The Energy Deposited in Resistive Walls from Coherent Synchrotron Radiation

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Abstract

We study coherent synchrotron radiation (CSR) in a vacuum chamber of rectangular cross section, treating a single bend and a following straight section. We wish to know the amount of energy transferred to the resistive walls of the chamber, being motivated by design issues for LCLS-II. The first-order effect of resistivity is obtained by computing the magnetic field with perfectly conducting boundary conditions, then using the resistive wall boundary condition to determine the electric factor in the Poynting vector at the wall. The magnetic field at perfectly conducting boundaries is obtained by numerical integration of the paraxial wave equation in the frequency domain, using a Fourier mode expansion in the vertical coordinate. In an application to the second bunch compressor for LCLS-II, the resistive wall energy deposit is totally negligible. Our paraxial field solver determines all six field components at all space-time points, works for short bunches, incorporates an effective new way to deal with a line charge source, and is fast in comparison to earlier codes. It has a potential for wide applications.

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I. WAVE EQUATION FOR THE SLOWLY VARYING AMPLITUDE IN ACCELERATOR COORDINATES

We work in standard accelerator coordinates (Frenet-Serret coordinates) defined in terms of a reference trajectory $\mathbf{R}_0(s)$ lying in a plane, and parametrized by its arc length s. Any spatial point in the laboratory system is represented as $\mathbf{R} = \mathbf{R}_0(s) + x\mathbf{n}(s) + y\mathbf{e}_y$ where $\mathbf{n}(s)$ and $\mathbf{e}_y$ are unit vectors normal to the unit tangent $\mathbf{t}(s) = \mathbf{R}_0'(s)$. To be definite we take the horizontal unit vector to be $\mathbf{n} = \mathbf{e}_y \times \mathbf{t}$.

The vacuum chamber is to have a rectangular form with planar surfaces at $y = \pm g$, thus with full height $h = 2g$. The vertical walls at

$$x = x_- \ , \quad x = x_+ \tag{1}$$

are either planar or cylindrical with constant radius of curvature, depending on s. This accommodates a beam centered at $x = y = 0$, following a sequence of straights and bends.

The electromagnetic boundary conditions for perfectly conducting walls are that the tangential component of $\mathbf{E}$ and the normal component of $\mathbf{H}$ should vanish. We shall meet these conditions on the top and bottom walls (that E_s, E_x, H_y should vanish) by making a Fourier development in y . Assuming a source of velocity βc, any field or source component will have a Fourier development as follows:

$$F(s,x,y,t) = \int_{-\infty}^{\infty} dk e^{ik(s-\beta ct)} \sum_{p=0}^{\infty} \phi_p^{(i)}(y) \hat{F}_p(k,s,x) \ . \tag{2}$$

The choice of the trigonometric function $\phi_p^{(i)}(y)$ to meet the boundary conditions at $y = \pm g$ depends on which field or source component is expanded. We have

$$\begin{aligned}
&\phi_p^{(1)}(y) = \sin(\alpha_p(y+g)) \ , \quad F = E_s,\ E_x,\ H_y,\ J_s,\ J_x,\ \rho \ , \\
&\phi_p^{(2)}(y) = \cos(\alpha_p(y+g)) \ , \quad F = H_s,\ H_x,\ E_y,\ J_y \ . \\
&\alpha_p = \pi p/h \ ,
\end{aligned} \tag{3}$$

where $\mathbf{J}$ and ρ are the current and charge densities of the beam. With these choices the Maxwell equations and boundary conditions are satisfied term-by-term in the sums over p. This follows from orthogonality,

$$\frac{1}{g}\int_{-g}^{g} \phi_p^{(i)}(y)\phi_q^{(j)}(y) dy = \delta_{ij}\delta_{pq} \ . \tag{4}$$

Note that the representation (2) is general, since any function of s and t may also be viewed as a function of $s - \beta ct$ and s. Superficially it suggests a picture of waves moving to the right only, but that is valid only when the amplitude $\hat{F}_p(k, s, x)$ is slowly varying as a function of s.

Next write the Maxwell equations using the standard expressions for divergence and curl in curvilinear coordinates. The metric tensor is diagonal with diagonal components

$$(g_s,\ g_x,\ g_y) = (\eta(x, s),\ 1,\ 1)\ , \quad \eta(x, s) = 1 + x\kappa(s) = 1 + x/R(s)\ , \tag{5}$$

where $\kappa(s)$ and $R(s)$ are the curvature and radius of curvature of the reference orbit at s. Substitute fields and sources in the form (2) and take the inverse Fourier transforms with respect to $z = s - \beta ct$ and y to obtain the following system (in SI units).

div $\mathbf{D} = \rho$:

$$ik\hat{E}_{sp} + \partial_s\hat{E}_{sp} + \partial_x(\eta\hat{E}_{xp}) - \alpha_p\eta\hat{E}_{yp} = \eta Z_o\hat{\rho}_p c\ , \tag{6}$$

div $\mathbf{B} = 0$:

$$ik\hat{H}_{sp} + \partial_s\hat{H}_{sp} + \partial_x(\eta\hat{H}_{xp}) + \alpha_p\eta\hat{H}_{yp} = 0\ , \tag{7}$$

curl $\mathbf{E} + \partial\mathbf{B}/\partial t = 0$:

$$\partial_x\hat{E}_{yp} - \alpha_p\hat{E}_{xp} - ik\beta Z_o\hat{H}_{sp} = 0\ , \tag{8}$$

$$\eta\alpha_p\hat{E}_{sp} - ik\hat{E}_{yp} - \partial_s\hat{E}_{yp} - ik\beta\eta Z_o\hat{H}_{xp} = 0\ , \tag{9}$$

$$ik\hat{E}_{xp} + \partial_s\hat{E}_{xp} - \partial_x(\eta\hat{E}_{sp}) - ik\beta\eta Z_o\hat{H}_{yp} = 0\ , \tag{10}$$

curl $\mathbf{H} - \partial\mathbf{D}/\partial t = \mathbf{J}$:

$$\partial_x\hat{H}_{yp} + \alpha_p\hat{H}_{xp} + ik\beta\hat{E}_{sp}/Z_o = \hat{J}_{sp}\ , \tag{11}$$

$$-\alpha_p\eta\hat{H}_{sp} - ik\hat{H}_{yp} - \partial_s\hat{H}_{yp} + ik\beta\eta\hat{E}_{xp}/Z_o = \eta\hat{J}_{xp}\ , \tag{12}$$

$$ik\hat{H}_{xp} + \partial_s\hat{H}_{xp} - \partial_x(\eta\hat{H}_{sp}) + ik\beta\eta\hat{E}_{yp}/Z_o = \eta\hat{J}_{yp}\ , \tag{13}$$

where $Z_o = \mu_o c = 1/(\epsilon_o c)$ is the impedance of free space.

These equations may be solved algebraically for all field components in terms of $\hat{E}_{yp}$ and $\hat{H}_{yp}$ and their derivatives, yielding the results

$$\hat{E}_{sp} = -\frac{1}{\gamma_p^2}\left[\frac{\alpha_p}{\eta}(ik\hat{E}_{yp} + \partial_s \hat{E}_{yp}) + i\beta k Z_o(\hat{J}_{sp} - \partial_x \hat{H}_{yp})\right] , \tag{14}$$

$$\hat{E}_{xp} = -\frac{1}{\gamma_p^2}\left[\alpha_p \partial_x \hat{E}_{yp} + i\beta k Z_o\big(\hat{J}_{xp} + \frac{1}{\eta}(ik\hat{H}_{yp} + \partial_s \hat{H}_{yp})\big)\right] , \tag{15}$$

$$Z_o\hat{H}_{sp} = -\frac{1}{\gamma_p^2}\left[-\alpha_p Z_o\big(\hat{J}_{xp} + \frac{1}{\eta}(ik\hat{H}_{yp} + \partial_s \hat{H}_{yp})\big) + i\beta k \partial_x \hat{E}_{yp}\right] , \tag{16}$$

$$Z_o\hat{H}_{xp} = -\frac{1}{\gamma_p^2}\left[Z_o\alpha_p(\hat{J}_{sp} - \partial_x \hat{H}_{yp}) - \frac{i\beta k}{\eta}(ik\hat{E}_{yp} + \partial_s \hat{E}_{yp})\right] , \tag{17}$$

$$\gamma_p^2 = (\beta k)^2 - \alpha_p^2 \ . \tag{18}$$

Moreover, $\hat{E}_{yp}$ and $\hat{H}_{yp}$ are obtained as solutions of two independent wave equations with sources. To derive the wave equations one may combine the transformed Maxwell equations as stated above, or proceed from the wave equations in Cartesian form and transform the differential operator to Frenet-Serret coordinates. The equation for $\hat{F}_p = (\hat{E}_{yp}, \hat{H}_{hp})$ with source $\hat{S}_p = \big(\hat{S}_{Ep},\ \hat{S}_{Hp}\big)$ is

$$-\frac{1}{\eta^2}\left[\left(2ik - \frac{\kappa' x}{\eta}\right)\frac{\partial \hat{F}_p}{\partial s} + \frac{\partial^2 \hat{F}_p}{\partial s^2}\right] = \frac{\partial^2 \hat{F}_p}{\partial x^2} + \frac{\kappa}{\eta}\frac{\partial \hat{F}_p}{\partial x} + \left[\gamma_p^2 - \frac{k^2}{\eta^2} - \frac{ik\kappa' x}{\eta^3}\right]\hat{F}_p - \hat{S}_p \ , \tag{19}$$

where

$$\hat{S}_{Ep} = Z_0(\alpha_p c\hat{\rho}_p - ik\hat{J}_{yp}) \ , \qquad \hat{S}_{Hp} = \frac{\kappa}{\eta}\hat{J}_{sp} + \frac{\partial \hat{J}_{sp}}{\partial x} + \frac{ik}{\eta}\hat{J}_{xp} \ . \tag{20}$$

The factor κ' in (19) is nonzero where the reference trajectory (which need not be an actual particle trajectory) changes from straight to curved or vice versa. If the change is abrupt at $s = s_0$ then κ' contains $\delta(s - s_0)$, and it is doubtful that the wave equation can be given a meaning in a neighborhood of that point. On the other hand, if we give $\kappa(s)$ a smooth transition over a distance comparable to a typical fringe field extent in a bending magnet, then each of the terms with κ' is small compared to the term immediately preceding it in (19). Accordingly we drop κ' terms but then allow κ to be a step function at bend-straight transitions, elsewhere in the equation. The solution $\hat{F}_p$ is required to be continuous at transitions. Integration of the equations with continuous κ but without κ' terms seems feasible, but has not yet been attempted.

We also drop the transverse currents $\hat{J}_{xp},\ \hat{J}_{yp}$ and proceed to the main approximation, which is to assume that the amplitude $\hat{F}_p$ in (2) is slowly varying as a function of s. We may state the criterion for slow variation in terms of a norm, for instance

$$\|f\| = \int |f(x)|dx \ , \tag{21}$$

where dependence of f on variables other than the transverse coordinate x is suppressed. Then the requirement in (2) is

$$\left\| \frac{\partial^2 \hat{F}_p}{\partial s^2} \right\| \ll 2k \left\| \frac{\partial \hat{F}_p}{\partial s} \right\| \ , \quad s_0 \le s \le s_1 \ , \tag{22}$$

over the interval of integration $[s_0, s_1]$. This does not make sense as $k \to 0$, but a property of CSR in a chamber is that values of k appreciably less than a "shielding cutoff" k_0 never occur. We shall actually monitor the condition (22) in our calculations, which as far as we know has not been done before in similar CSR studies. For convenience (22) is called the Slowly Varying Amplitude (SVA) Approximation or Paraxial Approximation. In our view the former name is more apt, since it reminds us of the only condition that need be enforced.

Now within a bend of constant bending radius R the simplified wave equation takes the form

$$\frac{\partial \hat{F}_p}{\partial s} = i\frac{(x+R)^2}{2kR^2}\left[\frac{\partial^2 \hat{F}_p}{\partial x^2} + \frac{1}{x+R}\frac{\partial \hat{F}_p}{\partial x} + \left(\gamma_p^2 - (\frac{kR}{x+R})^2\right)\hat{F}_p - \hat{S}_p\right] \ , \tag{23}$$

$$\hat{S}_{Ep} = Z_0 \alpha_p c \hat{\rho}_p \ , \qquad \hat{S}_{Hp} = \frac{1}{x+R}\hat{J}_{sp} + \frac{\partial \hat{J}_{sp}}{\partial x} \ . \tag{24}$$

The corresponding equation in a straight section is obtained in the limit $R \to \infty$ as

$$\frac{\partial \hat{F}_p}{\partial s} = \frac{i}{2k}\left[\frac{\partial^2 \hat{F}_p}{\partial x^2} - \alpha_p^2 \hat{F}_p - \hat{S}_p\right] \ , \tag{25}$$

$$\hat{S}_{Ep} = Z_0 \alpha_p c \hat{\rho}_p \ , \qquad \hat{S}_{Hp} = \frac{\partial \hat{J}_{sp}}{\partial x} \ . \tag{26}$$

These equations are sometimes described as "parabolic", but that is a misnomer. They are of Schrödinger type owing to the factor i, with mathematical properties different from those of a proper parabolic equation.

We next choose a simple factored form for the charge density of the beam, good enough for the present limited study but capable of being generalized. With the corresponding current density it is

$$\rho(s,x,y,t) = q\lambda(s-\beta ct)\delta(x)H(y) \ , \quad \mathbf{J}(s,x,y,t) = (\ \beta c\rho,\ 0,\ 0\ ) \ , \tag{27}$$

$$\int \lambda(z)dz = \int H(y)dy = 1 \ , \quad q = \int \rho(s,x,y,t)\eta(x,s)dsdxdy \ , \tag{28}$$

where q is the total charge. The continuity equation is satisfied. By (2),(3), and (4), the Fourier transform with respect to z and y is

$$\hat{\rho}_p = q\hat{\lambda}(k)H_p\delta(x) \ , \quad \hat{\lambda}(k) = \frac{1}{2\pi}\int e^{-ikz}\lambda(z)dz \ , \quad H_p = \frac{1}{g}\int_{-g}^{g}\sin(\alpha_p(y+g))H(y)dy \ . \tag{29}$$

For H we choose a Gaussian of zero mean and variance $\sigma_y \ll h$. Then H_p is zero for even p (as it is for any even function $H(y)$), while for odd p we have

$$H_p = (-1)^{(p-1)/2}\,\frac{1}{g}\exp\big(-\frac{1}{2}(\alpha_p\sigma_y)^2\big) \ . \tag{30}$$

Thus the sources in (24) become

$$\hat{S}_{Ep} = qZ_0\alpha_p c\hat{\lambda}(k)H_p\delta(x) \ , \quad \hat{S}_{Hp} = q\beta c\hat{\lambda}(k)H_p\big(\delta(x)/R + \delta'(x)\big) \ . \tag{31}$$

II. NUMERICAL SOLUTION OF THE SIMPLIFIED WAVE EQUATION

An elementary way to approach the solution of (23) or (25) is to discretize the right hand side on a grid in x-space, representing the x-derivatives by finite differences. The discretization involves values of the solution at the boundaries, $\hat{F}_p(k,s,x_\pm(s))$, which are to be fixed at values required by the boundary conditions at a perfect conductor. The equation is then regarded as a system of ordinary differential equations, with s as the independent variable, in the complex unknowns $\hat{F}_p(k,s,x_i)$, $i = 2,\cdots,N-1$. Here the x_i are the interior points of the x-grid. The system is treated as an initial value problem, the initial value being the s-independent solution in an infinite straight wave guide.

A. A Transformation to Regularize the Effective Source

There is an impediment to discretization, however, due to the $\delta(x)$ and $\delta'(x)$ in the source terms (31). By a change of the dependent variable, this source can be replaced by a new effective source which behaves as $\theta(x)$ at $x = 0$, where $\theta(x)$ is the Heaviside step function, equal to 0 for $x < 0$ and 1 otherwise. A second transformation gives the continuous function $x\theta(x)$ in the effective source, and successive transformations can make the source arbitrarily smooth.

For $\hat{F}_p = \hat{E}_{yp}$ the expression in square brackets on the right hand side of (23) or (25) can be written to emphasize the x-dependence as

$$\Phi = u'' + a(x)u' + b(x)u - \sigma\delta(x) \ , \quad u(x) = \hat{E}_{yp}(k,s,x), \quad \sigma = qZ_0\alpha_p c\hat{\lambda}(k)H_p \ . \tag{32}$$

Now define a new dependent variable $v(x)$ by

$$u(x) = \sigma x\theta(x) + v(x)\ , \tag{33}$$

and note that

$$\Phi = v'' + a(x)v' + b(x)v + \sigma(a(x) + xb(x))\theta(x)\ , \tag{34}$$

Since $\partial u/\partial s = \partial v/\partial s$, the differential equation for v is the same as that for u, except for the new source

$$\tilde{S}_{Ep} = -\sigma(a(x) + b(x)x)\theta(x)\ , \tag{35}$$

replacing $\sigma\delta(x)$. The new source is much more suitable for discretization, being piecewise continuous with a jump of $-\sigma a(0)$ at $x = 0$.

Numerical integration of the differential equations with this setup was found to be only partly successful. An instability was encountered at large p in some cases, a behavior that could be traced to the jump. To remove the jump we put

$$v(x) = -\frac{\sigma}{2}a(x)x^2\theta(x) + w(x)\ . \tag{36}$$

Now the term in v'' from the second derivative of $-x^2/2$ takes out the term $\sigma a(x)\theta(x)$ in (34). The source for w is proportional to $x\theta(x)$, since Φ takes the form

$$\Phi = w'' + aw' + bw - \sigma\big(2a' + a^2 - b + (a'' + aa' + ab)x/2\big)x\theta(x)\ . \tag{37}$$

Clearly, this process can be continued, the next transformation being

$$w(x) = \frac{\sigma}{6}\big(2a'(x) + a^2(x) - b(x)\big)x^3\theta(x) + \xi(x)\ , \tag{38}$$

which yields a source for ξ proportional to $x^2\theta(x)$.

A similar procedure works for the magnetic field, even though the source to start with is more singular. For $\hat{F}_p = \hat{H}_{yp}$ the expression in square brackets on the right hand side of (23) or (25) has the form

$$\Psi = u'' + a(x)u' + b(x)u - \tau(\delta(x)/R + \delta'(x))\ , \quad u(x) = \hat{H}_{yp}(k, s, x), \quad \tau = q\beta c\hat{\lambda}(k)H_p\ . \tag{39}$$

The first transformation to remove δ' also removes δ because of the special form of $a(x)$. Thus

$$u(x) = \tau\theta(x) + v(x)\ , \tag{40}$$

yields

$$\Psi = v'' + a(x)v' + b(x)V + \tau\big(b(x)\theta(x) + (a(x) - 1/R)\delta(x)\big)$$
$$= v'' + a(x)v' + b(x)v + \tau b(x)\theta(x) \ , \tag{41}$$

thus a new source

$$\tilde{S}_{Hp} = -\tau b(x)\theta(x) \ , \tag{42}$$

replacing $\tau(\delta(x)/R + \delta'(x))$. The δ drops out because in the bend $a(x)\delta(x) = \delta(x)/(x+R) = \delta(x)/R$ whereas in the straight $a = 0$ and $R = \infty$. As in the discussion above, the second transformation will be

$$v(x) = -\frac{1}{2}b(x)x^2\theta(x) + w(x) \ , \tag{43}$$

giving

$$\Psi = w'' + aw' + bw - \tau\big(2b' + ab + (b'' + ab' + b^2)x/2\big)x\theta(x) \ , \tag{44}$$

from which we see that the third transformation is

$$w(x) = \frac{\tau}{6}\big(2b'(x) + a(x)b(x)\big)x^3\theta(x) + \xi(x) \ . \tag{45}$$

Now let us summarize the net effect of two smoothing transformations, invoking the explicit forms of the coefficients a and b. The effective source for the smoothed field $w = (w_E, w_H)$ is denoted by $\tilde{S}_p = (\tilde{S}_{Ep}, \tilde{S}_{Hp})$, and kR is denoted by n, as is conventional in periodic problems where n is an integer. In the bend,

$$\hat{E}_{yp} = \sigma\left[1 - \frac{1}{2}\frac{x}{x+R}\right]x\theta(x) + w_E(x) \ ,$$
$$\tilde{S}_{Ep} = -\sigma\left[\gamma_p^2 - \frac{n^2-1}{(x+R)^2}\right]\left[1 - \frac{1}{2}\frac{x}{x+R}\right]x\theta(x) \ . \tag{46}$$

$$\hat{H}_{yp} = \tau\left[1 - \frac{1}{2}\left(\gamma_p^2 - \frac{n^2}{(x+R)^2}\right)x^2\right]\theta(x) + w_H(x) \ ,$$
$$\tilde{S}_{Hp} = \tau\left[\frac{1}{x+R}\left(\gamma_p^2 + \frac{3n^2}{(x+R)^2}\right) + \frac{1}{2}\left(- \frac{4n^2}{(x+R)^4} + \big(\gamma_p^2 - \frac{n^2}{(x+R)^2}\big)^2\right)x\right]x\theta(x) \ . \tag{47}$$

The corresponding formulas for the straight section, obtained in the limit $R \to \infty$, are

$$\hat{E}_{yp} = \sigma x\theta(x) + w_E(x) \ , \quad \tilde{S}_{Ep} = \sigma(\alpha_p^2 + k^2/\gamma^2)x\theta(x) \ , \tag{48}$$
$$\hat{H}_{yp} = \tau\big(1 + \frac{1}{2}\alpha_p^2x^2\big)\theta(x) + w_H(x) \ , \quad \tilde{S}_{Hp} = \frac{\tau}{2}(\alpha_p^2 + k^2/\gamma^2)^2x^2\theta(x) \ , \tag{49}$$

where γ is the Lorentz factor.

Note that $\hat{E}_{yp}$ and $\hat{H}_{yp}$ must be continuous at the transitions between bend and straight, while the corresponding $w_E,\ w_H$ are not continuous. This must be kept in mind in designing the algorithm for s-integration.

In (15) and (17) we have the factor $\hat{J}_{sp} - \partial_x \hat{H}_{yp}$, where $\hat{J}_{sp} = \tau\delta(x)$. Fortunately, the $\tau\delta(x)$ is cancelled by the term $\tau\partial_x\theta(x)$ in $\partial_x \hat{H}_{yp}$ as given by (47). Such a cancellation was noticed long ago in analytical models [4], but a good way to handle it in a numerical context was lacking before the present innovation.

B. Boundary Conditions at the Vertical Walls

With perfect conductivity the boundary conditions at the vertical walls are

$$\hat{E}_{yp}(k,s,x_\pm(s)) = \hat{E}_{sp}(k,s,x_\pm(s)) = 0\ , \quad \hat{H}_{xp}(k,s,x_\pm(s)) = 0\ . \tag{50}$$

From (14) and (17) we see that these conditions are met if $\hat{E}_{yp}$ satisfies a Dirichlet condition and $\hat{H}_{yp}$ a Neumann condition, namely

$$\hat{E}_{yp}(k,s,x_\pm(s)) = 0\ , \quad \partial_x \hat{H}_{yp}(k,s,x_\pm(s)) = 0\ . \tag{51}$$

The corresponding conditions on the smoothed fields w at the outer walls can be read off from (46) and (47). Thus in the bend,

$$w_E(x_-) = 0\ , \quad w_E(x_+) = b_E(x_+) := -\sigma\left[1 - \frac{1}{2}\frac{x_+}{x_+ + R}\right]x_+\ , \tag{52}$$

$$\partial_x w_H(x_-) = 0\ , \quad \partial_x w_H(x_+) = \partial_x b_H(x_+) := \tau\left[\gamma_p^2 - \frac{n^2}{(x_+ + R)^2} + \frac{n^2 x_+}{(x_+ + R)^3}\right]x_+\ , \tag{53}$$

and in the straight

$$w_E(x_-) = 0\ , \quad w_E(x_+) = -\sigma x_+\ , \tag{54}$$

$$\partial_x w_H(x_-) = 0\ , \quad \partial_x w_H(x_+) = -\tau\alpha_p^2 x_+\ . \tag{55}$$

C. Finite Difference Scheme

We suppose that the field values are interpolated by 4-th degree polynomials in x, and that derivatives are given by differentiating the interpolation. The 4-th degree interpolation

[16] of a function $f(x)$ on a grid $\{x_i\}_{i=1}^N$ with uniform cell size Δx is

$$f(x) = \sum_{j=-2}^{2} L(p,j) f(x_i + j\Delta x) + \epsilon \ , \qquad x = x_i + p\Delta x \ , \tag{56}$$

with Lagrange coefficients

$$\begin{aligned}
L(p,-2) &= \frac{1}{24}(p^2-1)p(p-2) \ , \\
L(p,-1) &= -\frac{1}{6}(p-1)p(p^2-4) \ , \\
L(p,0) &= \frac{1}{4}(p^2-1)(p^2-4) \ , \\
L(p,1) &= -\frac{1}{6}(p+1)p(p^2-4) \ , \\
L(p,-2) &= \frac{1}{24}(p^2-1)p(p+2) \ .
\end{aligned} \tag{57}$$

The error ϵ is $\mathcal{O}((\Delta x)^5)$, and is estimated in terms of the 5-th derivative [16]. For evaluation at interior points of the grid $x = x_k,\ k = 3, \cdots, N-2$ we take $i = k$ and $p = 0$ for centered interpolation, whereas at border points $x = x_k,\ k = 1, 2, N-1, N$ we take $i = 3, 3, N-2, N-2$ with $p = -2, -1, 1, 2$, respectively, for the necessary off-center interpolation.

Differentiating (56) with respect to $p\Delta x$ gives the formulas for derivatives. Define

$$L_1(p,j) = \frac{1}{\Delta x}\frac{\partial}{\partial p} L(p,j) \ , \quad L_2(p,j) = \frac{1}{(\Delta x)^2}\frac{\partial^2}{\partial p^2} L(p,j) \ . \tag{58}$$

Now we can write the discretized form of the wave equation (23) for $\hat{E}_{yp}$ as follows, in terms of the smooth field w_E:

$$\frac{\partial w_E(x_i)}{\partial s} = i\frac{(x_i+R)^2}{2kR^2}\left[D_2(x_i) + \frac{1}{x_i+R}D_1(x_i) + \left(\gamma_p^2 - \frac{n^2}{(x_i+R)^2}\right) w_E(x_i) - \tilde{S}_{Ep}(x_i)\right] \ , \quad i = 2, \cdots, N-1 \ , \tag{59}$$

where

$$\begin{aligned}
D_k(x_i) &= \sum_{j=-2}^{2} L_k(0,j) w_E(x_i + j\Delta x) \ , \quad i = 3, \cdots, N-2 \ , \\
D_k(x_2) &= \sum_{j=-2}^{2} L_k(-1,j) w_E(x_3 + j\Delta x) \ , \\
D_k(x_{N-1}) &= \sum_{j=-2}^{2} L_k(1,j) w_E(x_{N-2} + j\Delta x) \ .
\end{aligned} \tag{60}$$

In view of (52) the boundary values that appear in these sums are

$$w_E(x_1) = 0 \ , \quad w_E(x_N) = b_E(x_N) \ . \tag{61}$$

where the inner and outer boundaries are at $(x_-, x_+) = (x_1, x_N)$. The equation for $\hat{H}_{yp}$ in terms of the smooth field w_H has the same form, with the appropriate definitions from (47), except that the boundary values are expressed in terms of interior values by discretizing the Neumann conditions (53):

$$\partial w_H(x_1) \approx \sum_{j=-2}^{2} L_1(-2,j) w_H(x_3 + j\Delta x) = 0 \ ,$$
$$\partial w_H(x_N) \approx \sum_{j=-2}^{2} L_1(2,j) w_H(x_{N-2} + j\Delta x) = \partial_x b_H(x_N)$$

Solving for the boundary values we have

$$w_H(x_1) = -\frac{1}{L_1(-2,-2)} \sum_{j=-1}^{2} L_1(-2,j) w_H(x_3 + j\Delta x) \ , \tag{62}$$

$$w_H(x_N) = \frac{1}{L_1(2,2)} \Big[- \sum_{j=-2}^{1} L_1(2,j) w_H(x_{N-2} + j\Delta x) + \partial_x b_H(x_N) \Big] \tag{63}$$

In a straight section the discretized equation like (59) is

$$\frac{\partial w_E(x_i)}{\partial s} = \frac{i}{2k} \Big[D_2(x_i) - \alpha_p^2 w_e(x_i) - \tilde{S}_{Ep}(x_i) \Big], \quad i = 2, \cdots, N-1 \ , \tag{64}$$

with the definitions of (48), (60) and the boundary conditions of (54).

D. Initial Values for the Evolution in s

The system of linear differential equations (59) is to be solved as an initial value problem. Within a single bend it is autonomous; i.e., its coefficients are independent of s. We take the initial value for $s = 0$ at the beginning of the bend to be the steady-state field produced by the source in an infinitely long straight chamber. Thus the equation for an initial field $\hat{F}_p$ is (25) with $\partial \hat{F}_p / \partial s = 0$, or

$$\frac{\partial^2 \hat{F}_p}{\partial x^2} - \alpha_p^2 \hat{F}_p = \hat{S}_p \ . \tag{65}$$

Its general solution is a particular solution plus the general solution of the homogeneous equation,

$$\hat{F}_p(x) = A \exp(\alpha_p x) + B \exp(-\alpha_p x) + \int_{x_-}^{x} \sinh\big(\alpha_p (x - y)\big) \hat{S}_p(y) dy \ . \tag{66}$$

in which A and B must be chosen to meet the boundary conditions. With the notation defined in (32) and (39) we have $\hat{S}_{Ep}(x) = \sigma\delta(x)$, $\hat{S}_{Hp}(x) = \tau(\delta(x)/R + \delta'(x))$. Evaluating the integral in (66) and applying the boundary conditions (51) we find

$$\hat{E}_{yp} = \frac{\sigma}{\alpha_p}\left[-\frac{\sinh(\alpha_p x_+)}{\sinh(\alpha_p(x_+ - x_-))}\sinh(\alpha_p(x - x_-)) + \sinh(\alpha_p x)\theta(x)\right] , \tag{67}$$

$$\hat{H}_{yp} = \tau\left[-\frac{\sinh(\alpha_p x_+)}{\sinh(\alpha_p(x_+ - x_-))}\cosh(\alpha_p(x - x_-)) + \cosh(\alpha_p x)\theta(x)\right] . \tag{68}$$

The corresponding initial values of the other field components are derived from (14)-(17):

$$\hat{E}_{sp} = -\frac{ik\alpha_p}{\gamma_p^2}\frac{1}{(1+\beta)\gamma^2}\hat{E}_{yp} , \tag{69}$$

$$\hat{E}_{xp} = \frac{Z_0}{\beta}\hat{H}_{yp} , \tag{70}$$

$$\hat{H}_{sp} = 0 , \tag{71}$$

$$\hat{H}_{xp} = -\frac{\beta}{Z_0}\left[1 + \left(\frac{k}{\gamma\gamma_p}\right)^2\right]\hat{E}_{yp} , \tag{72}$$

where γ is the Lorentz factor. The mechanism for the expected small value of $\hat{E}_{sp}$ at large γ is the near cancellation of the terms from $\hat{E}_{yp}$ and $\hat{J}_{sp} - \partial_x\hat{H}_{yp}$ in (14). The cancellation becomes less precise during field evolution in the bend, but $\hat{E}_{sp}$ is still a small difference of two large terms.

A numerical difficulty arises in the application of (67) and (68) because of a close cancellation of large terms at large $x \approx x_+$. The increasing part of the second term in (67) or (68), namely $\exp(\alpha_p x)/2$, cancels against a part of the first term. By some rearrangement we can write the first term as $-\exp(\alpha_p x)/2$ plus a remainder, and the residual after cancellation is suitable for numerical evaluation. It takes the following forms:

$$\begin{aligned}
&\hat{E}_{yp} = -\frac{\sigma}{2\alpha_p}\left[e^{-\alpha_p x} + e^{\alpha_p x}(a + c + ac)\right] , \quad \hat{H}_{yp} = -\frac{\tau}{2}\left[-e^{-\alpha_p x} + e^{\alpha_p x}(b + c + bc)\right] , \\
&a = -e^{-2\alpha_p x_+} - e^{-2\alpha_p(x-x_-)} + e^{-2\alpha_p(x+x_+-x_-)} , \\
&b = -e^{-2\alpha_p x_+} + e^{-2\alpha_p(x-x_-)} + e^{-2\alpha_p(x+x_+-x_-)} , \\
&c = e^{-2\alpha_p(x_+-x_-)}/(1 - e^{-2\alpha_p(x_+-x_-)}) .
\end{aligned} \tag{73}$$

E. Evolution in s

Suppressing irrelevant variables we write the system of differential equations for evolution of $w = (w_E, w_H)$ as

$$\frac{dw}{ds} = f(w, s) \ , \tag{74}$$

where w and f are vectors with N complex components, and f is linear in w. For the approximation at $s = s^{(n)} = n\Delta s + s^{(0)}$ we write $w^{(n)} \approx w(s^{(n)})$, where the integration step Δs is allowed to be different in bends from what it is in straight sections. We adopt the leapfrog integration rule, based on the central difference approximation to the derivative:

$$\frac{w^{(n+1)} - w^{(n-1)}}{2\Delta s} = f(w^{(n)}, s^{(n)}) \ , \quad n = 1, 2, \cdots \tag{75}$$

To define $w^{(1)}$ for the first step we use Euler's rule,

$$\frac{w^{(1)} - w^{(0)}}{\Delta s} = f(w^{(0)}, s^{(0)}) \ . \tag{76}$$

As remarked above, the value of w at the end of a bend is not in general equal to the value of w at the beginning of a following straight, owing to a change in definition of w through source smoothing. Consequently, we use an Euler step to initialize a leapfrog integration in the straight, with the appropriate initial value defined by continuity of the physical (unsmoothed) field at the bend-straight transition.

Of course there are more powerful methods than the finite difference method for discretizing in x and the leapfrog method for s. We have chosen these simple schemes merely to make our strategies clear and to avoid complications in programming for this exploratory study. We have in fact compared results from a more sophisticated x-discretization using the Discontinuous Galerkin Method [17], as will be reported presently.

III. POYNTING FLUX AT THE WALLS

The Poynting vector $\mathbf{E} \times \mathbf{H}$ evaluated at a wall describes, through its outwardly directed normal component, the flow of energy into that wall, per unit area and per unit time. At a perfectly conducting wall $\mathbf{E}$ is normal to the wall while $\mathbf{H}$ is tangential, so the Poynting vector vanishes. The resistive wall boundary condition (A14) implies a tangential component of $\mathbf{E}$ at the wall and a non-zero energy flow. We can calculate this flow to lowest order from a knowledge of $\mathbf{H}_0$, the magnetic field computed for perfectly conducting walls. We replace $\mathbf{H}$ by $\mathbf{H}_0$ in both the second factor of the Poynting vector and in the boundary condition. In this approximation the Poynting vector at a point $\mathbf{r} = (s, x, y)$ on the wall is

$$\mathbf{E}\times\mathbf{H} = (1-i)\left(\frac{\beta Z_0}{2\sigma}\right)^{1/2}\int dk e^{ik(s-\beta ct)}k^{1/2}\mathbf{n}\times\hat{\mathbf{H}}_0(k,\mathbf{r})\times\int dk' e^{ik'(s-\beta ct)}\hat{\mathbf{H}}_0(k',\mathbf{r}) \ . \quad (77)$$

From here on we write $\hat{\mathbf{H}}$ for $\hat{\mathbf{H}}_0$ in accord with the notation of previous sections.

Since we are interested in the total energy loss we may integrate over t. Note that

$$\int_{-\infty}^{\infty} dt \exp(-i\beta ct(k+k')) = \frac{2\pi}{\beta c}\delta(k+k') \ , \quad \hat{\mathbf{H}}(-k,\mathbf{r}) = \hat{\mathbf{H}}(k,\mathbf{r})^* \ , \quad (78)$$

so that

$$\int_{-\infty}^{\infty} dt \ \mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t) = (1-i)\left(\frac{2Z_0}{\beta\sigma}\right)^{1/2}\frac{\pi}{c}\int dk \ k^{1/2}\Big(\mathbf{n}\times\hat{\mathbf{H}}(k,\mathbf{r})\Big)\times\hat{\mathbf{H}}(k,\mathbf{r})^* \quad (79)$$

Here the integrand has finite support in t because the fields follow the source, and are negligible for $|s-\beta ct|$ greater than some length L, the maximum range of wake or predecessor fields. Now notice that

$$\Big(\mathbf{n}\times\hat{\mathbf{H}}\Big)\times\hat{\mathbf{H}}^* = (\mathbf{n}\cdot\hat{\mathbf{H}}^*)\hat{\mathbf{H}} - (\hat{\mathbf{H}}\cdot\hat{\mathbf{H}}^*)\mathbf{n} \ = -(\hat{\mathbf{H}}\cdot\hat{\mathbf{H}}^*)\mathbf{n} \ , \quad (80)$$

since $\mathbf{H}$ satisfies the boundary condition for a perfect conductor, with zero normal component. Moreover, $(1-i)k^{1/2}$ goes into its complex conjugate as $k\to -k$, since $k^{1/2}\to i|k^{1/2}|$ as we have defined it in the complex plane in Appendix A. Then in view of (78) the integral on k is twice the real part of the integral on positive k and

$$\int_{-\infty}^{\infty} dt \ \mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t) = -\mathbf{n}\left(\frac{2Z_0}{\beta\sigma}\right)^{1/2}\frac{2\pi}{c}\int_0^{\infty} dk \ k^{1/2}\hat{\mathbf{H}}(k,\mathbf{r})\cdot\hat{\mathbf{H}}(k,\mathbf{r})^* \quad (81)$$

We see that the time-integrated energy flux is solely along the normal direction and is positive toward the wall at all points (since $\mathbf{n}$ is directed inward toward the vacuum).

Next we wish to integrate (81) over one transverse dimension at the walls; namely, over y at $x = x_\pm$ for vertical walls and over x at $y = \pm g$ for horizontal walls. By (2) and (3) the Fourier development in y is

$$\hat{H}(k,s,x,y) = \sum_{p(odd)=1}^{\infty}\Big(\mathbf{e}_s\phi_p^{(2)}(y)\hat{H}_{sp}(k,s,x) + \mathbf{e}_x\phi_p^{(2)}(y)\hat{H}_{xp}(k,s,x) + \mathbf{e}_y\phi_p^{(1)}(y)\hat{H}_{yp}(k,s,x)\Big) \ . \quad (82)$$

On the vertical walls this reduces to

$$\hat{H}(k,s,x_\pm,y)=\sum_{p(odd)=1}^{\infty}\left(\mathbf{e}_s\phi_p^{(2)}(y)\hat{H}_{sp}(k,s,x_\pm)+\mathbf{e}_y\phi_p^{(1)}(y)\hat{H}_{yp}(k,s,x_\pm)\right)\ . \tag{83}$$

while on the horizontal walls it becomes

$$\hat{H}(k,s,x,\pm g)=\pm\sum_{p(odd)=1}^{\infty}\left(\mathbf{e}_s\hat{H}_{sp}(k,s,x)+\mathbf{e}_x\hat{H}_{xp}(k,s,x)\right)\ . \tag{84}$$

At the vertical walls we can use the orthogonality of (4) to find the y-integral as

$$-\mathbf{n}\cdot\int_{-g}^{g}dy\int_{-\infty}^{\infty}dt\ \mathbf{E}(s,x_\pm,y,t)\times\mathbf{H}(s,x_\pm,y,t)=$$
$$\left(\frac{2Z_0}{\beta\sigma}\right)^{1/2}\frac{2g\pi}{c}\int_0^\infty dk\ k^{1/2}\sum_p\left(|\hat{H}_{sp}(k,s,x_\pm)|^2+|\hat{H}_{yp}(k,s,x_\pm)|^2\right) \tag{85}$$

At the horizontal walls the x-integral is

$$-\mathbf{n}\cdot\int_{x_-}^{x_+}dx\int_{-\infty}^{\infty}dt\ \mathbf{E}(s,x,\pm g,t)\times\mathbf{H}(s,x,\pm g,t)=$$
$$\left(\frac{2Z_0}{\beta\sigma}\right)^{1/2}\frac{2\pi}{c}\int_0^\infty dk\ k^{1/2}\sum_{p,p'}\int_{x_-}^{x_+}dx\left(\hat{H}_{sp}(k,s,x)\hat{H}^*_{sp'}(k,s,x)+\hat{H}_{xp}(k,s,x)\hat{H}^*_{xp'}(k,s,x)\right)\ . \tag{86}$$

To find the total energy deposited in the walls the expressions (85) and (86) must be integrated over s using the numerical solutions for the tangential $\mathbf{H}$ fields.

IV. TOTAL ENERGY RADIATED AND THE WAKE FIELD

Here we derive the formula for the total energy radiated, for comparison to the amount of energy absorbed in resistive walls. By conservation of energy this is just the negative of the work done on the beam by the longitudinal component of the electric field. The work done on an infinitesimal charge element $dQ=\rho(\mathbf{r},t)d\mathbf{r}$ in time dt is

$$dW=\rho(\mathbf{r},t)d\mathbf{r}E_s(\mathbf{r},t)\beta c dt\ ,\quad E_s(\mathbf{r},t)=\int dk\ e^{ik(s-\beta ct)}\sum_{p(odd)=1}^{\infty}\sin\alpha_p(y+g)\hat{E}_{sp}(k,\mathbf{r})\ . \tag{87}$$

Then the power radiated from all elements is

$$P=d\mathcal{E}/dt=-\beta c\int d\mathbf{r}\rho(\mathbf{r},t)E_s(\mathbf{r},t)\ , \tag{88}$$

and the energy radiated while the bunch center moves from $s=0$ to $s=\bar{s}$ is

$$\mathcal{E}(0,\bar{s})=-\beta c\int_0^{\bar{s}/\beta c}dt\int d\mathbf{r}\rho(\mathbf{r},t)E_s(\mathbf{r},t)\ , \tag{89}$$

For our simple model of the charge density in (27) we have

$$P=-q\beta c\int dsdxdy\lambda(s-\beta ct)\delta(x)H(y)\int dk\ e^{ik(s-\beta ct)}\sum_{p(odd)=1}^{\infty}\sin\alpha_p(y+g)\hat{E}_{sp}(k,s,x)=$$
$$=-q\beta cg\sum_p H_p\int dk\int ds\lambda(s-\beta ct)e^{ik(s-\beta ct)}\hat{E}_{sp}(k,s,0)\ . \tag{90}$$

The slowly varying amplitude $\hat{E}_{sp}(k,s,0)$ changes little over the length of the bunch, so that it may be replaced by $\hat{E}_{sp}(k,\beta ct,0)$ in (90). Then the s-integral gives just the conjugated Fourier transform of λ so that

$$P=-2\pi q\beta cg\sum_p H_p\int dk\ \hat{\lambda}_k^*\ \hat{E}_{sp}(k,\beta ct,0)=-4\pi q\beta cg\sum_p H_p\int_0^\infty dk\ \hat{\lambda}_k^*\ \hat{E}_{sp}(k,\beta ct,0)\ , \tag{91}$$

and

$$\mathcal{E}(0,\bar{s})=-4\pi q\beta cg\sum_p H_p\int_0^\infty dk\ \hat{\lambda}_k^*\ \int_0^{\bar{s}}ds\ \hat{E}_{sp}(k,s,0)\ . \tag{92}$$

For comparison to earlier work we are also interested in the longitudinal wake field,

$$W(z,s,x,y)=2\mathrm{Re}\int_0^\infty dk\ e^{ikz}\sum_p H_p\sin\alpha_p(y+g)\hat{E}_{sp}(k,s,x)\ ,\quad z=s-\beta ct \tag{93}$$

Evaluated at $x=0$ and integrated against the vertical charge distribution $H(y)$ this becomes

$$W(z,s)=2g\mathrm{Re}\int_0^\infty dk\ e^{ikz}\sum_p H_p^2\hat{E}_{sp}(k,s,0)\ . \tag{94}$$

For the parameters of our examples, H_p^2 from (30) is nearly equal to $1/g^2$ for the small values of p that appear in the calculations.

APPENDIX A: RESISTIVE WALL BOUNDARY CONDITION

We recall the derivation of the resistive wall boundary condition, adapting it to our particular context. We apply the wave equation obeyed by the field within the wall material, which is assumed to have magnetic permeability μ, electric permittivity ϵ, and conductivity σ, all independent of position and frequency. The basic assumption is that the variation of the field within the wall is by far the strongest in the direction normal to the wall. This picture can be checked *a posteriori* by first assuming it to be true, then deducing the consequent normal variation. This variation, characterized by a small penetration depth (skin depth) can be compared with estimates of variation in the tangential directions.

Invoking Ohm's Law $\mathbf{J} = \sigma\mathbf{E}$, we have the curl equations within the wall as

$$\text{curl } \mathbf{H} = \sigma\mathbf{E} + \epsilon\frac{\partial\mathbf{E}}{\partial \text{t}} \ , \tag{A1}$$

$$\text{curl } \mathbf{E} = -\mu\frac{\partial\mathbf{H}}{\partial \text{t}} \ . \tag{A2}$$

Next we take the Fourier transform with respect to t to obtain

$$\text{curl } \tilde{\mathbf{H}} = (\sigma - \text{i}\omega\epsilon)\tilde{\mathbf{E}} \ , \tag{A3}$$

$$\text{curl } \tilde{\mathbf{E}} = \text{i}\omega\mu\tilde{\mathbf{H}}, \tag{A4}$$

where

$$\tilde{F}(\omega,\mathbf{r}) = \frac{1}{2\pi}\int dt \ e^{i\omega t}F(\mathbf{r},t) \ , \quad \mathbf{r} = (s,x,y) \ . \tag{A5}$$

The term $-i\omega\epsilon$ from the displacement current is typically tiny in comparison to σ, and will be dropped henceforth.

We define a positive depth coordinate ξ, the distance from the beginning of the wall to an interior point of the wall medium, and a unit vector $\mathbf{n}$ normal to the wall and directed from the wall toward the vacuum. At the horizontal walls $\xi = \pm(y - g)$, whereas at vertical walls $\xi = \pm(x - x_\pm)$. Then with the assumption of dominant normal variation the gradient is represented as $\nabla = -\mathbf{n}\partial/\partial\xi$, so that

$$-\mathbf{n}\times\frac{\partial\tilde{\mathbf{H}}}{\partial\xi} = \sigma\tilde{\mathbf{E}} \ , \quad -\mathbf{n}\times\frac{\partial\tilde{\mathbf{E}}}{\partial\xi} = i\omega\mu\tilde{\mathbf{H}} \ , \tag{A6}$$

We can then eliminate $\tilde{\mathbf{E}}$ in (A6) by taking the curl of the first equation and substituting the second:

$$\mathbf{n}\times\frac{\partial}{\partial\xi}\left(\mathbf{n}\times\frac{\partial}{\partial\xi}\tilde{\mathbf{H}}\right) = \left(\mathbf{n}\cdot\frac{\partial^2\tilde{\mathbf{H}}}{\partial\xi^2}\right)\mathbf{n} - (\mathbf{n}\cdot\mathbf{n})\frac{\partial^2\tilde{\mathbf{H}}}{\partial\xi^2} = i\omega\mu\sigma\tilde{\mathbf{H}} \ . \tag{A7}$$

Since $\nabla\cdot\tilde{\mathbf{H}} = \partial(\mathbf{n}\cdot\tilde{\mathbf{H}})/\partial\xi = 0$, we have

$$\frac{\partial^2\tilde{\mathbf{H}}}{\partial\xi^2} + i\omega\mu\sigma\tilde{\mathbf{H}} = 0 \tag{A8}$$

The general solution of this harmonic equation with complex frequency is

$$\tilde{\mathbf{H}} = \mathbf{a}_+ \exp(\xi/\Delta) + \mathbf{a}_- \exp(-\xi/\Delta) \ , \quad \Delta = (i\omega\mu\sigma)^{-1/2} \ . \tag{A9}$$

The $\mathbf{a}_\pm$ depend only on coordinates other than ξ. Since the solution must decay at large $\xi > 0$ we retain only the second term and choose the branch of the square root so that $\mathrm{Re}\Delta > 0$, namely as

$$\Delta^{-1} = e^{-i\pi/4}(\mu\omega\sigma)^{-1/2} \ , \tag{A10}$$

where the square root in (A10) is positive at positive real ω. We define this root in the complex ω-plane with a cut on the positive real axis. It then acquires a factor of i in analytic continuation to negative ω, so that Δ^{-1} has positive real part at negative as well as positive ω. The conventional skin depth δ is defined by

$$\Delta^{-1} = (1-i)/\delta \ , \quad \delta = \left(\frac{2}{\mu\omega\sigma}\right)^{1/2} \ , \tag{A11}$$

so that the field decays by a factor $1/e$ in a distance δ.

By (A9) we have $\partial\tilde{\mathbf{H}}/\partial\xi = -\tilde{\mathbf{H}}/\Delta$ which when substituted in the first equation of (A6) yields

$$\tilde{\mathbf{E}} = (1-i)\left(\frac{\mu\omega}{2\sigma}\right)^{1/2} \mathbf{n} \times \tilde{\mathbf{H}} \ . \tag{A12}$$

Taking the limit $\xi \to 0$ in (A12) we have the resistive wall boundary condition, since there must be continuity with the fields in the vacuum.

The Fourier transform (A5) with respect to time is related to the transform (2) with respect to $s - \beta ct$ by the phase factor $\exp(-iks)/\beta c$, which cancels out in (A12). That is,

$$\hat{F}(k,s,x,y) = \frac{1}{2\pi}\int d(s-\beta ct)e^{-ik(s-\beta ct)}F(s,x,y,t) = \frac{\beta c}{2\pi}e^{iks}\int dt e^{ik\beta ct}F(s,x,y,t) \ . \tag{A13}$$

Thus with $\omega = \beta kc$ the boundary condition for the Fourier amplitudes used in this paper is

$$\hat{\mathbf{E}}(k,s,x,y) = (1-i)\left(\frac{\beta\mu ck}{2\sigma}\right)^{1/2} \mathbf{n} \times \hat{\mathbf{H}}(k,s,x,y) \ , \tag{A14}$$

at every point (s,x,y) on the boundary, with the unit normal $\mathbf{n}$ to the boundary directed toward the vacuum. With the good approximation $\mu = \mu_0$, which we adopt henceforth, the square root may be written in the convenient form $(\beta Z_0 k/2\sigma)^{1/2}$. Similarly the skin depth is $\delta = (2/\beta Z_0 k\sigma)^{1/2}$.

To test the assumption of dominant normal variation, consider the parameters for the example studied above, for the second bunch compressor in LCLS-II. The dominant frequencies are such that $5 \cdot 10^4 < kR < 10^6$, $R = 12.9$ m, and for aluminum we have $\sigma \approx 3.6 \cdot 10^7\ \Omega^{-1} m^{-1}$, while $Z_0 = 120\pi\ \Omega$. Thus the range of skin depth is about $4.5 \cdot 10^{-8}\mathrm{m} < \delta < 2 \cdot 10^{-7}\mathrm{m}$. Assuming that the variation of fields with s in the wall is comparable to that in the vacuum, it will be characterized by the rapidly varying factor $\exp(iks)$, hence with a length scale corresponding to the range of $1/k$. We then should compare δ to $1/k$ to test the assumption of dominant normal derivative. We have $7.5 \cdot 10^{-4} < \delta/k^{-1} < 3.4 \cdot 10^{-3}$, so the longitudinal variation indeed appears to be minor compared to the normal. On the vertical walls we can estimate the variation in the y-direction from the highest important mode in the Fourier development (2), which was found to be around $p = 5$ in the calculations. This mode has wavelength $2h/p$ with $h = 2$ cm in our example, so that the scale of variation is in the sub-centimeter range. For the x-variation at the horizontal walls we can refer to the numerical results as reported in Fig. showing variations on a sub-millimeter scale at the highest relevant k. To summarize, it appears that tangential field variations are slow compared to the normal variation due to the small skin depth, as was assumed in the derivation of (A14).

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[1] J. Schwinger, "On Radiation by Electrons in a Betatron", unpublished report (1945), transcribed by M. Furman and issued as Lawrence Berkeley Laboratory report

[2] J. S. Nodvick and D. S. Saxon, Phys. Rev. **96**, 180 (1954).

[3] L. I. Schiff, Rev. Sci. Instr. **17**, 6 (1946)..

[4] R. Warnock and P. Morton, SLAC-PUB-4562 (1988), later published in Part. Accel. **25**, 113 (1990).

[5] K.-Y. Ng, Part. Accel. **25**, 153 (1990). Ng's paper also appeared as a preprint around 1988.

[6] G. V. Stupakov and I. A. Kotelnikov, Phys. Rev. ST Accel. Beams **6**, 034401 (2003), *ibid.* **12**, 104401 (2009).

[7] T. Agoh, "Dynamics of Coherent Synchrotron Radiation by Paraxial Approximation", Doctoral Thesis, University of Tokyo, 2004.

[8] T. Agoh and K. Yokoya, Phys. Rev. ST Accel. Beams **7**, 054403 (2004).

[9] D. R. Gillingham and T. M. Antonsen, Jr., Phys. Rev. ST Accel. Beams **10**, 054402 (2007).

[10] D. Zhou, "Coherent Synchrotron Radiation and the Microwave Instability in Electron Storage Rings", Doctoral Thesis, The Graduate University for Advanced Studies, 2011.

[11] D. Zhou, K. Ohmi, K. Oide, L. Zang, and G. Stupakov, Japanese J. Appl. Phys. **51**, 016401 (2012).

[12] D. A. Bizzozero, R. Warnock, and J. A. Ellison, Proc. International Free Electron Laser Conference FEL 2014, Basel, Switzerland, August 25-29, 2014, paper TUP023.

[13] J. B. Murphy, S. Krinsky, and R. L. Gluckstern, Part. Accel. **57**, 9 (1997).

[14] R. Warnock, M. Venturini, J. A. Ellison, and R. Ruth, Phys. Rev. ST Accel. Beams **8**, 014402 (2005).

[15] R. Warnock and M. Venturini, Proc. 2003 Particle Accelerator Conf., paper RPP8061.

[16] "Handbook of Mathematical Functions", M. Abramowitz and I. A. Stegun, Eds. (National Bureau of Standards, 1964), §22.2.15.

[17] J. S. Hesthaven and T. Warburton, "Nodal Discontinuous Galerkin Methods", (Springer, 2008)

[18] R. L. Warnock, G. R. Bart, and S. Fenster, Part. Accel. **12**, 179 (1982), Appendix B.