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Polarization fields and phase space densities in storage rings: Stroboscopic averaging and the ergodic theorem[☆]

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Abstract

A class of orbital motions with volume preserving flows and with vector fields periodic in the "time" parameter θ is defined. Spin motion coupled to the orbital dynamics is then defined, resulting in a class of spin–orbit motions which are important for storage rings. Phase space densities and polarization fields are introduced. It is important, in the context of storage rings, to understand the behavior of periodic polarization fields and phase space densities. Due to the 2π time periodicity of the spin–orbit equations of motion the polarization field, taken at a sequence of increasing time values $\theta, \theta + 2\pi, \theta + 4\pi, \ldots$, gives a sequence of polarization fields, called the stroboscopic sequence. We show, by using the Birkhoff ergodic theorem, that under very general conditions the Cesàro averages of that sequence converge almost everywhere on phase space to a polarization field which is 2π-periodic in time. This fulfills the main aim of this paper in that it demonstrates that the tracking algorithm for stroboscopic averaging, encoded in the program SPRINT and used in the study of spin motion in storage rings, is mathematically well-founded. The machinery developed is also shown to work for the stroboscopic average of phase space densities associated with the orbital dynamics. This yields a large family of periodic phase space densities and, as an example, a quite detailed analysis of the so-called betatron motion in a storage ring is presented.

Keywords: Periodic spin field; Dynamical systems; Ergodic theory; Stroboscopic averaging; Spin polarization

1. Introduction

This paper explores certain mathematical matters relating to periodic particle and spin distributions in storage rings. Storage rings are large scale devices used to enable high energy fundamental particles such as electrons, positrons, protons and nuclei to be brought to collision in order to study the most basic properties of matter. Descriptions of storage rings can be found in standard text books and articles. See for example [21, 27,37,43,44]. However, to summarize, the common feature of a storage ring is that the electrically charged particles are confined to move in bunches on approximately circular orbits in a vacuum tube by combinations of electric and magnetic fields. The dimensions of a bunch are very small compared to the average radius of the ring, e.g., the proton–electron collider HERA at DESY has a circumference of 4 miles and bunch dimension of a few millimeters. For the purposes of this paper we ignore the emission of electromagnetic radiation by the particles and all collective effects, e.g., all interactions between the particles and the effects of the electric and magnetic fields set up in the vacuum pipe by the particles themselves. Then particle motion is determined just by the Lorentz force [35] and a Hamiltonian can be assigned. Since we are dealing with storage rings we take the orbital motion to be bounded.

Information from analysis of observations from particle collisions in storage rings is much enhanced if the beams are "spin polarized". Each electron, positron, proton or deuteron carries an intrinsic angular momentum called the "spin angular momentum" and there is an associated 3-vector called the "spin expectation value" $\vec{s}$ [24] which in this paper we simply call the "spin". This leads to the concept of a spin-valued

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function on phase space. The spin dynamics is governed by the Thomas–Bargmann–Michel–Telegdi (T–BMT) equation $d\vec{s}/dt = \vec{\Omega}(\vec{E}(t), \vec{B}(t), \vec{v}(t)) \times \vec{s}$ where $\vec{E}$, $\vec{B}$ and $\vec{v}$ are respectively the electric and magnetic fields and the velocity [35]. Thus the particle spins precess in the electric and magnetic fields and their lengths $|\vec{s}|$ are constant. The quantity of interest to experimenters using a storage ring is the beam polarization $\vec{P}$, i.e., the average of the normalized spins. To obtain this we first average only over the spin degrees of freedom to get the local polarization $\vec{P}_{\rm loc}$, i.e., the polarization at every point of phase space. Then the beam polarization is the phase space average of the local polarization. See Section 3.4.1. The above mentioned spin-valued functions on phase space will be called polarization fields and will be defined in Section 3.2.1. The local polarization is a polarization field. In a well set-up storage ring the phase space density ρ in bunches is periodic from turn to turn. The experimenters also desire that the local polarization is periodic from turn to turn. It is thus of great interest to find ways of calculating such one-turn periodic polarization fields. In order for the maximum information to be extracted from experiments, it is desirable that the beam polarization be as large as possible.

Although dynamical systems are often analyzed by taking time as the independent variable, this is usually not convenient for storage rings since there, the vacuum tube and the electric and magnetic guide fields have a fixed, approximately circular spatial layout. It is therefore standard practice to begin analysis by constructing the curvilinear "closed orbit", i.e., the orbit along which the particle motion is one-turn periodic. The equations of motion for the particles and their spins are then transformed into forms in which the angular distance, θ, along the closed orbit is taken as the independent variable and particle positions are defined with respect to the closed orbit. The one-turn periodicity of the positions of the electric and magnetic guide fields then becomes a 2π-periodicity in θ. The new equations of motion can be derived from an appropriate Hamiltonian and the three pairs of canonical variables are combined into a vector z with six components. For example, two of the pairs can describe transverse motion and one pair can describe longitudinal (synchrotron) motion within a bunch. One of this latter pair quantifies the deviation of the particle energy from the energy of a particle fixed at the center of a bunch and the other describes the time delay with respect to that particle (see e.g. [7]). With respect to the average radius of the closed orbit and the nominal particle energy, the canonical position variable and the energy variable are very small. Although it would be natural to choose an appropriate name for the new independent variable, we simply refer to θ as the "time". In the following we will, for convenience, dispense with the symbol "$\rightarrow$" over spin-like quantities.

Arguably the most general and reliable method for numerically calculating periodic polarization fields in storage rings involves "stroboscopic averaging" [32]. For this, one begins with an arbitrary polarization field, $\mathcal{S}(\theta, z)$, and computes the average of $\mathcal{S}(\theta, z)$, $\mathcal{S}(\theta + 2\pi, z)$, $\ldots$, $\mathcal{S}(\theta + 2\pi(N-1), z)$ which as a function of the integer N forms the "Cesàro sequence". One usually finds in practice that the Cesàro sequence converges for given (θ, z) and that it is, to numerical accuracy, 2π-periodic in θ at each z. The main accomplishments of this paper are precise definitions of the stroboscopic sequence, the Cesàro sequence and the stroboscopic average, the proof that the stroboscopic average is a 2π-periodic polarization field and our discussion of the convergence properties of the Cesàro sequence. An important tool in our work is the Birkhoff ergodic theorem, called henceforth the ergodic theorem. The rate of convergence of the Cesàro sequence is only briefly treated here (see Section 3.4.2) but it is discussed, for the case of integrable orbital systems, in [32].

A polarization field that is normalized and is 2π-periodic in θ at each z is now usually called an "invariant spin field". This is an important object for systematizing spin dynamics. For example the invariant spin field plays a role in the calculation of the maximum attainable equilibrium beam polarization and the maximum attainable time averaged beam polarization at each θ [4]. The invariant spin field is also the starting point for calculating the so-called amplitude dependent spin tune if the latter exists [3,4,7,17,18,33,39,42,45,46]. An invariant spin field is derived from a 2π-periodic polarization field via stroboscopic averaging by simply normalizing the stroboscopic average to unity.

The discovery that invariant spin fields could be approximated via stroboscopic averaging led to the creation of the computer code SPRINT [32,33,42] as a way of computing invariant spin fields for very high energy where other algorithms would fail or be impractical. SPRINT was then heavily used for computing the invariant spin field in a study of the feasibility of attaining high proton polarization in the electron–proton storage ring system, HERA [33,42]. Stroboscopic averaging can even handle exotic models where the invariant spin field is discontinuous in the orbit variables [9,10].

Although the numerical calculation of the invariant spin field by stroboscopic averaging represents a major advance in our ability to systematize spin motion in storage rings, there has been no detailed investigation of the Cesàro sequence and its convergence. Thus the bulk of the present paper consists of introducing mathematical definitions and stating and proving related theorems and propositions.

We begin in Section 2, with a discussion of the details of the orbital dynamics necessary for the discussion of the spin dynamics. To pave the way for the application of the ergodic theorem to the spin dynamics, we apply it first to the Cesàro sequence of phase space densities defined by the evolution of ensembles of particles. We thereby show that the stroboscopic average is a 2π-periodic phase space density and that the Cesàro sequence converges almost everywhere. A brief summary of Section 2 is given in Section 2.4 and in Appendix A, the results of Section 2 are applied to the linear one-degree-of-freedom time-periodic Hamiltonian case which is important for storage rings. In Section 3, we turn our attention to polarization fields. We first discuss the basics of spin motion and polarization fields, and then set up the problem in a form for application of the ergodic theorem. Its application yields our main results: the stroboscopic average of every polarization field is a 2π-periodic polarization field and, if the phase space is of finite measure, the Cesàro sequence converges almost everywhere. This fulfills the main aim of this paper in that it demonstrates that the tracking algorithm for stroboscopic averaging, encoded in the program SPRINT and used in the study of spin motion in storage rings, is mathematically well-founded. Finally, in Section 3.4, we discuss the physical significance of our results. Section 4 is a summary and brief discussion.

2. The orbital system and Liouville densities

The main purpose of this section is to set notation and to acquaint the reader with stroboscopic sequences, Cesàro sequences, stroboscopic averages and the ergodic theorem. We prove that the stroboscopic averages are bounded, time-periodic, $\mathcal{L}^1$ phase space densities. This is important since storage rings are only useful if the phase space density is approximately periodic turn to turn.

More specifically, in Section 2.1 we discuss the details of the orbital motion necessary for our paper. In Section 2.2 we introduce phase space densities, their stroboscopic sequences, Cesàro sequences, and stroboscopic averages. The ergodic theorem is stated in Section 2.3 and we illustrate its use in the simple context of phase space densities to construct periodic phase space densities by stroboscopic averaging. Also, as a concrete example, we discuss in Appendix A the linear one-degree-of-freedom time-periodic Hamiltonian system which is important since the betatron motion in a storage ring is a special subcase.

The material after the statement of the ergodic theorem (Theorem 2.4) is not needed for Section 3, in particular Lemma 2.5 and Theorem 2.6 are not needed. However the reader might find this material helpful as it is a simpler context for the application of the ergodic theorem.

2.1. The orbital motion

Consider the initial value problem

$$\dot{z} = f(\theta, z), \tag{2.1}$$

$$z(\theta_0) = z^0 \in \mathbb{R}^d, \tag{2.2}$$

where $f : \mathbb{R}^{d+1} \to \mathbb{R}^d$ is of class C^1, $f(\theta, \cdot)$ is divergence free ($\mathrm{Tr}[D_2 f(\theta, z)] = 0$), $f(\cdot, z)$ is 2π-periodic and θ_0 is an arbitrary initial time. The choice that f is of class C^1 is consistent with the fact that the electric and magnetic fields in storage rings are smooth. Throughout this paper D_k will denote the derivative with respect to the k-th argument, be it scalar or vector. For simplicity the variable θ is called "time". We want the results of this paper to apply to beam dynamics in storage rings (see Introduction, Section 3.4, Summary and Appendix A). In this case, θ plays the role of the so-called azimuthal variable and f is a Hamiltonian vector field 2π-periodic in θ with z being the vector of generalized position and momentum coordinates. However we do not assume that f is generated by a Hamiltonian.

We denote the solution of the initial value problem (2.1) and (2.2) by

$$z(\theta) = \varphi(\theta, \theta_0; z^0), \quad \theta \in I(\theta_0, z^0), \tag{2.3}$$

where

$$D_1\varphi(\theta, \theta_0; z^0) = f(\theta, \varphi(\theta, \theta_0; z^0)), \quad \varphi(\theta_0, \theta_0; z^0) = z^0, \tag{2.4}$$

and where $I(\theta_0, z^0)$ is the maximal interval of existence at (θ_0, z^0) (see Remark (1)). By the uniqueness of solutions, we have the basic identity

$$\varphi(\theta_2, \theta_1; \varphi(\theta_1, \theta_0; z^0)) = \varphi(\theta_2, \theta_0; z^0), \tag{2.5}$$

and in addition the periodicity of f ensures that

$$\varphi(\theta + 2\pi, \theta_0 + 2\pi; z^0) = \varphi(\theta, \theta_0; z^0), \quad \theta \in I(\theta_0, z^0). \tag{2.6}$$

Note that the periodicity of $f(\cdot, z)$ implies that if $z(\cdot)$ is a solution of (2.1) for all time then so is $z(\cdot + 2\pi)$. Note also that (2.5) holds subject to the constraint on the θ values imposed by the maximum intervals of existence.

Since $f(\theta, \cdot)$ is divergence free,

$$\det(D_3\varphi(\theta, \theta_0; z^0)) = 1, \quad \theta \in I(\theta_0, z^0). \tag{2.7}$$

Our only assumption in addition to the properties of f is the existence of an open nonempty set U such that

$$\varphi(2\pi, 0; U) = U. \tag{2.8}$$

Since we are dealing with a storage ring and neglect the emission of radiation and collective effects this is a reasonable assumption. In fact, integrable motion, which is often assumed in calculations, has this property (see Appendix A). We define $U_\theta := \varphi(\theta, 0; U)$ and from now on restrict the z^0 in (2.2) to U_{θ_0} and take the domain of φ to be $\mathbb{R} \times L$ where L is the set of admissible initial conditions (θ_0, z^0), i.e.,

$$L := \{(\theta, z) \in \mathbb{R}^{d+1} : z \in U_\theta\} = \bigcup_{\theta\in\mathbb{R}} \{\theta\} \times U_\theta. \tag{2.9}$$

From the invariance (2.8) of U under the period advance map $\varphi(2\pi, 0; \cdot)$, it is clear that $I(\theta_0, z^0) = \mathbb{R}$ if $z^0 \in U_{\theta_0}$ and that $\varphi(\theta, \theta_0; \cdot) : U_{\theta_0} \to U_\theta$ is a C^1-diffeomorphism onto U_θ. Note that U_θ is open in $\mathbb{R}^d$ and L is open in $\mathbb{R}^{d+1}$. It is also clear that $U_{\theta+2\pi} = U_\theta$ and that (2.4)–(2.7) hold whenever $z^0 \in U_{\theta_0}$.

We immediately draw an important conclusion from our assumptions. Let $(U_\theta, \mathcal{U}_\theta, \mu_d)$ be the measure space over the σ-algebra $\mathcal{U}_\theta$ of Borel subsets of U_θ with Lebesgue measure μ_d, then it follows from (2.7) and the transformation theorem for Lebesgue integrals (see for example [11, Section 19]) that each $\varphi(\theta, \theta_0; \cdot)$ is measure preserving, i.e., $\mu_d(\varphi(\theta_0, \theta; A)) = \mu_d(A)$ for all $A \in \mathcal{U}_\theta$. In the following we define $\mathcal{U} := \mathcal{U}_0$.

Remark.

(1) Let $I(t_0, z^0) \equiv (\alpha, \beta)$, then the standard continuation theorem for f in $C^1(\mathbb{R}^{d+1})$ gives that either $\beta = \infty$ or $|z(t)| \to \infty$ as $t \uparrow \beta$ and similarly for α (see, e.g., [14, Theorem 1.4] or [1,25]). □

2.2. Liouville densities on phase space and their stroboscopic average

2.2.1. Liouville densities and their basic properties

As motivation for our definition of Liouville densities, consider an ensemble of particles defined by a phase space density ρ which evolves under the time flow defined by (2.1). From particle conservation we have

$$\int_A \rho(0,z^0)\mathrm{d}\mu_d(z^0) = \int_{\varphi(\theta,0;A)} \rho(\theta,z)\mathrm{d}\mu_d(z)$$
$$= \int_A \rho(\theta,\varphi(\theta,0;z^0))\det(D_3\varphi(\theta,0;z^0))\mathrm{d}\mu_d(z^0)$$
$$= \int_A \rho(\theta,\varphi(\theta,0;z^0))\mathrm{d}\mu_d(z^0), \tag{2.10}$$

where $A \in \mathcal{U}$. The first equality expresses particle conservation, the second uses the transformation theorem for Lebesgue integrals and the third follows from (2.7). Since A is an arbitrary element of $\mathcal{U}$ it follows from (2.10) that, for μ_d-almost every $z \in U_\theta$, we have $\rho(0,z) = \rho(\theta,\varphi(\theta,0;z))$ so that

$$\rho(\theta,z) = \rho\left(0,\varphi(0,\theta;z)\right). \tag{2.11}$$

Since sets of measure zero are unimportant we are free to specify $\rho(0,\cdot)$ and define ρ everywhere by (2.11). Eq. (2.11) motivates our definition of a Liouville density (LD).

Definition 2.1 (*Liouville Density*). Let $g : U \to [0,\infty)$ be a bounded function in $\mathcal{L}^1(U,\mathcal{U},\mu_d)$. A function $\rho_g : L \to [0,\infty)$ is called a "Liouville density" (LD) if

$$\rho_g(\theta,z) = g\left(\varphi(0,\theta;z)\right). \tag{2.12}$$

The function g will be called the 'generator' of ρ_g. □

For the definition of $\mathcal{L}^1(U,\mathcal{U},\mu_d)$, see Remark (5). Clearly, every LD is a bounded function since every upper bound of g is an upper bound of ρ_g. Note that for $z \in U_\theta$, $\varphi(0,\theta;z) \in U$ so that ρ_g is well defined. Moreover it is easy to check by (2.5) and (2.12), that at times θ_1 and θ_2

$$\rho_g(\theta_2,z) = \rho_g\left(\theta_1,\varphi(\theta_1,\theta_2;z)\right). \tag{2.13}$$

Since (θ_2,z) and $(\theta_1,\varphi(\theta_1,\theta_2;z))$ are on the same solution curve of (2.1) it follows from (2.13) that ρ_g is constant along solution curves.

Of course, by setting $A = U$ in (2.10) one obtains the obvious results for an LD ρ_g that $\rho_g(\theta,\cdot) \in \mathcal{L}^1(U_\theta,\mathcal{U}_\theta,\mu_d)$ and

$$\int_U g\mathrm{d}\mu_d = \int_{U_\theta} \rho_g(\theta,z)\mathrm{d}\mu_d(z). \tag{2.14}$$

It is convenient not to require these integrals to be 1. However we say that an LD ρ_g is "normalized" if $\int_U g\mathrm{d}\mu_d = 1$ (then, clearly, $\int_{U_\theta} \rho_g(\theta,z)\mathrm{d}\mu_d(z) = 1$).

We call an LD ρ_g "2π-periodic" iff $\rho_g(\theta+2\pi,z) = \rho_g(\theta,z)$. Thus if ρ_g is 2π-periodic, (2.13) gives

$$\rho_g(\theta,z) = \rho_g(\theta,\varphi(\theta,\theta+2\pi;z)). \tag{2.15}$$

Note that ρ_g is 2π-periodic iff g satisfies the fixed point equation

$$g(z) = g\left(\varphi(0,2\pi;z)\right). \tag{2.16}$$

As we will see, (2.16) is the invariance property in the ergodic theorem (see also Remark (3)).

More generally, we call ρ_g "2π-periodic in measure" if for every θ the equality: $\rho_g(\theta+2\pi,z) = \rho_g(\theta,z)$ holds for μ_d-almost every z. In that case there may be no z such that $\rho_g(\cdot,z)$ is 2π-periodic, however the 'average' $\int_{U_\theta} \rho_g(\theta,z)F(\theta,z)\mathrm{d}\mu_d(z)$ over a 2π-periodic 'observable' F is a 2π-periodic function of θ. Note also that ρ_g is 2π-periodic in measure iff g satisfies the fixed point equation (2.16) for μ_d-almost every z. The optimal situation in a storage ring is for the physically relevant LD (see Section 3.4.1) to be 2π-periodic or, at least, 2π-periodic in measure. Note however that, in this paper, those LD's which are 2π-periodic in measure but not 2π-periodic play only a minor role.

Liouville densities in accelerators are discussed in e.g. [20, 22,23], [27, Section 2.5] and their importance for polarized beams in storage rings will be further revealed in Section 3.4.1.

Remark.

(2) If the LD ρ_g is C^1 then it satisfies the first order PDE

$$\frac{\partial}{\partial\theta}\rho_g + \nabla_z \cdot (\rho_g f(\theta,z)) = 0. \tag{2.17}$$

In the more general case where $\mathrm{Tr}[D_2 f] \neq 0$, Eq. (2.10) motivates the definition $\rho_g(\theta,z) = g\left(\varphi(0,\theta;z)\right)\exp\left(-\int_0^\theta \mathrm{Tr}[D_2 f(\theta',\varphi(\theta',\theta;z))]\mathrm{d}\theta'\right)$ since the Wronskian $\mathcal{W}(\theta,z^0) := \det(D_3\varphi(\theta,0;z^0))$ satisfies $D_1\mathcal{W} = a\mathcal{W}$ where $a(\theta,z^0) := \mathrm{Tr}[D_2 f\left(\theta,\varphi(\theta,0;z^0)\right)]$. Thus this more general definition of ρ_g satisfies (2.17) if f and g are sufficiently smooth. The PDE (2.17) is often called the "Liouville equation" (or the "Vlasov equation" in the collective case). □

2.2.2. The stroboscopic and Cesàro sequences of a Liouville density

In this section we define a stroboscopic sequence of the LD ρ_g and the associated Cesàro sequence, and derive properties of the Cesàro sequence.

Definition 2.2 (*Stroboscopic Sequence and Cesàro Sequence*). Let ρ_g be an LD. The "stroboscopic sequence of ρ_g" consists of the functions $\rho_g(\cdot+2\pi n,\cdot) : L \to [0,\infty)$ where $n = 0,1,\ldots$. Let $\rho_g^N : L \to [0,\infty)$ be defined by

$$\rho_g^N(\theta,z) := \frac{1}{N}\sum_{n=0}^{N-1} \rho_g(\theta+2\pi n,z)$$
$$= \frac{1}{N}\sum_{n=0}^{N-1} g\left(\varphi(0,\theta+2\pi n;z)\right), \tag{2.18}$$

where $N = 1,2,\ldots$. The sequence $\{\rho_g^N\}_{N=1}^\infty$ is called the "Cesàro sequence of ρ_g" and the ρ_g^N are called "Cesàro averages of ρ_g". □

In the special case when ρ_g is 2π-periodic, we have $\rho_g^N = \rho_g$. In the general case, we claim that ρ_g^N is an LD for each N.

We first argue that each element of the stroboscopic sequence is an LD, i.e., that shifting the time by $2\pi n$ in the LD ρ_g gives an LD if n is an integer. In fact, since $\rho_g(2\pi n, z) = g(\varphi(0, 2\pi n; z))$ we see that $g(\varphi(0, 2\pi n; \cdot))$ is an $[0, \infty)$-valued, bounded function in $\mathcal{L}^1(U, \mathcal{U}, \mu_d)$ and we have, by (2.5), (2.6) and (2.12),

$$\begin{aligned} & g(\varphi(0, 2\pi n; \varphi(0, \theta; z))) \\ &\quad = g(\varphi(0, 2\pi n; \varphi(2\pi n, \theta + 2\pi n; z))) \\ &\quad = g(\varphi(0, \theta + 2\pi n; z)) \\ &\quad = \rho_g(\theta + 2\pi n, z), \end{aligned} \tag{2.19}$$

which proves that $\rho_g(\cdot + 2\pi n, \cdot)$ is an LD which is generated by $g(\varphi(0, 2\pi n; \cdot))$ so that each element of the stroboscopic sequence is an LD. It follows from (2.18) and (2.19) that, for every N, ρ_g^N is an LD which is generated by $(1/N)\sum_{n=0}^{N-1} g(\varphi(0, 2\pi n; \cdot))$. Since ρ_g^N is an LD then by definition

$$\rho_g^N(\theta, z) = \rho_g^N(0, \varphi(0, \theta; z)). \tag{2.20}$$

Having used (2.6) in (2.19) we see that the 2π-periodicity of $f(\cdot, z)$ is essential in proving that each element ρ_g^N of the stroboscopic sequence is an LD.

In addition $\int_{U_\theta} \rho_g^N(\theta, z)\mathrm{d}\mu_d(z) = \int_U g\mathrm{d}\mu_d$. To see this, note that by (2.14) and (2.18)

$$\begin{aligned} \int_{U_\theta} \rho_g^N(\theta, z)\mathrm{d}\mu_d(z) &= \int_U \rho_g^N(0, \cdot)\mathrm{d}\mu_d \\ &= \frac{1}{N}\sum_{n=0}^{N-1}\int_U g(\varphi(0, 2\pi n; \cdot))\,\mathrm{d}\mu_d. \end{aligned}$$

Since $\varphi(0, 2\pi n; \cdot)$ is measure preserving and $\varphi(0, 2\pi n; U) = U$, the stated result follows.

2.2.3. *The stroboscopic average of a Liouville density*

We now discuss some convergence properties of the Cesàro sequence. Let $U_\theta^g \subset U_\theta$ be the set on which $\rho_g^N(\theta, \cdot)$ converges, i.e.,

$$U_\theta^g := \{z \in U_\theta : \lim_{N\to\infty} \rho_g^N(\theta, z) \text{ exists}\}. \tag{2.21}$$

Clearly $U_\theta^g \in \mathcal{U}_\theta$ and our goal is to show that this is large in measure.

From (2.20) $\rho_g^N(0, \varphi(0, \theta; z))$ converges on the same set, thus $\rho_g^N(0, z)$ converges for $z \in \varphi(0, \theta; U_\theta^g)$ and so

$$U_\theta^g = \varphi(\theta, 0; U_0^g). \tag{2.22}$$

Since $\varphi(0, \theta; \cdot)$ is measure preserving, (2.22) gives $\mu_d(U_\theta^g) = \mu_d(U_0^g)$. Note that since g is bounded, the sequence $\{\rho_g^N(\theta, z)\}_{N=1}^\infty$ is bounded. Thus the limit, if it exists, is always a real number. In this paper, the term "converges" without a qualifier will always mean *pointwise converges*. The boundedness of the sequence also assures that for every pair (θ, z) a convergent subsequence exists but we will not use this fact.

Definition 2.3 (*Stroboscopic Average*). Let ρ_g be an LD. Then we call the function $\hat{\rho}_g : L \to [0, \infty)$, defined by $\hat{\rho}_g(\theta, z) := \lim_{N\to\infty} 1_{U_\theta^g}(z)\rho_g^N(\theta, z)$, the "stroboscopic average of ρ_g". □

Here 1_A denotes the indicator function of the set A, i.e., $1_A(z) = 1$ if $z \in A$ and 0 otherwise; thus $\hat{\rho}_g(\theta, z)$ is the limit of $\rho_g^N(\theta, z)$ if it exists and zero otherwise. It follows from the definition that every LD ρ_g has a unique stroboscopic average $\hat{\rho}_g$ which is a bounded and nonnegative function and that $\hat{\rho}_g(\theta, \cdot)$ is $\mathcal{U}_\theta$-$\mathcal{R}$-measurable where $\mathcal{R}$ is the σ-algebra of Borel subsets of $\mathbb{R}$. Note that $\rho_g^N(\theta, z)$ converges for all $(\theta, z) \in L$ as $N \to \infty$ iff $U_0^g = U$. In particular this is true if ρ_g is 2π-periodic. By (2.22)

$$1_{U_\theta^g}(z) = 1_{U_0^g}(\varphi(0, \theta; z)), \tag{2.23}$$

and, since ρ_g^N is an LD, we obtain that the function $1_{U_\theta^g}(z)\rho_g^N(\theta, z)$ in Definition 2.3 is an LD. Thus the stroboscopic average $\hat{\rho}_g$ is the limit of a sequence of LD's.

The main issues now are whether $\hat{\rho}_g$ is a 2π-periodic LD and whether U_θ^g is of full measure in U_θ. Since $1_{U_\theta^g}(z)\rho_g^N(\theta, z)$ is an LD we have

$$1_{U_\theta^g}(z)\rho_g^N(\theta, z) = 1_{U_0^g}(\varphi(0, \theta; z))\,\rho_g^N(0, \varphi(0, \theta; z)).$$

Then by taking $N \to \infty$, we obtain

$$\hat{\rho}_g(\theta, z) = \hat{\rho}_g(0, \varphi(0, \theta; z)). \tag{2.24}$$

Thus if $\hat{\rho}_g(0, \cdot)$ is in $\mathcal{L}^1(U, \mathcal{U}, \mu_d)$ then $\hat{\rho}_g$ is an LD. If U is of finite measure then this is true by the dominated convergence theorem (see for example [11, Section 15]).

That $\hat{\rho}_g$ is a 2π-periodic LD and that U_θ^g is of full measure in U_θ will be shown in Section 2.3 as a consequence of the ergodic theorem. A subset of a measure space will be said to have *full measure* if its complement has measure zero. Thus, in the case of a finite measure space, a set is of full measure iff it has the measure of the underlying measure space.

2.3. *Applying the ergodic theorem to the Cesàro sequence of a Liouville density*

In this section we prove that the stroboscopic average of every LD is a 2π-periodic LD. Our analysis will also show that for the generally nonperiodic ρ_g in (2.12) its Cesàro sequence converges almost everywhere, i.e., that U_θ^g is of full measure. These results follow easily from an application of the ergodic theorem, which we now state.

An $\mathcal{M}$-$\mathcal{M}$-measurable map T on a measure space $(M, \mathcal{M}, m)$ is said to be measure preserving if $m(T^{-1}A) = m(A)$ for all A in $\mathcal{M}$. A set $A \in \mathcal{M}$ satisfying $T^{-1}(A) = A$ is said to be an invariant set and $\mathcal{I} := \{A \in \mathcal{M} : T^{-1}(A) = A\}$ is the σ-algebra of T-invariant sets in $\mathcal{M}$. We now can state:

Theorem 2.4 (*Ergodic Theorem*). *Let $T : M \to M$ be a measure preserving map on the σ-finite measure space $(M, \mathcal{M}, m)$ and let $b \in \mathcal{L}^1(M, \mathcal{M}, m)$. Then an element $\check{b}$ of*

$\mathcal{L}^1(M, \mathcal{I}, m) \subset \mathcal{L}^1(M, \mathcal{M}, m)$ exists such that, for m-almost every z,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} b(T^n(z)) = \check{b}(z), \tag{2.25}$$

with $\int_M |\check{b}|\mathrm{d}m \le \int_M |b|\mathrm{d}m$. If m is a probability measure then the sequence $(1/N)\sum_{n=0}^{N-1} b(T^n(\cdot))$ converges in $\mathcal{L}^1$ (convergence in the mean) to $\check{b}$ and $\check{b}$ is a version of the conditional expectation $E(b|\mathcal{I})$. □

There are many proofs of the ergodic theorem in the literature. See for example [12,13,19]. Note that in Theorem 2.4 and in the following the phrase 'm-almost every z' always relates to $(M, \mathcal{M}, m)$, not to $(M, \mathcal{I}, m)$.

Remarks.

(3) Since $\check{b}$ in Theorem 2.4 is $\mathcal{I}$-measurable, $\check{b}$ is invariant under T. In fact even the reverse holds, i.e., a $\mathcal{M}$-measurable $\check{b}$ is $\mathcal{I}$-measurable iff $\check{b}(z) = \check{b}(T(z))$ [13, Section 6.4].

(4) Note that in the σ-finite case $\check{b} \in \mathcal{L}^1(M, \mathcal{M}, m)$ even though the convergence may not be $\mathcal{L}^1$.

(5) We define for a measure space $(M, \mathcal{M}, m)$ the real vector space $\mathcal{L}^p(M, \mathcal{M}, m)$ in the same way as in [11, Section 14]. Then $b \in \mathcal{L}^p(M, \mathcal{M}, m)$ means that b is a real-valued $\mathcal{M}$-$\mathcal{R}$-measurable function which is p-fold m-integrable, i.e. $\int_M |b|^p dm < \infty$ where $1 \le p < \infty$. Note that b is a function, not an equivalence class of functions. For brevity we sometimes call a function "measurable" if it is clear which σ-algebras are involved.

(6) Recall that the two defining properties of $E(b|\mathcal{I}) : M \to \mathbb{R}$ are that $E(b|\mathcal{I})$ is $\mathcal{I}$-$\mathcal{R}$-measurable and that $\int_A E(b|\mathcal{I})dm = \int_A bdm$ for all $A \in \mathcal{I}$. □

We now apply the ergodic theorem to ρ_g in (2.12). We define $T := \varphi(0, 2\pi; \cdot)$ so that $T : U \to U$ is a measure preserving C^1-diffeomorphism onto U. In addition we take $M = U, \mathcal{M} = \mathcal{U}$, m to be the Lebesgue measure, and we set $b = g$ so that $\check{b} = \check{g}$. Thus $\mathcal{I} = \{A \in \mathcal{U} : \varphi(2\pi, 0; A) = A\}$.

Note that, by (2.5) and (2.6), $\varphi(0, 2\pi n; \cdot) = T^n$. Thus, by (2.18),

$$\rho_g^N(0, z) = \frac{1}{N} \sum_{n=0}^{N-1} g(\varphi(0, 2\pi n; z)) = \frac{1}{N} \sum_{n=0}^{N-1} g(T^n(z)). \tag{2.26}$$

By (2.26) $(1/N)\sum_{n=0}^{N-1} g(T^n(z))$ converges as $N \to \infty$ iff $z \in U_0^g$. Hence, by Theorem 2.4, U_0^g is of full measure. Thus applying (2.26) and Theorem 2.4 again, we obtain, for μ_d-almost every $z \in U$

$$\hat{\rho}_g(0, \cdot) = \check{g}. \tag{2.27}$$

Since $\check{g} \in \mathcal{L}^1(U, \mathcal{I}, \mu_d), \mathcal{L}^1(U, \mathcal{I}, \mu_d) \subset \mathcal{L}^1(U, \mathcal{U}, \mu_d)$ and since $\hat{\rho}_g(0, \cdot)$ is $\mathcal{U}$-$\mathcal{R}$-measurable, we see by (2.27) that $\hat{\rho}_g(0, \cdot) \in \mathcal{L}^1(U, \mathcal{U}, \mu_d)$.

It is easy to see that $U_0^g \in \mathcal{I}$ and $\hat{\rho}_g(0, \cdot) \in \mathcal{L}^1(U, \mathcal{I}, \mu_d)$ as follows. From (2.26)

$$\rho_g^N(0, T(z)) = \frac{1}{N} \sum_{n=0}^{N-1} g(T^{n+1}(z)) = \frac{1}{N}(g(T^N(z)) - g(z)) + \rho_g^N(0, z). \tag{2.28}$$

Since g is bounded it follows from (2.28) that $z \in U_0^g$ iff $T(z) \in U_0^g$. Thus $U_0^g \in \mathcal{I}$, therefore $1_{U_0^g}(T(z)) = 1_{U_0^g}(z)$. Hence by (2.28)

$$1_{U_0^g}(T(z))\rho_g^N(0, T(z)) = \frac{1_{U_0^g}(z)}{N}(g(T^N(z)) - g(z)) + 1_{U_0^g}(z)\rho_g^N(0, z). \tag{2.29}$$

Since $\hat{\rho}_g(0, \cdot)$ is $\mathcal{U}$-$\mathcal{R}$-measurable, the $\mathcal{I}$-$\mathcal{R}$-measurability of $\hat{\rho}_g(0, \cdot)$ follows from the relation $\hat{\rho}_g(0, z) = \hat{\rho}_g(0, T(z))$ for all z and this equality holds for $z \in U$ from (2.29) and from Definition 2.3. Because $\hat{\rho}_g(0, \cdot)$ is $\mathcal{I}$-$\mathcal{R}$-measurable and since $\hat{\rho}_g(0, \cdot) \in \mathcal{L}^1(U, \mathcal{U}, \mu_d)$, it follows that $\hat{\rho}_g(0, \cdot) \in \mathcal{L}^1(U, \mathcal{I}, \mu_d)$.

Also, we have by (2.27) and Theorem 2.4 that $\int_U \hat{\rho}_g(0, \cdot)\mathrm{d}\mu_d = \int_U |\hat{\rho}_g(0, \cdot)|\mathrm{d}\mu_d = \int_U |\check{g}|\mathrm{d}\mu_d \le \int_U |g|\mathrm{d}\mu_d = \int_U g\mathrm{d}\mu_d$.

Now let $\mu_d(U) = 1$, i.e., let $(U, \mathcal{U}, \mu_d)$ be a probability space. Thus $\check{g} = E(g|\mathcal{I})$ and, by (2.27),

$$\int_A \hat{\rho}_g(0, \cdot)\mathrm{d}\mu_d = \int_A \check{g}\mathrm{d}\mu_d = \int_A g\mathrm{d}\mu_d, \tag{2.30}$$

for $A \in \mathcal{I}$. Since $\hat{\rho}_g(0, \cdot)$ is $\mathcal{I}$-$\mathcal{R}$-measurable, we conclude that $\hat{\rho}_g(0, \cdot) = E(g|\mathcal{I})$. If $\mu_d(U)$ is finite but arbitrary then we can repeat this argument by using the probability measure $\mu_d/\mu_d(U)$ and again obtain (2.30). Thus (2.30) holds whenever $\mu_d(U)$ is finite.

We have thus proved the following lemma about the $\theta = 0$ section:

Lemma 2.5. *Let ρ_g be an LD and $\hat{\rho}_g(0, \cdot)$ be defined as in Definition 2.3. Then $U_0^g \in \mathcal{I} = \{A \in \mathcal{U} : \varphi(2\pi, 0; A) = A\}$ and it is of full measure in U. Moreover $\hat{\rho}_g(0, \cdot)$ is in $\mathcal{L}^1(U, \mathcal{I}, \mu_d) \subset \mathcal{L}^1(U, \mathcal{U}, \mu_d)$ and satisfies the condition $\int_U \hat{\rho}_g(0, \cdot)\mathrm{d}\mu_d \le \int_U g\mathrm{d}\mu_d$ and the invariance property $\hat{\rho}_g(0, z) = \hat{\rho}_g(0, \varphi(0, 2\pi; z))$.*

If $\mu_d(U) = 1$ then $\hat{\rho}_g(0, \cdot) = E(g|\mathcal{I})$ and $\int_U \hat{\rho}_g(0, \cdot)\mathrm{d}\mu_d = \int_U g\mathrm{d}\mu_d$. The latter equality holds whenever $\mu_d(U)$ is finite. □

Note that, having used (2.6), we see that the 2π-periodicity of $f(\cdot, z)$ is essential for our proof of Lemma 2.5.

With Lemma 2.5 we easily obtain:

Theorem 2.6. *Let ρ_g be an LD. Then its stroboscopic average $\hat{\rho}_g$ is a 2π-periodic LD, U_θ^g is of full measure in U_θ and $U_{\theta+2\pi}^g = U_\theta^g$. Furthermore*

$$\int_{U_\theta} \hat{\rho}_g(\theta, z)\mathrm{d}\mu_d(z) \le \int_U g\mathrm{d}\mu_d. \tag{2.31}$$

Equality holds if $\mu_d(U) < \infty$.

Recall that in Lemma 2.5 and Theorem 2.6 μ_d is the Lebesgue measure.

Proof of Theorem 2.6. As pointed out in Section 2.2.3, the stroboscopic average $\hat{\rho}_g$ is a bounded and nonnegative function. Because, by Lemma 2.5, $\hat{\rho}_g(0,\cdot) \in \mathcal{L}^1(U,\mathcal{U},\mu_d)$ and since $\hat{\rho}_g \geq 0$, it follows from (2.24) that $\hat{\rho}_g$ is an LD.

To show that the LD $\hat{\rho}_g$ is 2π-periodic we first note that since

$$\rho_g^N(\theta+2\pi,z) - \rho_g^N(\theta,z) = \frac{1}{N}\left(\rho_g(\theta+2\pi N,z) - \rho_g(\theta,z)\right), \tag{2.32}$$

and since ρ_g is bounded, we have $z \in U_\theta^g$ iff $z \in U_{\theta+2\pi}^g$, i.e., $U_{\theta+2\pi}^g = U_\theta^g$. Thus we conclude from (2.32) that

$$1_{U_{\theta+2\pi}^g}(z)\rho_g^N(\theta+2\pi,z) - 1_{U_\theta^g}(z)\rho_g^N(\theta,z) = \frac{1_{U_\theta^g}(z)}{N}\left(\rho_g(\theta+2\pi N,z) - \rho_g(\theta,z)\right).$$

By taking the limit as $N \to \infty$ we obtain $\hat{\rho}_g(\theta,z) = \hat{\rho}_g(\theta+2\pi,z)$.

Because by Lemma 2.5, U_0^g is of full measure in U and since $\varphi(0,\theta;\cdot)$ is measure preserving, U_θ^g is, by (2.22), of full measure in U_θ.

It follows from Lemma 2.5 that $\int_U \hat{\rho}_g(0,\cdot)\mathrm{d}\mu_d \leq \int_U g\mathrm{d}\mu_d$. Since $\hat{\rho}_g$ is an LD, (2.14) gives $\int_U \hat{\rho}_g(0,\cdot)\mathrm{d}\mu_d = \int_{U_\theta} \hat{\rho}_g(\theta,z)\mathrm{d}\mu_d(z)$ and (2.31) follows. For finite measure μ_d we have from Lemma 2.5 that $\int_U \hat{\rho}_g(0,\cdot)\mathrm{d}\mu_d = \int_U g\mathrm{d}\mu_d$. Therefore equality holds in (2.31). □

2.4. Summary and discussion

An initial ensemble of particles, g, is given and assumed to evolve according to the flow φ of f by $\rho_g(\theta,z) = g(\varphi(0,\theta;z))$. An open nonempty set U which is invariant under the flow is assumed to exist. For each g, we have constructed a 2π-periodic Liouville density $\hat{\rho}_g(\theta,z) = \lim_{N\to\infty} 1_{U_\theta^g}(z)\rho_g^N(\theta,z)$ where $\rho_g^N(\theta,z) = (1/N)\sum_{n=0}^{N-1}\rho_g(\theta+2\pi n,z)$ (see (2.18)) and where $U_\theta^g = \varphi(\theta,0;U_0^g)$ is of full measure as we proved in Theorem 2.6 using the ergodic theorem, Theorem 2.4.

The definition of the flow φ, the invariant set U and the Liouville density are given in (2.4) and (2.8) and Definition 2.1 respectively. The stroboscopic average $\hat{\rho}_g$ of ρ_g is defined in Definition 2.3 and the convergence set U_θ^g of $\rho_g^N(\theta,\cdot)$ is defined in (2.21). The main results of Section 2 are Lemma 2.5 and Theorem 2.6 and the main tool is the ergodic theorem. In Lemma 2.5 we have constructed an $\mathcal{L}^1(U,\mathcal{I},\mu_d)$ fixed point solution of (2.16), namely $\hat{\rho}_g(0,\cdot)$; in fact, the fixed point condition is the invariance property which is equivalent to $\mathcal{I}$-measurability. Theorem 2.6 is proved using Lemma 2.5 and yields the following. If $\mu_d(U) < \infty$ and g is normalized then $\hat{\rho}_g$ is normalized. If g is normalized and $\mu_d(U) = \infty$, the strict inequality may hold in (2.31) (see for example [40,41]) in which case $\hat{\rho}_g$ is not normalized. In fact $\hat{\rho}_g(\theta,\cdot)$ could be zero almost everywhere, however, if not, it can be normalized by simply multiplying by a constant.

The long time stability of 2π-periodic Liouville densities is an important issue for storage rings but we have not investigated this.

In Appendix A, we apply our theory to the important case of linearized particle motion in a storage ring, the so-called integrable betatron motion in one degree of freedom. However, it is important to note that our theory does not require integrability.

3. The spin–orbit system and polarization fields

We remind the reader of the brief overview of this section in the last paragraph of the Introduction. This section runs parallel to Section 2 in the main. After introducing the spin–orbit motion in Section 3.1 we turn our attention to polarization fields. Because Liouville densities are simpler than polarization fields, the proof of Lemma 2.5 shows in a nutshell how the ergodic theorem applies to stroboscopic sequences. Therefore the proof of Lemma 2.5 may help the reader to go through the more complicated application of the ergodic theorem to the Cesàro sequences of polarization fields (see the proof of Lemma 3.6).

3.1. The spin–orbit motion

Now that we have tackled the purely orbital system and illustrated the basic ideas, we are ready to apply our techniques to the spin–orbit system, which is more complicated. As we explained in Section 1, spin motion is governed by the T–BMT equation [35]. The equations of the spin–orbit motion can then be written as

$$\dot{z} = f(\theta,z), \qquad z(\theta_0) = z^0 \in \mathbb{R}^d, \tag{3.1}$$

$$\dot{s} = \mathcal{A}(\theta,z)s, \qquad s(\theta_0) = s^0 \in \mathbb{R}^3, \tag{3.2}$$

where the 3×3 matrix $\mathcal{A}$ which represents the rotation vector $\vec{\Omega}$ in the T–BMT equation, is skew-symmetric ($\mathcal{A}^{\mathrm{T}} = -\mathcal{A}$) and real. Moreover, $\mathcal{A}(\cdot,z)$ is 2π-periodic and $\mathcal{A} : \mathbb{R}^{d+1} \to \mathbb{R}^6$ is of class C^1. The choice that $\mathcal{A}$ is of class C^1 is consistent with the fact that the electric and magnetic fields in storage rings are smooth.

Eq. (3.1) describes the same orbital motion as in Section 2 so that we continue to restrict z^0 to U_{θ_0}. Thus $z(\theta) = \varphi(\theta,\theta_0;z^0)$ for all time and (2.4)–(2.7) hold; also $U = \varphi(2\pi,0;U)$. Because of the linearity of (3.2) in s the solutions of (3.1) and (3.2) exist for all time since $z^0 \in U_{\theta_0}$.

From now on we restrict the s^0 in (3.2) to $V := \{s \in \mathbb{R}^3 : |s| < a\}$ with $a > 0$, where $|\cdot|$ is the Euclidean norm. Let $w = \binom{z}{s}$ and $f_{so}(\theta,w) = \binom{f(\theta,z)}{\mathcal{A}(\theta,z)s}$, then from (3.1) and (3.2) we obtain

$$\dot{w} = f_{so}(\theta,w), \qquad w(\theta_0) = w^0 \in U_{\theta_0} \times V. \tag{3.3}$$

We denote the solution of (3.3) by $w(\theta) = W(\theta,\theta_0;w^0)$ where the domain of W is $\mathbb{R}\times L\times V$ where L is defined in (2.9). Note that the periodicity of f_{so} implies that if $w(\cdot)$ is a solution of (3.3) then so is $w(\cdot+2\pi)$. In analogy with (2.5) and (2.6) we have

$$W(\theta_3, \theta_2; W(\theta_2, \theta_1; w)) = W(\theta_3, \theta_1; w), \tag{3.4}$$

$$W(\theta_2 + 2\pi, \theta_1 + 2\pi; w) = W(\theta_2, \theta_1; w). \tag{3.5}$$

Since $f_{so}(\theta, \cdot)$ is divergence free,

$$\det(D_3 W(\theta, \theta_0; w^0)) = 1. \tag{3.6}$$

From our assumptions on f, φ and $\mathcal{A}$ and noting that each $U_\theta \times V$ is open in $\mathbb{R}^{d+3}$, it is clear that $W(\theta, \theta_0; \cdot)$ is a C^1-diffeomorphism from $U_{\theta_0} \times V$ onto $U_\theta \times V$. We take as our measure space $(U_\theta \times V, \mathcal{U}_\theta \otimes \mathcal{V}, \mu_d \times \mu_3)$ where $\mathcal{V}$ is the collection of Borel subsets of V, $\mathcal{U}_\theta \otimes \mathcal{V}$ is the product σ-algebra of $\mathcal{U}_\theta$ and $\mathcal{V}$ and μ_3 is Lebesgue measure restricted to $\mathcal{V}$. It follows from (3.6) and the transformation theorem for Lebesgue integrals that each $W(\theta, \theta_0; \cdot)$ is measure preserving. Furthermore, $U \times V$ is an invariant set in the sense that $U \times V = W(2\pi, 0; U \times V)$.

Inserting the orbital dynamics into (3.2) yields

$$\dot{s} = \mathcal{A}(\theta, \varphi(\theta, \theta_0; z^0))s, \qquad s(\theta_0) = s^0. \tag{3.7}$$

Because of the smoothness of $\mathcal{A}$ and φ and the linearity of (3.7), solutions exist for all time and can be written as

$$s(\theta) = \Psi(\theta, \theta_0; z^0)s^0, \tag{3.8}$$

where Ψ is a function on $\mathbb{R} \times L$, called the "spin transfer matrix", and is defined by

$$D_1 \Psi(\theta, \theta_0; z^0) = \mathcal{A}(\theta, \varphi(\theta, \theta_0; z^0)) \Psi(\theta, \theta_0; z^0), \tag{3.9}$$

$$\Psi(\theta_0, \theta_0; z^0) = I_{3\times3}. \tag{3.10}$$

Note that $\Psi(\theta, \theta_0; \cdot)$ is of class C^1 on U_{θ_0}. Since $\mathcal{A}$ is real skew-symmetric, Ψ is real, $\Psi\Psi^{\mathrm{T}} = I_{3\times3}$ and $\det(\Psi) = 1$. Thus $\Psi \in SO(3)$ and $|s(\theta)| = |s^0|$ for all θ as mentioned in Section 1. It is clear that $W(\theta, \theta_0; w^0) = \begin{pmatrix} \varphi(\theta, \theta_0; z^0) \\ \Psi(\theta, \theta_0; z^0)s^0 \end{pmatrix}$. It follows from (3.4) and (3.5) that for all $z^0 \in U_{\theta_0}$

$$\Psi(\theta_2, \theta_1; \varphi(\theta_1, \theta_0; z^0)) \Psi(\theta_1, \theta_0; z^0) = \Psi(\theta_2, \theta_0; z^0), \tag{3.11}$$

$$\Psi(\theta + 2\pi, \theta_0 + 2\pi; z^0) = \Psi(\theta, \theta_0; z^0). \tag{3.12}$$

Taking $\theta_2 = \theta_0$ in (3.11) we obtain

$$\begin{aligned} \Psi^{-1}(\theta_1, \theta_0; z^0) &= (\Psi(\theta_1, \theta_0; z^0))^{\mathrm{T}} \\ &= \Psi(\theta_0, \theta_1; \varphi(\theta_1, \theta_0; z^0)). \end{aligned} \tag{3.13}$$

It is interesting to note that the $\Psi(\theta_0 + 2\pi n, \theta_0; \cdot)$ form a measurable $SO(3)$-cocycle over the orbital $\mathbb{Z}$-action on U_{θ_0} where n varies over the integers [28,29]. This is reflected in the identity (3.11).

3.2. *Polarization fields and their stroboscopic sequence*

3.2.1. *Polarization fields and their basic properties*

In this section we will outline the basic properties of polarization fields mentioned in Section 1 and we will define them rigorously below. Their significance in beam physics is discussed in Section 3.4. Consider an initial assignment of spins $G : U \to \mathbb{R}^3$, i.e., a spin attached to every phase space point. Under the flow of (3.3), the point z^0 and its attached spin $G(z^0)$ evolve to $z(\theta) = \varphi(\theta, 0; z^0)$ and $s(\theta) = \Psi(\theta, 0; z^0)G(z^0)$ at time θ. Let $\mathcal{S}_G(\theta, z)$ be the spin which is at the point $z = z(\theta)$ at time θ. Then

$$\mathcal{S}_G(\theta, z) = \Psi(\theta, 0; \varphi(0, \theta; z))\, G(\varphi(0, \theta; z)). \tag{3.14}$$

Definition 3.1 (*Polarization Field*). We call a function $\mathcal{S}_G : L \to \mathbb{R}^3$ a "polarization field", if it satisfies (3.14) where $G \in \mathcal{M}_b(\mathcal{U}, \mathcal{R}^3)$. □

Here $\mathcal{M}_b(\mathcal{U}_\theta, \mathcal{R}^3)$ is the set of bounded $\mathcal{U}_\theta$-$\mathcal{R}^3$-measurable functions where $\mathcal{R}^3$ is the σ-algebra of Borel subsets of $\mathbb{R}^3$ and as before $\mathcal{U}_0 = \mathcal{U}$. The function G will be called the 'generator' of $\mathcal{S}_G$. Note that by (3.14) and by the fact that $\varphi(0, \theta; \cdot)$ and the nine components of $\Psi(\theta, 0; \cdot)$ are measurable functions, $\mathcal{S}_G$ is a bounded function and $\mathcal{S}_G(\theta, \cdot)$ is in $\mathcal{M}_b(\mathcal{U}_\theta, \mathcal{R}^3)$. Of course, $\mathcal{S}_G(0, \cdot) = G$. Using (2.5) and (3.11), the relation between the values of the polarization field at times θ_1 and θ_2 is given by

$$\mathcal{S}_G(\theta_2, z) = \Psi(\theta_2, \theta_1; \varphi(\theta_1, \theta_2; z))\mathcal{S}_G(\theta_1, \varphi(\theta_1, \theta_2; z)). \tag{3.15}$$

The polarization field has two very different properties: the "dynamical" condition (3.14) and "regularity" conditions. In contrast to the dynamical condition, the regularity conditions are to a certain extent a matter of convenience. For example in this paper we choose weak conditions, namely that $G \in \mathcal{M}_b(\mathcal{U}, \mathcal{R}^3)$, because they serve us well when we come to stroboscopic averaging in later sections. However, one sometimes drops the condition that G is bounded and on other occasions one assumes that G is of class C^k for some nonnegative integer k. In Section 3.4.4 we discuss what happens if one drops the regularity conditions.

It is desirable that, in a storage ring with a polarized beam, the physically relevant polarization fields (see Section 3.4.1) be 2π-periodic, i.e., $\mathcal{S}_G(\theta + 2\pi, z) = \mathcal{S}_G(\theta, z)$ or, at least, 2π-periodic in measure. We say that a polarization field $\mathcal{S}_G$ is "2π-periodic in measure" if for every θ the equality $\mathcal{S}_G(\theta + 2\pi, z) = \mathcal{S}_G(\theta, z)$ holds for μ_d-almost every z in U_θ. Note that if $\mathcal{S}_G$ is 2π-periodic then (3.15) gives

$$\begin{aligned} \mathcal{S}_G(\theta, z) = {} & \Psi(\theta + 2\pi, \theta; \varphi(\theta, \theta + 2\pi; z)) \\ & \times \mathcal{S}_G(\theta, \varphi(\theta, \theta + 2\pi; z)). \end{aligned} \tag{3.16}$$

Spin fields and invariant spin fields play an important role in the theory of polarized beam physics and we now define them.

Definition 3.2 (*Spin Field and Invariant Spin Field*). A polarization field $\mathcal{S}_G$ with $|\mathcal{S}_G(\theta, z)| = 1$ is called a "spin field". A spin field is called an "invariant spin field" if it is 2π-periodic in measure. □

Thus a polarization field $\mathcal{S}_G$ is a spin field iff $|G(z)| = 1$. Clearly 2π-periodic spin fields are invariant spin fields as mentioned already in Section 1 and, in this paper, the emphasis is on 2π-periodic invariant spin fields.

The concept of the invariant spin field was introduced into polarized beam physics in the 1970s [17,18], but at that time mathematical conditions like the regularity conditions were not used explicitly. Moreover, the concept arose within a Hamiltonian integrable description of spin–orbit motion.

If $\mathcal{S}_G$ is a 2π-periodic polarization field, then from (3.14) G must satisfy the basic fixed point or functional equation

$$G(z) = \Psi(2\pi, 0; \varphi(0, 2\pi; z))\, G(\varphi(0, 2\pi; z)). \tag{3.17}$$

Recall the analogous fixed point equation (2.16) and note the similarity to homological equations in dynamical systems. This important equation was, at least in the context of polarized beam physics, probably first exploited by Yokoya and by Balandin and Golubeva [2,47]. 2π-periodic polarization fields are also discussed in [7,8,17,18,33,39,42,45,48]. Note that (3.17) is also a *sufficient* condition for the 2π-periodicity of $\mathcal{S}_G$ since

$$\begin{aligned}
\mathcal{S}_G(\theta + 2\pi, z) &= \Psi(\theta + 2\pi, 2\pi; \varphi(2\pi, \theta + 2\pi; z)) \\
&\quad \times \mathcal{S}_G(2\pi, \varphi(2\pi, \theta + 2\pi; z)) \\
&= \Psi(\theta, 0; \varphi(0, \theta; z))\mathcal{S}_G(2\pi, \varphi(0, \theta; z)) \\
&= \Psi(\theta, 0; \varphi(0, \theta; z)) \\
&\quad \times \Psi(2\pi, 0; \varphi(0, 2\pi; \varphi(0, \theta; z))) \\
&\quad \times G(\varphi(0, 2\pi; \varphi(0, \theta; z))) \\
&= \Psi(\theta, 0; \varphi(0, \theta; z))G(\varphi(0, \theta; z)) \\
&= \mathcal{S}_G(\theta, z),
\end{aligned} \tag{3.18}$$

where we used (3.15) in the first equality, (2.6) and (3.12) in the second, (3.14) in the third and fifth equalities and (3.17) in the fourth equality. We conclude that a polarization field $\mathcal{S}_G$ is 2π-periodic iff G satisfies (3.17) for $z \in U$. It is also easy to show that a polarization field $\mathcal{S}_G$ is 2π-periodic in measure iff G satisfies (3.17) for μ_d-almost z in U.

If the polarization field $\mathcal{S}_G$ is of class C^1 then, due to (2.4), (3.9) and (3.14), it satisfies the first order PDE

$$D_1\mathcal{S}_G + (D_2\mathcal{S}_G) f(\theta, z) = \mathcal{A}(\theta, z)\mathcal{S}_G.$$

The following remarks will be useful.

Remarks.

(7) Let $\mathcal{S}_G$ be a polarization field and let, for every $z \in U$, $G(z) \neq 0$. Since

$$|\mathcal{S}_G(\theta, z)| = |G(\varphi(0, \theta; z))|, \tag{3.19}$$

we have $\mathcal{S}_G(\theta, z) \neq 0$. Since $G/|G|$ is in $\mathcal{M}_b(\mathcal{U}, \mathcal{R}^3)$ and since by (3.14) and (3.19)

$$\frac{\mathcal{S}_G(\theta, z)}{|\mathcal{S}_G(\theta, z)|} = \Psi(\theta, 0; \varphi(0, \theta; z))\, \frac{G(\varphi(0, \theta; z))}{|G(\varphi(0, \theta; z))|},$$

we obtain that $\mathcal{S}_G/|\mathcal{S}_G|$ is a spin field generated by $G/|G|$. If $\mathcal{S}_G$ is 2π-periodic then $\mathcal{S}_G/|\mathcal{S}_G|$ is an invariant spin field.

(8) If $\mathcal{S}_G$ is a polarization field and ρ_g is an LD (recall Section 2.2.1) then $\rho_g\mathcal{S}_G$ is a polarization field. This can be demonstrated in a way similarly to that in Remark (7). Thus if a 2π-periodic spin field and a large number of 2π-periodic LD's exists, one has a large number of 2π-periodic polarization fields. □

3.2.2. The stroboscopic and Cesàro sequences of a polarization field

In this section we define a stroboscopic sequence of the polarization field $\mathcal{S}_G$ and the associated Cesàro sequence, and derive properties of the Cesàro sequence.

Definition 3.3 (*Stroboscopic Sequence and Cesàro Sequence*). Let $\mathcal{S}_G$ be a polarization field. The "stroboscopic sequence of $\mathcal{S}_G$" consists of the functions $\mathcal{S}_G(\cdot + 2\pi n, \cdot) : L \to \mathbb{R}^3$ where $n = 0, 1, \ldots$. Let $\mathcal{S}_G^N : L \to \mathbb{R}^3$ be defined by

$$\begin{aligned}
\mathcal{S}_G^N(\theta, z) &:= \frac{1}{N}\sum_{n=0}^{N-1} \mathcal{S}_G(\theta + 2\pi n, z) \\
&= \frac{1}{N}\sum_{n=0}^{N-1} \Psi(\theta + 2\pi n, 0; \varphi(0, \theta + 2\pi n; z)) \\
&\quad \times G(\varphi(0, \theta + 2\pi n; z)),
\end{aligned} \tag{3.20}$$

where $N = 1, 2, \ldots$. The sequence $\{\mathcal{S}_G^N\}_{N=1}^\infty$ is called the "Cesàro sequence of $\mathcal{S}_G$" and the $\mathcal{S}_G^N$ are called "Cesàro averages of $\mathcal{S}_G$". □

Of course, in the special case when $\mathcal{S}_G$ is 2π-periodic, we have $\mathcal{S}_G^N = \mathcal{S}_G$. In the general case we claim that $\mathcal{S}_G^N$ is a polarization field for each N. We first argue that each element of the stroboscopic sequence is a polarization field, i.e., that shifting the time by $2\pi n$ in the polarization field $\mathcal{S}_G$ gives a polarization field if n is an integer. In fact, $\Psi(2\pi n, 0; \varphi(0, 2\pi n; \cdot))\, G(\varphi(0, 2\pi n; \cdot))$ is in $\mathcal{M}_b(\mathcal{U}, \mathcal{R}^3)$ and we have, by (2.5), (2.6), (3.11), (3.12) and (3.14),

$$\begin{aligned}
&\Psi(\theta, 0; \varphi(0, \theta; z))\,\Psi(2\pi n, 0; \varphi(0, 2\pi n; \varphi(0, \theta; z))) \\
&G(\varphi(0, 2\pi n; \varphi(0, \theta; z))) = \Psi(\theta, 0; \varphi(0, \theta; z)) \\
&\quad \times \Psi(2\pi n, 0; \varphi(0, \theta + 2\pi n; z))G(\varphi(0, \theta + 2\pi n; z)) \\
&\quad = \Psi(\theta + 2\pi n, 2\pi n; \varphi(0, \theta; z)) \\
&\quad \times \Psi(2\pi n, 0; \varphi(0, \theta + 2\pi n; z))G(\varphi(0, \theta + 2\pi n; z)) \\
&\quad = \Psi(\theta + 2\pi n, 0; \varphi(0, \theta + 2\pi n; z))G(\varphi(0, \theta + 2\pi n; z)) \\
&\quad = \mathcal{S}_G(\theta + 2\pi n, z),
\end{aligned} \tag{3.21}$$

which proves that $\mathcal{S}_G(\cdot + 2\pi n, \cdot)$ is a polarization field which is generated by $\Psi(2\pi n, 0; \varphi(0, 2\pi n; \cdot))\, G(\varphi(0, 2\pi n; \cdot))$. Thus each element of the stroboscopic sequence is a polarization field. It follows from (3.20) and (3.21) that, for every N, $\mathcal{S}_G^N$ is a polarization field generated by $(1/N)\sum_{n=0}^{N-1} \Psi(2\pi n, 0; \varphi(0, 2\pi n; \cdot))\, G(\varphi(0, 2\pi n; \cdot))$. Since $\mathcal{S}_G^N$ is a polarization field then by definition

$$\mathcal{S}_G^N(\theta, z) = \Psi(\theta, 0; \varphi(0, \theta; z))\, \mathcal{S}_G^N(0, \varphi(0, \theta; z)). \tag{3.22}$$

3.2.3. The stroboscopic average of a polarization field

We now discuss some convergence properties of the Cesàro sequence. Let $\tilde{U}_\theta^G \subset U_\theta$ be the set on which $\mathcal{S}_G^N(\theta, \cdot)$ converges, i.e.,

$$\tilde{U}_\theta^G := \{z \in U_\theta : \lim_{N\to\infty} \mathcal{S}_G^N(\theta, z) \text{ exists}\}, \tag{3.23}$$

Clearly $\tilde{U}_\theta^G \in \mathcal{U}_\theta$ and our goal is to show that this is large in measure.

From (3.22) $\mathcal{S}_G^N(0, \varphi(0, \theta; z))$ converges on the same set, thus $\mathcal{S}_G^N(0, z)$ converges for $z \in \varphi(0, \theta; \tilde{U}_\theta^G)$ and so

$$\tilde{U}_\theta^G = \varphi(\theta, 0; \tilde{U}_0^G). \tag{3.24}$$

Since $\varphi(0,\theta;\cdot)$ is measure preserving, (3.24) gives $\mu_d(U_\theta^g) = \mu_d(U_0^g)$. Note that since G is bounded, the sequence $\{\mathcal{S}_G^N(\theta,z)\}_{N=1}^\infty$ is bounded. Thus the limit, if it exists, is always in $\mathbb{R}^3$. Furthermore since

$$\mathcal{S}_G^N(\theta+2\pi,z) - \mathcal{S}_G^N(\theta,z) = \frac{1}{N}\left(\mathcal{S}_G(\theta+2\pi N,z) - \mathcal{S}_G(\theta,z)\right), \tag{3.25}$$

and since $\mathcal{S}_G$ is bounded, we have $z \in \tilde{U}_\theta^G$ iff $z \in \tilde{U}_{\theta+2\pi}^G$, i.e., $\tilde{U}_{\theta+2\pi}^G = \tilde{U}_\theta^G$. The latter equality and (3.24) imply that $\tilde{U}_0^G = \varphi(2\pi,0;\tilde{U}_0^G)$, i.e. $\tilde{U}_0^G \in \mathcal{I}$.

Definition 3.4 (*Stroboscopic Average*). Let $\mathcal{S}_G$ be a polarization field. Then we call the function $\hat{\mathcal{S}}_G : L \to \mathbb{R}^3$, defined by $\hat{\mathcal{S}}_G(\theta,z) := \lim_{N\to\infty} 1_{\tilde{U}_\theta^G}(z)\mathcal{S}_G^N(\theta,z)$, the "stroboscopic average of $\mathcal{S}_G$". $\square$

It follows from the definition that $\hat{\mathcal{S}}_G(\theta,z)$ is the limit of $\mathcal{S}_G^N(\theta,z)$ if it exists and zero otherwise. Furthermore every polarization field $\mathcal{S}_G$ has a unique stroboscopic average $\hat{\mathcal{S}}_G$ which is a bounded function and $\hat{\mathcal{S}}_G(\theta,\cdot)$ is in $\mathcal{M}_b(\mathcal{U}_\theta,\mathcal{R}^3)$. We also note that $\mathcal{S}_G^N(\theta,z)$ converges for all $(\theta,z) \in L$ as $N \to \infty$ iff $\tilde{U}_0^G = U$. In particular this is true if $\mathcal{S}_G$ is 2π-periodic. It is also clear that $\tilde{U}_0^G = U$ if $\hat{\mathcal{S}}_G$ has no zeros. We note by (3.24) that

$$1_{\tilde{U}_\theta^G}(z) = 1_{\tilde{U}_0^G}\left(\varphi(0,\theta;z)\right), \tag{3.26}$$

and, since $\mathcal{S}_G^N$ is a polarization field, we see that the function $1_{\tilde{U}_\theta^G}(z)\mathcal{S}_G^N(\theta,z)$ in Definition 3.4 is a polarization field. Thus the stroboscopic average $\hat{\mathcal{S}}_G$ is the limit of a sequence of polarization fields.

In Section 3.3 it will be shown, as a consequence of the ergodic theorem, that $\tilde{U}_\theta^G$ is of full measure in U_θ if U has finite measure. We now show that $\hat{\mathcal{S}}_G$ is a 2π-periodic polarization field. Since $1_{\tilde{U}_\theta^G}(z)\mathcal{S}_G^N(\theta,z)$ is a polarization field we have

$$1_{\tilde{U}_\theta^G}(z)\mathcal{S}_G^N(\theta,z) = \Psi\left(\theta,0;\varphi(0,\theta;z)\right)1_{\tilde{U}_0^G}\left(\varphi(0,\theta;z)\right)\mathcal{S}_G^N\left(0,\varphi(0,\theta;z)\right).$$

By taking $N \to \infty$, we obtain

$$\hat{\mathcal{S}}_G(\theta,z) = \Psi\left(\theta,0;\varphi(0,\theta;z)\right)\hat{\mathcal{S}}_G\left(0,\varphi(0,\theta;z)\right). \tag{3.27}$$

Thus and since $\hat{\mathcal{S}}_G(0,\cdot) \in \mathcal{M}_b(\mathcal{U},\mathcal{R}^3)$ we find that $\hat{\mathcal{S}}_G$ is a polarization field. To show that it is 2π-periodic, we conclude from (3.25) that

$$1_{\tilde{U}_{\theta+2\pi}^G}(z)\mathcal{S}_G^N(\theta+2\pi,z) - 1_{\tilde{U}_\theta^G}(z)\mathcal{S}_G^N(\theta,z) = \frac{1_{\tilde{U}_\theta^G}(z)}{N}\left(\mathcal{S}_G(\theta+2\pi N,z) - \mathcal{S}_G(\theta,z)\right), \tag{3.28}$$

where we also used the fact that $\tilde{U}_{\theta+2\pi}^G = \tilde{U}_\theta^G$. Taking $N \to \infty$ in (3.28) and using the fact that $\mathcal{S}_G$ is bounded we find that $\hat{\mathcal{S}}_G(\theta+2\pi,z) = \hat{\mathcal{S}}_G(\theta,z)$ so that the stroboscopic average is 2π-periodic. We have thus proved:

Theorem 3.5. *Let $\mathcal{S}_G$ be a polarization field. Then its stroboscopic average $\hat{\mathcal{S}}_G$ is a 2π-periodic polarization field. Furthermore $\tilde{U}_{\theta+2\pi}^G = \tilde{U}_\theta^G$ and $\tilde{U}_0^G \in \mathcal{I} = \{A \in \mathcal{U} : \varphi(2\pi,0;A) = A\}$.* $\square$

Having used (2.6) and (3.12) in (3.21) we see that the 2π-periodicity of $f(\cdot,z)$ and $\mathcal{A}(\cdot,z)$ is essential in our proof. Note also that we did not need the ergodic theorem in our proof of Theorem 3.5 since for a polarization field we do not have to demonstrate that its components are integrable.

Remark.

(9) It follows from Theorem 3.5 that $\hat{\mathcal{S}}_G(0,z)$ satisfies the fixed point Eq. (3.17), i.e.,

$$\hat{\mathcal{S}}_G(0,z) = \Psi\left(2\pi,0;\varphi(0,2\pi;z)\right)\hat{\mathcal{S}}_G\left(0,\varphi(0,2\pi;z)\right). \tag{3.29}$$

Recalling Remark (3), $\hat{\mathcal{S}}_G(0,\cdot)$ is $\mathcal{I}$-$\mathcal{R}^3$-measurable iff

$$\hat{\mathcal{S}}_G(0,\cdot) = \hat{\mathcal{S}}_G\left(0,\varphi(0,2\pi;\cdot)\right). \tag{3.30}$$

In the trivial case where $\mathcal{A} = 0$ (hence $\Psi = I_{3\times3}$), (3.29) implies (3.30) and thus $\hat{\mathcal{S}}_G(0,\cdot)$ is $\mathcal{I}$-$\mathcal{R}^3$-measurable. However, if $\mathcal{A} \neq 0$ then $\Psi \neq I_{3\times3}$ so that (3.30) does not hold whence $\hat{\mathcal{S}}_G(0,\cdot)$ is not $\mathcal{I}$-$\mathcal{R}^3$-measurable. $\square$

3.3. Applying the ergodic theorem to the Cesàro sequence of a polarization field

With Theorem 3.5 we see that the stroboscopic average of a polarization field has the important property that it is a 2π-periodic polarization field. In this section we use the ergodic theorem to prove that the Cesàro sequence converges (at least) almost everywhere. In particular, we will prove Theorem 3.9 which states that if U has finite measure then, for every θ, $\tilde{U}_0^G$ is of full measure. It follows that the sequence $\mathcal{S}_G^N(\theta,z)$ converges for μ_d-almost every z to $\hat{\mathcal{S}}_G(\theta,z)$ as $N \to \infty$.

By (3.24) the convergence of a Cesàro sequence $\mathcal{S}_G^N$ is determined by the convergence of the sequence $\mathcal{S}_G^N(0,\cdot)$. Therefore the following preparatory lemma studies the sequence $\mathcal{S}_G^N(0,\cdot)$ in detail. With

$$G_N(z) := \mathcal{S}_G^N(0,z) = \frac{1}{N}\sum_{n=0}^{N-1}\Psi(2\pi n,0;\varphi(0,2\pi n;z))G(\varphi(0,2\pi n;z)), \tag{3.31}$$

we obtain

Lemma 3.6. *Let U be of finite measure and G in $\mathcal{M}_b(\mathcal{U},\mathcal{R}^3)$. Then $\tilde{U}_0^G$ is of full measure in U, i.e., for μ_d-almost every z,*

$$\lim_{N\to\infty} G_N(z) = \hat{\mathcal{S}}_G(0,\cdot). \tag{3.32}$$

Proof of Lemma 3.6. The lemma will be proved with the aid of two propositions.

We begin by defining the measure preserving period advance map associated with (3.3). The period advance map $\mathcal{P} : U \times V \to U \times V$ in the $\theta = 0$ section and its inverse are given by

$$\mathcal{P}(w) = W(2\pi, 0; w), \qquad \mathcal{P}^{-1}(w) = W(0, 2\pi; w). \tag{3.33}$$

Of course, $\mathcal{P}$ and $\mathcal{P}^{-1}$ are measure preserving C^1-diffeomorphisms onto $U \times V$. This follows from Section 3.1 and since $U_{2\pi} = U$.

The key to our proof is the application of the ergodic theorem (Theorem 2.4) to the function $b : U \times V \to \mathbb{R}$ defined by

$$b(w) := s^{\mathrm{T}} G(z). \tag{3.34}$$

Since U is of finite measure and $G \in \mathcal{M}_b(\mathcal{U}, \mathcal{R}^3)$ we have $b \in \mathcal{L}^1(U \times V, \mathcal{U} \otimes \mathcal{V}, \mu_d \times \mu_3)$. With this b and with $T = \mathcal{P}^{-1}, \mathcal{M} = \mathcal{U} \otimes \mathcal{V}, m = \mu_d \times \mu_3$, Theorem 2.4 can be used to prove the first proposition, namely Proposition 3.7 below.

The b in (3.34) is not an obvious choice but we need a scalar function of w and this is a simple one. Furthermore $s^{\mathrm{T}} \mathcal{S}_G(\theta, z)$ is conserved along solutions of (3.3) as is easily seen using (3.14) and the relations $z = \varphi(\theta, 0; z^0), s = \Psi(\theta, \theta_0; z^0)s^0$ and it has the formal structure of a "spin action" (see e.g. [7, 45]). However the only justification which matters is that the ansatz (3.34) works.

Proposition 3.7. *There exists a function $\check{b}$ in $\mathcal{L}^1(U \times V, \mathcal{U} \otimes \mathcal{V}, \mu_d \times \mu_3)$ and a set $E \in \mathcal{U} \otimes \mathcal{V}$ of full measure such that, for each $w = (z, s) \in E$,*

$$\lim_{N\to\infty} s^{\mathrm{T}} G_N(z) = \check{b}(w).$$

Proof of Proposition 3.7. Clearly $\mathcal{P}^{-n}(w) = W(0, 2\pi n; w) = \begin{pmatrix} \varphi(0, 2\pi n; z) \\ \Psi(0, 2\pi n; z)s \end{pmatrix}$ and thus

$$\begin{aligned} b(\mathcal{P}^{-n}(w)) &= (\Psi(0, 2\pi n; z)s)^{\mathrm{T}} G(\varphi(0, 2\pi n; z)) \\ &= s^{\mathrm{T}} [\Psi^{\mathrm{T}}(0, 2\pi n; z) G(\varphi(0, 2\pi n; z))] \\ &= s^{\mathrm{T}} [\Psi(2\pi n, 0; \varphi(0, 2\pi n; z))\, G(\varphi(0, 2\pi n; z))], \end{aligned}$$

where the third equality follows from (3.13). Therefore by (3.31) $s^{\mathrm{T}} G_N(z) = \frac{1}{N} \sum_{n=0}^{N-1} b(\mathcal{P}^{-n}(w))$ and, because b is in $\mathcal{L}^1(U \times V, \mathcal{U} \otimes \mathcal{V}, \mu_d \times \mu_3)$, Proposition 3.7 follows from Theorem 2.4. $\square$

To go from the $\check{b}$ of Proposition 3.7 to the $\hat{\mathcal{S}}_G(0, \cdot)$ of Lemma 3.6 we need a technical lemma.

Proposition 3.8. *Let $F \in \mathcal{U} \otimes \mathcal{V}$ be of full measure and define the section of F at $s \in V$ by $F^s := \{z \in U : (z, s) \in F\}$. Then the following hold.*

(a) *There exists a $V_F \subset V$ of full measure such that, for $s \in V_F$, $\mu_d(F^s) = \mu_d(U)$, i.e., for μ_3-almost every $s \in V$, F^s is of full measure in U.*

(b) *There exist three linearly independent vectors s^1, s^2, s^3 in V_F.*

Proof of Proposition 3.8. Clearly $1_{F^s}(z) = 1_F(z, s)$ so that $F^s \in \mathcal{U}$. Thus

$$\mu_d(F^s) = \int_U 1_{F^s}(z) \mathrm{d}\mu_d(z) = \int_U 1_F(z, s) \mathrm{d}\mu_d(z), \tag{3.35}$$

and $\mu_d(F^s)$ is a $\mathcal{V}$-$\mathcal{R}$-measurable and bounded function of s. By (3.35) and the Fubini theorem $\int_V \mu_d(F^s) \mathrm{d}\mu_3(s) = \int_{U \times V} 1_F \mathrm{d}(\mu_d \times \mu_3) = (\mu_d \times \mu_3)(F) = \mu_d(U)\mu_3(V) = \int_V \mu_d(U) \mathrm{d}\mu_3$. Therefore $\int_V (\mu_d(U) - \mu_d(F^s)) \mathrm{d}\mu_3(s) = 0$. Since the integrand is nonnegative, a) follows (see for example [11, Section 13]). If (b) were not true then all s in V_F would lie in a plane in $\mathbb{R}^3$ and we would obtain the contradiction that $\mu_3(V_F) = 0$. $\square$

We now complete the proof of Lemma 3.6. The set E of Proposition 3.7 is of full measure so we can apply Proposition 3.8 to it. Choose s^1, s^2, s^3 as in Proposition 3.8(b) and let $U^1 := E^{s^1} \cap E^{s^2} \cap E^{s^3}$. Then $\mu_d(U^1) = \mu_d(U)$. If $z \in U^1$ then $(z, s^k) \in E$ and, by Proposition 3.7, $(s^k)^{\mathrm{T}} G_N(z)$ converges to $\check{b}(z, s^k)$ as $N \to \infty$. Let $A = [s^1, s^2, s^3]^{\mathrm{T}}$ then

$$G_N(z) = A^{-1} A G_N(z) = A^{-1} \begin{pmatrix} (s^1)^{\mathrm{T}} G_N(z) \\ (s^2)^{\mathrm{T}} G_N(z) \\ (s^3)^{\mathrm{T}} G_N(z) \end{pmatrix},$$

which converges, when $z \in U^1$, to $A^{-1} \begin{pmatrix} \check{b}(z, s^1) \\ \check{b}(z, s^2) \\ \check{b}(z, s^3) \end{pmatrix}$. Since U^1 is of full measure, we have shown that $G_N(z)$ converges for μ_d-almost every z, i.e., $\tilde{U}_0^G$ is of full measure. $\square$

It is now easy to prove the main result of this section.

Theorem 3.9. *Let $\mathcal{S}_G$ be a polarization field and U have finite measure. Then, for every θ, $\tilde{U}_\theta^G$ is of full measure in U_θ, i.e., the sequence $\mathcal{S}_G^N(\theta, z)$ converges for μ_d-almost every $z \in U_\theta$ to $\hat{\mathcal{S}}_G(\theta, z)$ as $N \to \infty$. Moreover, every component of $\mathcal{S}_G^N(\theta, \cdot)$ is in $\mathcal{L}^1(U_\theta, \mathcal{U}_\theta, \mu_d)$ and converges to the corresponding component of $\hat{\mathcal{S}}_G(\theta, \cdot)$ in $\mathcal{L}^1$ and each component of $\hat{\mathcal{S}}_G(\theta, \cdot)$ is in $\mathcal{L}^1(U_\theta, \mathcal{U}_\theta, \mu_d)$.*

Proof of Theorem 3.9. Because of Lemma 3.6, $\tilde{U}_0^G$ is of full measure in U. Since $\varphi(0, \theta; \cdot)$ is measure preserving, (3.24) gives $\mu_d(U_\theta^g) = \mu_d(U_0^g) = \mu_d(U)$.

Since $\mathcal{S}_G^N(\theta, \cdot)$ is a bounded function and U is of finite measure, every component of $\mathcal{S}_G^N(\theta, \cdot)$ is in $\mathcal{L}^1(U_\theta, \mathcal{U}_\theta, \mu_d)$. Thus, by Definition 3.4, and as a consequence of the dominated convergence theorem, every component of $\mathcal{S}_G^N(\theta, \cdot)$ converges in $\mathcal{L}^1$ to the corresponding component of $\hat{\mathcal{S}}_G(\theta, \cdot)$. In particular, each component of $\hat{\mathcal{S}}_G(\theta, \cdot)$ is in $\mathcal{L}^1(U_\theta, \mathcal{U}_\theta, \mu_d)$. $\square$

Remarks.

(10) In the trivial case where $\mathcal{A} = 0$ (hence $\Psi = I_{3\times 3}$) one has

$$\mathcal{S}_G^N(0, z) = \frac{1}{N} \sum_{n=0}^{N-1} G(\varphi(0, 2\pi n; z)), \tag{3.36}$$

so that in this case the proof of Lemma 3.6 is straightforward—one just applies Theorem 2.4 to the components of G! However if $\mathcal{A} \neq 0$ then $\Psi \neq I_{3\times 3}$ so that one cannot apply Theorem 2.4 to the components of G because (3.36) does not hold. Thus if $\mathcal{A} \neq 0$ an alternative has to be found. Our above approach is to apply Theorem 2.4 to the function b in (3.34) and then use Proposition 3.8(b) to prove Lemma 3.6. Accordingly, the proof of Lemma 2.5 is considerably simpler than that of Lemma 3.6.

(11) With Remark (10) we see that ρ_g and $\mathcal{S}_G$ behave very differently under stroboscopic averaging, i.e. the ergodic theorem has very different meanings for the orbit and spin motion. Thus it is not surprising that the convergence speeds of the Cesàro sequences are different. See also Section 3.4.2.

(12) It is clear by Remark (7) and by Theorem 3.5 how one constructs an invariant spin field if $\hat{\mathcal{S}}_G$ has no zeros. One can show more generally that an invariant spin field exists if $\hat{\mathcal{S}}_G(0,\cdot)$ is nonzero μ_d-almost everywhere. On the other hand we do not know if such a $\hat{\mathcal{S}}_G$ always exists. Nevertheless our experience suggests that invariant spin fields always exist. However, as we know from [4, Section 6], there are examples where no C^1 invariant spin field exists.

(13) If a positive constant ξ exists such that, for all N and z, $(\mathcal{S}_G(0,z))^{\mathrm{T}}\mathcal{S}_G^N(0,z) \geq \xi$ then, trivially, $\hat{\mathcal{S}}_G(0,\cdot)$ has no zeros on $\tilde{U}_0^G$. This criterion plays an important role in the computation of invariant spin fields with the computer code SPRINT [32,33,42] and these computations show that the criterion is very often fulfilled.

(14) The simplest example for f with U of finite measure is the betatron motion of Appendix A. However, since f is much more general, our spin–orbit system can cover situations where the orbital motion is nonintegrable. The typical nonintegrable examples we have in mind for f with U of finite measure are orbital motions which obey the *Moser twist theorem* and where U is bounded. $\square$

3.4. Several aspects of polarization fields including applications to the physics of polarized beams

In this section we outline various aspects of polarization fields and their Cesàro sequences and in particular we elaborate on our comments in Section 1 on the significance of our work for polarized beam physics in storage rings. See also the reference list and Section I in [4]. The basics of polarized beam physics are discussed in [27, Section 2.7], [33].

3.4.1. Long time averages of polarization

The statistical properties of a beam of spin-1/2 particles (e.g. protons, electrons, positrons, muons) can be encoded in a one-particle quantum mechanical density matrix [24]. A particle beam in a storage ring represents a highly mixed quantum state. The orbital density $\rho = \rho_g$ and P_{loc} are then obtained by a Wigner–Weyl transform in the semiclassical approximation [5,6]. The local polarization is a polarization field and it is the local average of the normalized vector $s/|s|$ at each (θ, z), so that $|P_{\mathrm{loc}}| \leq 1$. These things are discussed for example in [30].

The orbital density is a normalized LD as defined in Section 2.2.1. In order to apply Theorem 3.9, we assume that U has finite measure. We also assume that the beam is in "orbital equilibrium", i.e. that the orbital density ρ_g is 2π-periodic in measure. The polarization $P(\theta)$ of the whole beam, i.e., the beam polarization, is defined as

$$P(\theta) := \int_{U_\theta} \rho_g(\theta, z) P_{\mathrm{loc}}(\theta, z)\mathrm{d}\mu_d(z). \tag{3.37}$$

Recalling from Remark (8) that $\rho_g P_{\mathrm{loc}}$ is a polarization field, the integral in (3.37) is well defined because $\rho_g(\theta,\cdot)P_{\mathrm{loc}}(\theta,\cdot)$ is bounded and U has finite measure. Since $\rho_g P_{\mathrm{loc}}$ is a polarization field it has a generator G, i.e., $\rho_g P_{\mathrm{loc}} = \mathcal{S}_G$ where $G(z) = \rho_g(0,z)P_{\mathrm{loc}}(0,z)$.

If the beam is in "spin equilibrium", i.e. if P_{loc} is 2π-periodic in measure, then clearly $\mathcal{S}_G$ is a polarization field 2π-periodic in measure and the beam polarization is 2π-periodic by (3.37).

If the beam is not in spin equilibrium, one considers the time average of the beam polarization. Then, although the local polarization is not 2π-periodic in measure, the time average of the beam polarization exists and is 2π-periodic as we now show. The time average at some θ, usually corresponding to the position of a particle physics experiment in the ring, is

$$\bar{P}(\theta) := \lim_{N\to\infty}(1/N)\sum_{n=0}^{N-1} P(\theta + 2\pi n). \tag{3.38}$$

Using the theory we have developed, we first show that $\bar{P}$ exists. We conclude from (3.37) that

$$\begin{aligned}
&\frac{1}{N}\sum_{n=0}^{N-1} P(\theta + 2\pi n)\\
&\quad= \frac{1}{N}\sum_{n=0}^{N-1}\int_{U_{\theta+2\pi n}} \rho_g(\theta+2\pi n, z)P_{\mathrm{loc}}(\theta+2\pi n, z)\mathrm{d}\mu_d(z)\\
&\quad= \frac{1}{N}\sum_{n=0}^{N-1}\int_{U_{\theta+2\pi n}} \mathcal{S}_G(\theta+2\pi n, z)\mathrm{d}\mu_d(z)\\
&\quad= \int_{U_\theta} \mathcal{S}_G^N(\theta, z)\mathrm{d}\mu_d(z).
\end{aligned} \tag{3.39}$$

Clearly the integrals in (3.39) are well defined. By Theorem 3.9, the components of $\mathcal{S}_G^N(\theta,\cdot)$ converge in $\mathcal{L}^1$ to the corresponding components of $\hat{\mathcal{S}}_G(\theta,\cdot)$ so that by (3.39) (see also [11, Theorem 15.1]) $\lim_{N\to\infty}(1/N)\sum_{n=0}^{N-1} P(\theta + 2\pi n) = \lim_{N\to\infty}\int \mathcal{S}_G^N(\theta,z)\mathrm{d}\mu_d(z) = \int \hat{\mathcal{S}}_G(\theta,z)\mathrm{d}\mu_d(z)$. Thus $\bar{P}$ exists.

It is easy to show that $\bar{P}$ is 2π-periodic once one has established that P is bounded. The latter holds since

$$\begin{aligned}
|P(\theta)| &\leq \int_{U_\theta} \rho_g(\theta,z)|P_{\mathrm{loc}}(\theta,z)|\mathrm{d}\mu_d(z)\\
&\leq \int_{U_\theta} \rho_g(\theta,z)\mathrm{d}\mu_d(z) = 1,
\end{aligned}$$

where we used the fact that $|P_{\mathrm{loc}}| \leq 1$ and that the LD ρ_g is normalized.

For particle physics experiments it is important to maximize $|\bar{P}(\theta)|$ as a function of P_{loc}. This is discussed in [4, Section I], [33].

3.4.2. The stroboscopic averaging algorithm and SPRINT

We now describe a simple and transparent spin–orbit tracking algorithm for calculating the stroboscopic average of polarization fields.

Consider a polarization field $\mathcal{S}_G$ i.e., fix the generator G. If $z \in \tilde{U}^G_\theta$ and if N is sufficiently large then $\mathcal{S}^N_G(\theta, z)$ is a good approximation of $\hat{\mathcal{S}}_G(\theta, z)$. Now consider $\mathcal{S}^N_G(0, z^0)$ where we hold $z^0 \in \tilde{U}^G_0$ fixed. Then

$$\begin{aligned}\mathcal{S}^N_G&(0, z^0)\\ &= \frac{1}{N}\sum_{n=0}^{N-1} \Psi(2\pi n, 0; \varphi(0, 2\pi n; z^0))G(\varphi(0, 2\pi n; z^0))\\ &= \frac{1}{N}\sum_{n=0}^{N-1} (\Psi(-2\pi n, 0; z^0))^{\mathrm{T}} G(\varphi(-2\pi n, 0; z^0)), \end{aligned} \tag{3.40}$$

where we used (3.20) in the first equality and (2.6), (3.12) and (3.13) in the second. With (3.40) we see that in order to compute $\mathcal{S}^N_G(0, z^0)$ we only need to compute G at the set of points $\{\varphi(-2\pi n, 0; z^0)\}^{N-1}_{n=0}$ and multiply by the matrices in the set $\{(\Psi(-2\pi n, 0; z^0))^{\mathrm{T}}\}^{N-1}_{n=0}$. This clearly reduces the task of computing $\mathcal{S}^N_G(0, z^0)$ to the backtracking of the trajectory, defined by the initial value problem

$$\begin{aligned}&\dot{z} = f(\theta, z), \qquad z(0) = z^0,\\ &D_1\Psi(\theta, 0; z^0) = \mathcal{A}(\theta, z)\Psi(\theta, 0; z^0),\\ &\Psi(0, 0; z^0) = I_{3\times 3},\end{aligned} \tag{3.41}$$

over $N-1$ turns backward around the storage ring. Moreover, it is simple to calculate $\mathcal{S}^N_G(0, z)$ for $z = \varphi(2\pi n, 0; z^0)$ by further forward or backward tracking using, for sufficiently large N, the relations

$$\begin{aligned}\mathcal{S}^N_G(0, \varphi(2\pi n, 0; z^0)) &\approx \mathcal{S}^N_G(2\pi n, \varphi(2\pi n, 0; z^0))\\ &= \Psi(2\pi n, 0; z^0)\mathcal{S}^N_G(0, z^0),\end{aligned} \tag{3.42}$$

where in the equality we used the fact that $\mathcal{S}^N_G$ is a polarization field and where in the approximation we used (3.25) and that $\varphi(2\pi n, 0; z^0) \in \tilde{U}^G_0$. To summarize, by tracking a single trajectory of the system (3.41) and with (3.40), (3.42), we can compute $\mathcal{S}^N_G(0, z)$ for all points z of the form $z = \varphi(2\pi k, 0; z^0)$ where k is an integer.

Of course, one then repeats this procedure with a sufficiently dense grid of z^0's until a sufficiently dense portrait of the function $\mathcal{S}^N_G(0, \cdot)$ has been obtained. Then one can use (3.22) to compute the functions $\mathcal{S}^N_G(\theta, \cdot)$ for a sufficiently dense grid of θ's in the interval $[0, 2\pi]$. These are the key ideas in [32] and they led to the stroboscopic averaging algorithm in SPRINT [8, 33,42]. In SPRINT, G is usually taken to be a constant function so that the calculations are further simplified.

Stroboscopic averaging was originally motivated by the need to calculate invariant spin fields since, as explained in Section 1 and in Section I in [4], invariant spin fields are key objects for systematizing spin motion in storage rings. See also Section 3.4.3 below. An invariant spin field is derived from a 2π-periodic polarization field obtained by stroboscopic averaging by simply normalizing, as outlined in Remark (12), assuming that the stroboscopic average has no zeros. Stroboscopic averaging provides a general and powerful means to calculate invariant spin fields as well as to inquire about their existence. Hence it represents a major addition to the tool kit for describing polarized beams in storage rings. For example, it can be applied to nonintegrable orbital motion and for integrable orbital motion it even works on orbital resonance. See also Section 3.4.5. For references to algorithms for computing invariant spin fields see [27].

Remark.

(15) We now briefly comment on the rate of convergence. A high convergence speed for a Cesàro sequence is of course crucial to the success of the tracking algorithm outlined in this section and since SPRINT has been in operation much experience has been accumulated. One finds that if the orbital motion is integrable the convergence is usually linear in N in the sense that for a given z^0 there exists a real constant c such that for all N

$$\left|\frac{\mathcal{S}^N_G(0, z^0)}{|\mathcal{S}^N_G(0, z^0)|} - \frac{\mathcal{S}^{N_f}_G(0, z^0)}{|\mathcal{S}^{N_f}_G(0, z^0)|}\right| \leq \frac{c}{N}, \tag{3.43}$$

where $\mathcal{S}^N_G(0, z^0)$ and $\mathcal{S}^{N_f}_G(0, z^0)$ are assumed to be nonzero and $\mathcal{S}^{N_f}_G(0, z^0)$ is assumed to be approximately $\hat{\mathcal{S}}_G(0, z^0)$. Experience for high energy storage rings shows that a few thousand turns usually suffice, i.e. $N_f = 10\,000$ is an appropriate order of magnitude. The linear convergence law (3.43) was derived analytically, by using plausibility assumptions, in [32]. This is a subject for further investigation.

Note that for LD's there seems to exist no linear law analogous to (3.43). For the convergence speed of the Cesàro sequences of LD's, see for example [40, Section 3.2]. Recall also Remark (11). □

3.4.3. The uniqueness of invariant spin fields

While, as pointed out in Remark (8), there may be a large number of 2π-periodic polarization fields, this does not imply that there is a large number of 2π-periodic spin fields and we address this issue here.

For integrable orbital motion, we can approach the uniqueness of the stroboscopic average of a polarization field by appealing to existing results on the uniqueness of the invariant spin field. For integrable orbital motion the domains U_θ decompose into tori and for every torus a "spin tune" can be defined which determines if the system is on "spin–orbit resonance" (the spin tune, as a function of the torus, is the "amplitude dependent spin tune" mentioned in Section 1). Note that the ellipses $E_{\varepsilon,\theta}$ in Appendix A are the tori for the betatron motion. In particular, if for a given torus the system is not on orbital resonance and not on spin–orbit resonance, we know that a 2π-periodic spin field on that torus is unique up to a sign if it is of class C^1 [4]. One expects quite generally that all invariant spin fields on such a nonresonant torus are essentially identical in a measure-theoretic sense, i.e. modulo μ_d-nullsets. It thus follows from Remark (12) that in the above situation all stroboscopic averages, which have no zero, point into the essentially unique direction given by the invariant spin field on the given torus. Calculations with SPRINT have provided a large amount of numerical evidence for this behavior. While we do not dwell further on the issue of uniqueness in the present paper, let us mention that the procedure would rest upon Fourier analysis in the orbital angle as one can see for the C^1 case in [4]. The only additional ingredient in the present context, where we don't have invariant spin fields which are C^1 in the orbital angle, is Parseval's equality [31].

The uniqueness in the case of nonintegrable orbital motion is more complicated to deal with and to our knowledge has not been seriously studied. Of course, since SPRINT is a tracking algorithm, it can be used to look closely at this issue.

3.4.4. The "filling-up" method

We now comment on the consequences of relaxing the regularity conditions of Section 3.2.1.

It is trivial that polarization fields exist in great numbers since every bounded and measurable G gives a polarization field $\mathcal{S}_G$. In other words, the generator G has only to fulfill the regularity conditions. However the existence of invariant spin fields is, except for trivial spin–orbit systems, a nontrivial matter. For example, Theorem 3.5 does not guarantee that $\hat{\mathcal{S}}_G$ has no zeros. If $\mathcal{S}_G$ is a polarization field 2π-periodic in measure then, as we know from Section 3.2.1, G is not only bounded and measurable but also has to fulfill the dynamical condition (3.17) almost everywhere. Conversely, we also know from Section 3.2.1 that $\mathcal{S}_G$ is 2π-periodic in measure if G satisfies (3.17) almost everywhere. As far as we are aware, it is, by analytical means, nontrivial to discover whether a G exists which solves (3.17) almost everywhere and which is nonzero almost everywhere, given our regularity conditions on G. However, with stroboscopic averaging one has a numerical means to investigate the matter of existence. If we relax the regularity conditions it is known that the dynamical condition (3.17) allows a large class of normalized solutions. This can be shown with the so-called "filling-up" method [34]. Although this method does not solve the above problem of existence of an appropriate G, it does give further insight into the nature of invariant spin fields. Thus we will now explain it.

For brevity we only treat the case of nonperiodic orbital motion in the sense that, for every $z^0 \in U$ and every nonzero integer n, $\varphi(2\pi n, 0; z^0) \neq z^0$. Defining $M(z^0) := \bigcup_{n\in\mathbb{Z}}\{\varphi(2\pi n, 0; z^0)\}$, we know, by the axiom of choice, that a subset $\tilde{M}$ of U exists such that

$$U = \bigcup_{z^0\in\tilde{M}} M(z^0), \tag{3.44}$$

and such that, if $z^0, z^1 \in \tilde{M}$ and $z^0 \neq z^1$, then $M(z^0)\cap M(z^1) = \emptyset$. Thus for every $z \in U$ we have a unique integer $n(z)$ and a unique point $z^0(z) \in \tilde{M}$ such that $z = \varphi(2\pi n(z), 0; z^0(z))$ and such that, if $z \in \tilde{M}$, $z^0(z) = z$. Of course, for every $z \in \tilde{M}$, we have $n(z) = 0$ and it easily follows from (2.5) and (2.6) that for every $z \in U$ and every integer k we have $z^0(\varphi(2\pi k, 0; z)) = z^0(z)$ and $n(\varphi(2\pi k, 0; z)) = n(z) + k$. We now use the axiom of choice again to choose a spin vector $s(z)$ for every $z \in \tilde{M}$ and we define a function $G : U \to \mathbb{R}^3$ by

$$G(z) := \Psi(2\pi n(z), 0; z^0(z))s(z^0(z)). \tag{3.45}$$

Using (3.11) and (3.12) and the above mentioned properties of $z^0(z)$ and $n(z)$ it follows that G, defined by (3.45), satisfies (3.17), whence $\mathcal{S}_G(\theta, z)$ defined by (3.14) is 2π-periodic in θ. Of course if s does not vanish everywhere then neither does G. Moreover G is normalized if s is. Since the function s is basically arbitrary, we have shown that a large supply of normalized functions $G : U \to \mathbb{R}^3$ exists which satisfy (3.17). It is clear that if a 2π-periodic polarization field $\mathcal{S}_G$ exists, then the function s can be chosen such that (3.45) holds, since, if $z \in \tilde{M}$, then $G(z) = s(z)$. However (3.45) alone gives no hints for choosing s in a way such that the G in (3.45) is $\mathcal{U}$-$\mathcal{R}^3$-measurable, i.e., fulfills the regularity conditions. Thus the filling-up method does not solve the problem of the existence of 2π-periodic spin fields and it is also easy to see that it also does not solve the problem of existence of invariant spin fields.

Note that the problem of the existence of normalized 2π-periodic LD's is, at least in the case of a U of finite measure, considerably simpler. In fact if U is of finite measure then, there exists a constant positive g, such that ρ_g is a 2π-periodic normalized LD.

3.4.5. The map formalism

Although in this paper we have chosen to work with flows, we will now comment on the use of one-turn maps.

In this paper we have restricted our discussion to functions f_{so} of class C^1 in order to be consistent with the situation in real storage rings and the treatment in [4]. But in practice, simulations are often made with the approximation that the magnets have hard edged magnetic fields or are represented by thin lenses so that the requirement that $f_{so}(\theta, w)$ be of class C^1 with respect to θ must be replaced by the weaker requirement that it is only piecewise C^1 in θ. It is then usually convenient to analyze the spin motion using one-turn maps instead of differential equations and to study the spin–orbit motion on the $\theta = 0$ section, having chosen the time origin so that $f_{so}(0, \cdot)$ is C^1. The dynamics is then determined by $\mathcal{P}$ and the definitions of LD and polarization field are modified accordingly. For example, the 2π-periodicity in measure becomes a one-turn periodicity in measure and the definition of the 2π-periodic polarization field is essentially replaced by (3.17). Thus [4] and this paper can be easily reformulated in terms of maps. One thereby obtains essentially the same theory as with the flow formalism.

An interesting example where f_{so} is only piecewise C^1 is provided by the following important model. In this model orbital motion is integrable and has just one degree of freedom and the ring includes two thin lens Siberian Snakes which thus provide two discontinuities in the θ dependence of the $\mathcal{A}$ in (3.2). This model is often called the "single resonance model with two Siberian Snakes" [9,10]. Outside the snakes, $\mathcal{A}$ corresponds to the model discussed in Section VII in [4]. For the single resonance model with two Siberian Snakes one can write a simple analytical expression for $\mathcal{P}$. Thus analysis is especially simple and spin–orbit tracking simulations can be made to run very efficiently. As always, when $\mathcal{A}$ is piecewise C^1 in θ, $\mathcal{P}$ is C^1.

We expect for systems where $\mathcal{P}$ is C^1 that an invariant spin field exists which is C^1. See the models in [4]. However, there are exceptional models where this is not true. For example when the single resonance model with two Siberian Snakes is on orbital resonance at a so-called "snake resonance tune" and for most values of the remaining parameters of the model, invariant spin fields exist but none of them is C^1. See Fig. 12 in [10] and the accompanying discussion.

4. Summary and discussion

In Section 3 of this paper, we consider an initial assignment of spins, G, which evolves according to the flow W of f_{so} by $\mathcal{S}_G(\theta, z) = \Psi(\theta, 0; \varphi(0, \theta; z))\, G(\varphi(0, \theta; z))$. We call $\mathcal{S}_G$ a polarization field. For each G, we have constructed a 2π-periodic polarization field $\hat{\mathcal{S}}_G(\theta, z) = \lim_{N\to\infty} 1_{\tilde{U}_\theta^G}(z)\mathcal{S}_G^N(\theta, z)$ where $\mathcal{S}_G^N(\theta, z) = (1/N)\sum_{n=0}^{N-1} \mathcal{S}_G(\theta + 2\pi n, z)$ and $\tilde{U}_\theta^G \subset U_\theta$ is of full measure when $\mu_d(U) < \infty$, as we proved using the ergodic theorem. This is the main result of our paper.

The flow W is defined after (3.3) and the polarization field in Definition 3.1. The stroboscopic average $\hat{\mathcal{S}}_G$ of $\mathcal{S}_G$ is defined in Definition 3.4 and the convergence set $\tilde{U}_\theta^G$ of $\mathcal{S}_G^N(\theta, \cdot)$ is defined in (3.23). The main results of this section are Theorems 3.5 and 3.9. In Theorem 3.5 we prove that $\hat{\mathcal{S}}_G$ is a 2π-periodic polarization field but obtain no information on the measure of $\tilde{U}_\theta^G$. In Theorem 3.9 we use the ergodic theorem to show that $\tilde{U}_\theta^G$ is of full measure in U_θ. As in the orbital case the stroboscopic average may be zero almost everywhere.

An important problem left unsolved in this paper is that of the conditions under which the stroboscopic average of a polarization field has no zeros. This problem is closely related with the problem of existence of the invariant spin field since (see Remark (12)) a stroboscopic average which has almost no zeros implies the existence of an invariant spin field. Another problem which we did not investigate is the long time stability of 2π-periodic polarization fields.

In the proof of Lemma 3.6, which is central to the proof of Theorem 3.9, the reader can see how we have overcome the main technical obstacle in this paper which is the fact (see also Remark (10)) that the ergodic theorem cannot be applied "directly" to polarization fields because they have three, and not just one, components. In contrast the proof of Lemma 2.5 applies the ergodic theorem "directly" since Liouville densities are scalar quantities.

Section 2 laid the framework for the orbital part of Section 3 and it also discussed the ergodic theorem and its use in a simple setting to prepare the reader for its use in Section 3. Section 2.4 gives a summary and Appendix A applies the theory to an important example from beam dynamics.

Section 3.4 covers several aspects of polarization fields including applications to the physics of polarized beams. In Section 3.4.1 we applied Theorem 3.9 to the polarization of the whole beam. Section 3.4.3 introduces the reader to the important issue of the uniqueness of invariant spin fields and Section 3.4.4 discusses a side issue. The final Section 3.4.5 outlines the important extension of our work from the flow formalism of [4] to the map formalism.

With this paper we have established that the simulation code SPRINT, outlined in Section 3.4.2, computes periodic polarization fields, i.e., functions which, in particular, obey regularity conditions. This fact was not clear when SPRINT was first created. While numerical work with SPRINT has involved integrable and bounded orbital motion, our work here does not require integrability. In addition, SPRINT has been applied to discontinuous functions f and $\mathcal{A}$. Thus it would be interesting to extend SPRINT to nonintegrable orbital motion and to extend our approach to discontinuous functions f and $\mathcal{A}$.

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Appendix A. Appendix on the example of betatron motion

We will now illustrate the machinery of Section 2 with an important model of linearized particle motion in a storage ring. The main result of this section is the explicit construction of the stroboscopic average in Proposition A.1.

A.1. Basic properties

As explained in Section 1, a storage ring has a *closed orbit* which is a periodic curve in $\mathbb{R}^3$ defined by dipole magnets. A storage ring is designed, using magnets, so that the motion of particles starting near the closed orbit is stable. In Frenet–Serret coordinates defined on the closed orbit the transverse linearized motion is given by a pair of *Hill equations* (thus $d = 4$). Restricting to one degree of freedom one has $d = 2$ and one Hill equation,

$$\dot{z} = f(\theta, z) := (z_2, -k(\theta)z_1)^{\mathrm{T}}, \tag{A.1}$$

where k is a 2π-periodic C^1 function determined by the quadrupole magnets which serve to focus the beam.

Eq. (A.1) is a special case of the more general problem which we will treat in this section, namely

$$\dot{z} = f(\theta, z) := \mathcal{J} D_1 H(z, \theta)^{\mathrm{T}} = \mathcal{J} S(\theta) z, \tag{A.2}$$

defined by the Hamiltonian $H(z, \theta) = \frac{1}{2} z^{\mathrm{T}} S(\theta) z$ (see also [38]). Here S is a 2π-periodic, symmetric, 2×2 matrix of class C^1, $\mathcal{J} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $\mathcal{J}S$ is sometimes called a Hamiltonian matrix [26]. In the case of (A.1), $S(\theta) = \mathrm{diag}(k(\theta), 1)$.

The principal solution matrix (PSM) Φ is defined by

$$D_1\Phi(\theta, \theta_0) = \mathcal{J}S(\theta)\Phi(\theta, \theta_0), \quad \Phi(\theta_0, \theta_0) = I, \tag{A.3}$$

and is symplectic, i.e., $\Phi^{\mathrm{T}}\mathcal{J}\Phi = \mathcal{J}$, due to the symmetry of S. The general solution (2.3) can be written as

$$\varphi(\theta, \theta_0; z^0) = \Phi(\theta, \theta_0)z^0. \tag{A.4}$$

The f given by (A.2) fulfills the conditions imposed in Section 2.1. In particular, $U = \mathbb{R}^2$ is an open nonempty set satisfying (2.8). We will obtain a more appropriate U shortly.

The *period advance matrix* for the θ_0 section is given by

$$P(\theta_0) := \Phi(\theta_0 + 2\pi, \theta_0). \tag{A.5}$$

Due to (2.6) we have $\Phi(\theta + 2\pi, \theta_0 + 2\pi) = \Phi(\theta, \theta_0)$, so that P is 2π-periodic as it must be. Also, we note that P is symplectic and that it satisfies

$$\dot{P}(\theta) = \mathcal{J}S(\theta)P(\theta) - P(\theta)\mathcal{J}S(\theta). \tag{A.6}$$

Since the trace of the rhs of (A.6) is zero, $\mathrm{Tr}[P]$ is independent of θ. We assume that S is such that the solutions of (A.2) are stable, i.e., the elliptic case where $|\mathrm{Tr}[P]| < 2$. Thus there are two values of $\nu \in (0, 2\pi)$ such that $\mathrm{Tr}[P] = 2\cos\nu$. In addition $P_{12}(\theta)$ is nonzero, for if it were zero then $P_{11}(\theta)P_{22}(\theta)$ would be 1 and we would have the contradiction that $|\mathrm{Tr}[P]| = |P_{11}(\theta) + 1/P_{11}(\theta)| \geq 2$. Since $\sin\nu$ is nonzero and unique up to a sign we fix it by choosing it to have the same sign as $P_{12}(\theta)$. In summary, there is a unique value of $\nu \in (0, 2\pi)$ such that $\mathrm{Tr}[P] = 2\cos\nu$ and $P_{12}(\theta)\sin\nu > 0$.

Remark.

(16) Note that $P(\theta)$ is a 2π-periodic solution of (A.6). We do not claim that all solutions of (A.6) are 2π-periodic nor is this required. $\square$

Given ν we now construct, for every θ, an important representation of the symplectic matrix $P(\theta)$. Since $\sin\nu \neq 0$ we can define functions α and β by $P_{11}(\theta) =: \cos\nu + \alpha(\theta)\sin\nu$ and $P_{12}(\theta) =: \beta(\theta)\sin\nu$. Noting that $\mathrm{Tr}[P] = 2\cos\nu$ and that $\det P = 1$, we obtain the representation

$$P(\theta) = I\cos\nu + J(\theta)\sin\nu, \quad J(\theta) := \begin{pmatrix} \alpha(\theta) & \beta(\theta) \\ -\gamma(\theta) & -\alpha(\theta) \end{pmatrix}, \tag{A.7}$$

where $\gamma(\theta) := (1 + \alpha^2(\theta))/\beta(\theta)$. It is clear that $\beta(\theta), \gamma(\theta) > 0$ and that the functions α, β, γ are uniquely determined by P and are 2π-periodic and C^1. Of course P determines ν and it is interesting to note that it has two distinct and θ-independent eigenvalues: $\cos\nu \pm \mathrm{i}\sin\nu$. From (A.6) and (A.7) J satisfies

$$\dot{J} = \mathcal{J}S(\theta)J - J\mathcal{J}S(\theta), \tag{A.8}$$

and, since $J^2 = -I$,

$$P = \exp(\nu J). \tag{A.9}$$

We are now in a position to derive the most important property of the Hamiltonian system (A.2), namely a conservation law. It is easy to see that the PSM satisfies

$$\Phi^{\mathrm{T}}(\theta, \theta_0)C(\theta)\Phi(\theta, \theta_0) = C(\theta_0), \quad \text{where} \quad C(\theta) := \mathcal{J}^{\mathrm{T}}J(\theta) = \begin{pmatrix} \gamma(\theta) & \alpha(\theta) \\ \alpha(\theta) & \beta(\theta) \end{pmatrix}, \tag{A.10}$$

by simply showing that the derivative of the lhs with respect to θ is zero. Since solutions of (A.2), with $z(\theta_0) = z^0$, are given by $z(\theta) = \Phi(\theta, \theta_0)z^0$, it follows that

$$z^{\mathrm{T}}C(\theta)z = (z^0)^{\mathrm{T}}C(\theta_0)z^0. \tag{A.11}$$

In particular

$$e(z, \theta) := z^{\mathrm{T}}C(\theta)z = \gamma(\theta)z_1^2 + 2\alpha(\theta)z_1z_2 + \beta(\theta)z_2^2 \tag{A.12}$$

is a constant of the motion, i.e., $e(z, \theta)$ is constant along solutions of (A.2).

Since $C(\theta)$ is symmetric and positive definite, then for each θ, the set of points $z \in \mathbb{R}^2$ for which

$$\varepsilon = e(z, \theta) = \gamma(\theta)z_1^2 + 2\alpha(\theta)z_1z_2 + \beta(\theta)z_2^2, \quad (0 \leq \varepsilon < \infty) \tag{A.13}$$

forms an ellipse which we denote by $E_{\varepsilon,\theta}$. Since e is a constant of the motion, we have

$$\varphi(\theta, \theta_0; E_{\varepsilon,\theta_0}) = E_{\varepsilon,\theta}. \tag{A.14}$$

Because $E_{\varepsilon,\theta+2\pi} = E_{\varepsilon,\theta}$, we obtain

$$\varphi(\theta_0 + 2\pi n, \theta_0; E_{\varepsilon,\theta_0}) = E_{\varepsilon,\theta_0}. \tag{A.15}$$

In particular, the motion under (A.2) in each θ_0 section is confined to a single ellipse. It follows that

$$U := \bigcup_{0 \leq \varepsilon' < \varepsilon} E_{\varepsilon',0} = \{z \in \mathbb{R}^2 : e(z, 0) < \varepsilon\} \tag{A.16}$$

is, for every $\varepsilon > 0$, a nonempty open invariant set satisfying (2.8), i.e., $\Phi(2\pi, 0)U = U$. Furthermore by (A.14), $U_\theta = \Phi(\theta, 0)U = \varphi(\theta, 0; U) = \bigcup_{0 \leq \varepsilon' < \varepsilon} \varphi(\theta, 0; E_{\varepsilon',0}) = \bigcup_{0 \leq \varepsilon' < \varepsilon} E_{\varepsilon',\theta} = \{z \in \mathbb{R}^2 : e(z, \theta) < \varepsilon\}$.

Finally, we construct the PSM Φ. Consider the symplectic matrix transformation

$$\zeta = T(\theta)z, \qquad T^{\mathrm{T}}\mathcal{J}T = \mathcal{J} \tag{A.17}$$

which gives

$$\dot{\zeta} = \mathcal{J}(R(\theta) + \hat{S}(\theta))\zeta, \quad \text{where } R := -\mathcal{J}\dot{T}T^{-1} \quad \text{and} \quad \hat{S} := T^{-\mathrm{T}}ST^{-1}. \tag{A.18}$$

It follows from the symplecticity of T that R and $\hat{S}$ are symmetric whence $\mathcal{J}(R + \hat{S})$ is a Hamiltonian matrix. Our aim is to find T so that (A.18) can be integrated and we proceed by looking for T such that $\zeta^{\mathrm{T}}\zeta$ is a constant of the motion, i.e., $\zeta^{\mathrm{T}}\zeta = z^{\mathrm{T}}T^{\mathrm{T}}Tz$ is constant. From (A.11), this will be true if $T^{\mathrm{T}}T = C$. This gives $T_{11}^2 + T_{21}^2 = \gamma$, $T_{11}T_{12} + T_{21}T_{22} = \alpha$, $T_{12}^2 + T_{22}^2 = \beta$ and from symplecticity, $T_{11}T_{22} - T_{12}T_{21} = 1$.

We find that these can be solved by taking $T_{12} = 0$ and we obtain

$$T = \begin{pmatrix} \beta^{-\frac{1}{2}} & 0 \\ \alpha\beta^{-\frac{1}{2}} & \beta^{\frac{1}{2}} \end{pmatrix}. \tag{A.19}$$

From the definitions in (A.18) a short calculation using (A.8) gives $R + \hat{S} = (S_{22}/\beta)I$ and clearly the solution of (A.18) for $\zeta(\theta_0) = \zeta^0$ is

$$\zeta = \exp(\psi(\theta,\theta_0)\mathcal{J})\zeta^0, \qquad \psi(\theta,\theta_0) = \int_{\theta_0}^{\theta} \frac{S_{22}(\theta')}{\beta(\theta')}\mathrm{d}\theta'. \tag{A.20}$$

Since $z = T^{-1}(\theta)\zeta = T^{-1}(\theta)\exp(\psi(\theta,\theta_0)\mathcal{J})T(\theta_0)z^0$ the PSM for (A.2) is

$$\Phi(\theta,\theta_0) = T^{-1}(\theta)\exp(\psi(\theta,\theta_0)\mathcal{J})T(\theta_0). \tag{A.21}$$

Remark.

(17) We could have proceeded by looking for T with $T_{12} = 0$ such that $R + \hat{S} = \omega I$ for scalar ω, and the result would be the same. □

It follows from (A.21) that $P(\theta_0) = \exp(\hat{\nu}J(\theta_0))$ where $\hat{\nu} = \int_0^{2\pi} S_{22}/\beta\mathrm{d}\theta$ and we have used the fact that

$$T^{-1}\mathcal{J}T = J. \tag{A.22}$$

Thus

$$\nu = \int_0^{2\pi} \frac{S_{22}(\theta)}{\beta(\theta)}\mathrm{d}\theta \bmod 2\pi. \tag{A.23}$$

Since the transformation (A.17) is measure preserving and since the set TU_θ is the interior of the circle with radius $\sqrt{\varepsilon}$ centered at the origin, we have $\mu_2(U_\theta) = \pi\varepsilon$, which is independent of θ.

Remark.

(18) In the case of (A.1), we have derived the Courant–Snyder description of betatron motion (see also [27, Section 2.1.1]) which was developed for describing particle motion in circular accelerators in the 1950s. In this case, e is called the Courant–Snyder invariant to honor the important contribution of Courant and Snyder in their classic paper [15]. An interesting discussion of various approaches to (A.11) can be found in [16]. We have followed the spirit of [38] in our study of (A.2).

The calculation of α and β is central to the design of storage rings and algorithms for their construction can be found in standard accelerator physics texts [21]. From the approach above it is clear that J can be determined by calculating P as defined in (A.5). Another approach would be to solve (A.8) with an appropriate initial condition subject to $\gamma(\theta_0) := (1 + \alpha^2(\theta_0))/\beta(\theta_0)$. In this context it would be interesting to understand the set of all solutions to (A.8) and in particular the subset given by $\gamma(\theta_0) := (1 + \alpha^2(\theta_0))/\beta(\theta_0)$. This reduces to the study of $\Phi(\theta,0)Z\Phi(0,\theta)$ which for an arbitrary constant matrix Z is the general solution of (A.8). □

A.2. Computing the stroboscopic average of an arbitrary LD

From Theorem 2.6 the stroboscopic average of every LD is a 2π-periodic LD. Here, we construct an explicit formula for $\hat{\rho}_g$ where U in (A.16) is the domain of g. Note that U depends on $\varepsilon > 0$.

We begin by constructing a candidate for $\hat{\rho}_g$. From (2.18) we have

$$\rho_g^N(\theta,z) = \frac{1}{N}\sum_{n=0}^{N-1} g\left(\Phi(0,\theta+2\pi n)z\right). \tag{A.24}$$

Note that for every $z \in U_\theta$ there is a unique ε' in $[0,\varepsilon)$ such that z belongs to $E_{\varepsilon',\theta}$ whence by (A.14) the sequence of points $\Phi(0,\theta+2\pi n)z$ belongs to the ellipse $E_{\varepsilon',0}$. Thus for fixed (θ,z) the sequence (A.24) samples points on the ellipse $E_{\varepsilon',0}$ and so $\hat{\rho}_g(\theta,z)$ can depend only on values of g on that ellipse. That this is the case will be seen explicitly in (A.28) and (A.29).

Because the stroboscopic average $\hat{\rho}_g$ is an LD, (2.24) holds and so we only need to determine $\hat{\rho}_g(0,\cdot)$. By (A.9) and (A.22) we have that $\Phi(0,2\pi n) = P^{-n}(0) = T^{-1}(0)\exp(-n\nu\mathcal{J})T(0)$ and so by (A.24)

$$\begin{aligned}\rho_g^N(0,z) &= \frac{1}{N}\sum_{n=0}^{N-1} g\left(\exp(-n\nu J(0))z\right) \\ &= \frac{1}{N}\sum_{n=0}^{N-1} g(T^{-1}(0)\exp(-n\nu\mathcal{J})T(0)z).\end{aligned} \tag{A.25}$$

If $z \in U$, then z lies on an ellipse $E_{\varepsilon',0}$ and, by (A.15), the points $\exp(-n\nu J(0))z$ lie on that ellipse. Accordingly the points $\zeta_n = \exp(-n\nu\mathcal{J})T(0)z$ lie on the circle with radius ε' centered at the origin and if $\nu/2\pi$ is rational the average is easily computed as Proposition A.1(b) will show. If $\nu/2\pi$ is irrational then the ζ_n lie densely on the circle and we can apply the Weyl equidistribution theorem [36, Section 3] if g is continuous. It also follows, by the continuity of the function $e(\cdot,0)$, that if $\nu/2\pi$ is irrational and $z \in E_{\varepsilon',0}$, $x \in \mathbb{R}$ then all points $\exp(-xJ(0))z$ lie on $E_{\varepsilon',0}$ (in particular if $z \in U$ then $\exp(-xJ(0))z \in U$). We conclude that if $\nu/2\pi$ is irrational and g is continuous then $\rho_g^N(0,z)$ converges for all $z \in U$ and

$$\hat{\rho}_g(0,z) = \lim_{N\to\infty}\rho_g^N(0,z) = \bar{g}(z), \tag{A.26}$$

where the function $\bar{g} : U \to [0,\infty)$ is defined by $\bar{g}(z) := (1/2\pi)\int_0^{2\pi} g(\exp(-xJ(0))z)\mathrm{d}x$ and where we have used (A.22). From (2.24) and (A.26)

$$\hat{\rho}_g(\theta,z) = \frac{1}{2\pi}\int_0^{2\pi} g\left(\exp(-xJ(0))\Phi(0,\theta)z\right)\mathrm{d}x, \tag{A.27}$$

and this is our candidate for the stroboscopic average. In fact the above is a proof that it is, in the case where g is continuous. Note that since $\exp(-xJ(0))z \in U$ if $\nu/2\pi$ is irrational and $z \in U$, it is clear that the integrals in the definition of $\bar{g}$ and in (A.27) are well defined if $\nu/2\pi$ is irrational. This may seem odd but recall from (A.7) that J and ν are intimately related. Note that in this appendix we abbreviate $\mathrm{d}\mu_1(x)$ and $\mathrm{d}\mu_2(x)$ by $\mathrm{d}x$.

If g is not continuous we cannot apply the equidistribution theorem but we nevertheless know from Lemma 2.5 that $\rho_g^N(0,z)$ converges for μ_2-almost every z in U and so we expect for irrational $\nu/2\pi$ that (A.26) holds for μ_2-almost every z in U. The following proposition, which is a consequence of the ergodic theorem, confirms this expectation.

Proposition A.1. *Let f be given by (A.2) in the elliptic case, i.e., $|\mathrm{Tr}[P]| < 2$ and let $U := \{z \in \mathbb{R}^2 : e(z,0) < \varepsilon\}$ where $\varepsilon > 0$. Then for every LD ρ_g the following hold.*

(a) *If $\nu/2\pi$ is irrational then, for μ_2-almost every $z \in U_\theta$,*

$$\hat{\rho}_g(\theta,z) = \frac{1}{2\pi}\int_0^{2\pi} g\left(\exp(-xJ(0))\Phi(0,\theta)z\right)\mathrm{d}x, \quad \text{(A.28)}$$

where Φ is the PSM and J is given by (A.7).

(b) *If $\nu/2\pi$ is rational, i.e., $\nu/2\pi = q/p$ with integers $p > 0$ and q, then*

$$\hat{\rho}_g(\theta,z) = \frac{1}{p}\sum_{n=0}^{p-1} g\left(\exp(-n\nu J(0))\Phi(0,\theta)z\right). \quad \text{(A.29)}$$

Remark. That U is nonempty and open, and satisfies (2.8) was shown in Appendix A.1. Without loss of generality we take $\varepsilon = 1/\pi$ so that μ_2 is a probability measure.

Proof of Proposition A.1(a). The proof follows easily once we prove that the function $\bar{g}$, defined after (A.26), is a version of $E(g|\mathcal{I})$, where $\mathcal{I} := \{A \in \mathcal{U} : \varphi(2\pi,0;A) = A\} = \{A \in \mathcal{U} : P(0)A = A\}$. We also showed earlier in this section that the integral in (A.28) is well defined.

We first show that $\bar{g}$ is $\mathcal{I}$-$\mathcal{R}$-measurable. Since g is bounded and $\mathcal{U}$-$\mathcal{R}$-measurable we easily find that $\bar{g}$ is $\mathcal{U}$-$\mathcal{R}$-measurable. Thus the proof is complete if we show that $\bar{g}$ is invariant. Now $\exp(-xJ(0)) = T^{-1}(0)\exp(-x\mathcal{J})T(0)$ and $P(0) = T^{-1}(0)\exp(\nu\mathcal{J})T(0)$ whence

$$\begin{aligned}\bar{g}(P(0)z) &= \frac{1}{2\pi}\int_0^{2\pi} g(T^{-1}(0)\exp((\nu-x)\mathcal{J})T(0)z)\mathrm{d}x\\ &= \frac{1}{2\pi}\int_0^{2\pi} g(T^{-1}(0)\exp(-x\mathcal{J})T(0)z)\mathrm{d}x\\ &= \bar{g}(z),\end{aligned} \quad \text{(A.30)}$$

where in the second equality we used the fact that $\exp(-x\mathcal{J})$ is 2π-periodic in x.

Secondly we need to show that for $A \in \mathcal{I}$

$$\int_A \bar{g}(z)\mathrm{d}z = \int_A g(z)\mathrm{d}z. \quad \text{(A.31)}$$

Integrating $\bar{g}$ over A and using the Fubini theorem we have

$$\int_A \bar{g}(z)\mathrm{d}z = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x, \quad \text{(A.32)}$$

where $f : [0,2\pi] \to \mathbb{R}$ is defined by

$$\begin{aligned}f(x) &:= \int_A g\left(\exp(-xJ(0))z\right)\mathrm{d}z\\ &= \int_{\exp(-xJ(0))A} g(z)\mathrm{d}z,\end{aligned} \quad \text{(A.33)}$$

and where in the second equality of (A.33) we used the transformation theorem for Lebesgue integrals. To prove (A.31) we need two lemmas which we prove after completing the proof of Proposition A.1.

Lemma A.2. *If $h \in \mathcal{L}^1(U,\mathcal{U},\mu_2)$ is a continuous function, $A \in \mathcal{I}$ and $x \in [0,2\pi]$ then*

$$\int_{\exp(-xJ(0))A} h(z)\mathrm{d}z = \int_A h(z)\mathrm{d}z. \quad \text{(A.34)}$$

Lemma A.3. *For all x in $[0,2\pi]$ and $A \in \mathcal{I}$*

$$f(x) = \int_A g(z)\mathrm{d}z. \quad \text{(A.35)}$$

We conclude from (A.35) that $\frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x = \int_A g(z)\mathrm{d}z$ whence, by (A.32), it follows that (A.31) holds. This completes the proof that $\bar{g}$ is a version of $E(g|\mathcal{I})$.

Now, since $\bar{g}$ is a version of $E(g|\mathcal{I})$ and since, by Lemma 2.5, $\hat{\rho}_g(0,\cdot)$ is also a version of $E(g|\mathcal{I})$, we conclude that (A.26) holds for μ_1-almost every z in U. Thus by Theorem 2.6, $\hat{\rho}_g$ is an LD so that (A.28) holds for μ_2-almost every $z \in U_\theta$. □

Proof of Proposition A.1(b). Let $\nu/2\pi$ be rational, i.e., $\nu/2\pi = q/p$ with integers $p > 0$ and q. We showed earlier in this section that $\exp(-n\nu J(0))z \in U$ if $z \in U$ whence the sum in (A.29) is well defined. Let

$$g_n := g\left(\exp(-n\nu J(0))z\right) = g(T^{-1}(0)\exp(-n\nu\mathcal{J})T(0)z),$$

then $g_{n+p} = g_n$ and

$$\begin{aligned}\rho_g^N(0,z) &= \frac{1}{N}\left(\sum_{n=0}^{p-1} + \cdots + \sum_{n=(l-1)p}^{lp-1} + \sum_{n=lp}^{N-1}\right) g_n\\ &= \frac{1}{N} l \sum_{n=0}^{p-1} g_n + \frac{1}{N}\sum_{n=lp}^{N-1} g_n,\end{aligned} \quad \text{(A.36)}$$

where l is the integer part of $(N-1)/p$. Clearly the second term on the rhs goes to zero as $N \to \infty$ and $\lim_{N\to\infty}(l/N) = 1/p$ whence $\hat{\rho}_g(0,z) = (1/p)\sum_{n=0}^{p-1} g(\exp(-n\nu J(0))z)$. Since $\hat{\rho}_g$ is an LD, we conclude that (A.29) holds. □

We now prove the two lemmas.

Proof of Lemma A.2. Clearly

$$\int_A h\left(\exp(-xJ(0))z\right)\mathrm{d}z = \int_{\exp(-xJ(0))A} h(z)\mathrm{d}z. \quad \text{(A.37)}$$

Since h is continuous, it follows that $h\left(\exp(-xJ(0))z\right)$ is continuous in x whence by (A.37) and a lemma on parameter dependent integrals (see e.g. [11, Lemma 16.1]) the lhs of (A.34) is continuous in x.

For every integer n we have by (A.9) and (A.22)

$$\begin{aligned}A &= P^n(0)A = \exp(n\nu J(0))A\\ &= T^{-1}(0)\exp(n\nu\mathcal{J})T(0)A.\end{aligned} \quad \text{(A.38)}$$

Since $\nu/2\pi$ is irrational it follows from (A.38) that for a dense set of x-values in $[0,2\pi]$ we have $\exp(-xJ(0))A = A$. It follows that for a dense set of x-values in $[0, 2\pi]$, (A.34) holds. Since the lhs of (A.34) is continuous in x, the claim follows. □

Proof of Lemma A.3. Let $g_1, g_2, \dots$ be a sequence of continuous functions in $\mathcal{L}^1(U, \mathcal{U}, \mu_2)$ converging in $\mathcal{L}^1$ to g as $n \to \infty$, i.e., $\int_U |g_n(z) - g(z)|\mathrm{d}z \to 0$ as $n \to \infty$ (such a sequence exists due to [11, Theorem 29.14]). By Lemma A.2 we have for all $x \in [0, 2\pi]$

$$\int_{\exp(-xJ(0))A} g_n(z)\mathrm{d}z = \int_A g_n(z)\mathrm{d}z. \tag{A.39}$$

Therefore

$$\begin{aligned} &\left| f(x) - \int_A g(z)\mathrm{d}z \right| \\ &= \left| \int_{\exp(-xJ(0))A} (g(z) - g_n(z))\mathrm{d}z + \int_A (g_n(z) - g(z))\mathrm{d}z \right| \\ &\leq \int_{\exp(-xJ(0))A} |g(z) - g_n(z)|\mathrm{d}z + \int_A |g_n(z) - g(z)|\mathrm{d}z \\ &\leq 2\int_U |g_n(z) - g(z)|\mathrm{d}z, \end{aligned} \tag{A.40}$$

whence $f(x) = \int_A g(z)\mathrm{d}z$. Thus (A.35) holds for all x in $[0, 2\pi]$. □

Remark.

(19) It follows from the proof of Proposition A.1a that the function $\bar{g}$ satisfies the fixed point Eq. (2.16) and is the generator of a 2π-periodic LD (these facts even hold when $\nu/2\pi$ is rational). Thus if $\nu/2\pi$ is irrational then, by Proposition A.1a, the 2π-periodic LD's $\rho_{\bar{g}}$ and $\hat{\rho}_g$ are almost equal, i.e., for μ_2-almost every $z \in U_\theta$, $\rho_{\bar{g}}(\theta, z) = \hat{\rho}_g(\theta, z)$. □

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