) that $\widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)},SO_3(2)}(f)$ is in $RED_{SO_3(2),per}(\lambda_{\mathcal{SOT}(d)})$.

To prove the converse direction let $f \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3)/SO_3(2))$ such that $\widehat{MAIN}_{\lambda_{\mathcal{SOT}(d)},SO_3(2)}(f)$ is invariant under the group $\Phi_{\omega,A}(\mathbb{Z})$. Thus, by Theorem 9.5a, I obtain (9.70) for every $\phi \in \mathbb{R}^d$. I now define $G \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$ by $G := F \circ f$. It follows from (9.31), (9.65),(9.70) that, for every $\phi \in \mathbb{R}^d$,

$$\begin{aligned} & A(\phi)G(\phi) = L^{(3D)}(A(\phi); G(\phi)) = L^{(3D)}(A(\phi); F(f(\phi))) \\ & = F(L_{SO(3)/SO_3(2)}(A(\phi); f(\phi))) = F(f(L_\omega(1;\phi)) = G(L_\omega(1;\phi)) \,. \quad \text{(F.130)} \end{aligned}$$

It follows from (F.130) and Proposition 6.3 that $\mathcal{S}_G$ is an ISF of (ω, A). □

Appendix G

Subject index for spin-orbit tori

Please note the following definitions and abbreviations used in this part of the thesis:

- R^T (transpose of matrix R), iff (means: if and only if).
- Section 6.1: $\mathbb{Z}$, 3×3 unit matrix $I_{3\times3}$, Euclidean norm $|\cdot|$, (ω, A), $\Psi_{\omega,A}$ $\mathcal{SOT}(d,\omega)$, $\mathcal{SOT}(d)$, $\mathcal{SOT}$, 2π–periodic function on $\mathbb{R}^k$, spin-orbit torus, orbital tune vector, orbital trajectory, spin trajectory, spin trajectory over ϕ_0, spin-orbit trajectory, n-turn spin transfer matrix.
- Section 6.2: $L_{\omega,A}$, L_ω, $\rho_{\mathcal{SOT}(d)}$.
- Section 6.3: $L^{(PF)}_{\omega,A}$, $\mathbb{S}^2$, polarization field, generator of polarization field, invariant polarization field, spin field, invariant spin field (ISF).
- Section 7.1: $L_T, R_{d,\omega}$, $\sim_{d,\omega}$, transfer field.
- Section 7.2: $\mathcal{T}(d,\omega)$, $\mathcal{AT}(d,\omega)$, $\mathcal{WT}(d,\omega)$, fractional part $\lfloor x \rfloor$ of a real number x, trivial spin-orbit torus, almost trivial spin-orbit torus, weakly trivial spin-orbit torus.

- Section 7.3: $\mathcal{CB}(d,\omega)$, $\mathcal{ACB}(d,\omega)$, $\mathcal{WCB}(d,\omega)$, coboundary, almost coboundary, weak coboundary.

- Section 7.4: $\Xi_1(\omega, A)$, spin tune of first kind, on spin-orbit resonance of first kind, off spin-orbit resonance of first kind.

- Section 7.6: ISF-conjecture.

- Section 7.7: $\mathcal{J}'$.

- Section 8.2: Simple precession frame (SPF) over ϕ_0, differential phase function, integral phase function.

- Section 8.3: $\Xi_2(\omega, A)$, $\Xi_2(\omega, A, \phi_0)$, $\sim_\omega$, $[\nu]_\omega$, uniform precession frame (UPF) over ϕ_0, uniform precession precession rate (UPR).

- Section 8.4: Well-tuned, ill-tuned, spin tune of second kind, spin-orbit resonance of second kind, on spin-orbit resonance of second kind, off spin-orbit resonance of second kind.

- Section 8.6: Y^s_χ, Y^{half}_χ, $\Xi^s_1(\omega, A)$.

- Section 9.2: $L^{(T)}_\omega$, $L^{(T)}_{\omega,A}$, $p_{5,d}$, $\Psi'_{\omega,A}$.

- Section 9.3: $\lambda_{\mathcal{SOT}(d)}$.

- Appendix B: e_G, (X, L), $COC(X, G, H)$, left G-action, G-action, right G-action, free right G-action, translation function of a free right G-action, left G-space, G-space, right G-space, topological group, topological left G-space, topological G-space, topological right G-space, G-map, conjugate, extension of left G-space, extension of G-space, extension of right G-space, extension of topological left G-space, extension of topological G-space, extension of topological right G-space, skew product of left G-space, skew product of G-space, skew product of right G-space, skew product of topological left G-space, skew

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