A unified treatment of spin-orbit systems using tools distilled from the theory of principal bundles*

K. Heinemann[b], D. P. Barber[a], J. A. Ellison[b] and Mathias Vogt[a]

[a] Deutsches Elektronen–Synchrotron, DESY, 22607 Hamburg, Germany

[b] Department of Mathematics and Statistics, The University of New Mexico, Albuquerque, New Mexico 87131, U.S.A.

December 21, 2014

Abstract

We return to our study [BEH] of invariant spin fields and spin tunes for polarized beams in storage rings but in contrast to the continuous-time treatment in [BEH], we now employ a discrete-time formalism, beginning with the Poincaré maps of the continuous time formalism. We then substantially extend our toolset and generalize the notions of invariant spin field and invariant frame field. We revisit some old theorems and prove several theorems believed to be new. In particular we study two transformation rules, one of them known and the other new, where the former turns out to be an $SO(3)$-gauge transformation rule. We then apply the theory to the dynamics of spin-1/2 and spin-1 particle bunches and their density matrix functions, describing semiclassically the particle-spin content of bunches. Our approach thus unifies the spin-vector dynamics from the T-BMT equation with the spin-tensor dynamics and other dynamics. This unifying aspect of our approach relates the examples elegantly and uncovers relations between the various underlying dynamical systems in a transparent way. As in [BEH], the particle motion is integrable but we now allow for nonlinear particle motion on each torus.

Since this work is inspired by notions from the theory of bundles, we also provide insight into the underlying bundle-theoretic aspects of the well-established concepts of invariant spin field, spin tune and invariant frame field. Thus the group theoretical notions hidden in [BEH] and in its forerunners [DK73, Yo2] will be exhibited. Since we neglect, as is usual, the Stern-Gerlach force, the underlying principal bundle is of product form so that we can present the theory in a fashion which does not use bundle theory at all. Nevertheless we occasionally mention the bundle-theoretic meaning of our concepts and we also mention the similarities with the geometrical approach to Yang-Mills Theory.

*PACS numbers: 02.20.Bb,02.40.Re,03.65.Sq,05.45.-a,29.27.Hj

Contents

1 Introduction

In [BEH] we undertook an extensive study of the concept of spin tune in storage rings on the basis of the Thomas–Bargmann–Michel–Telegdi (T–BMT) equation [Ja] of spin precession. This naturally included a discussion of the invariant spin field and the invariant frame field. This work is a sequel to [BEH] and is based largely on mathematical concepts and ideas in the PhD Thesis [He2] of the first author (KH), where a method from Dynamical-Systems Theory is exploited to distil some essential features of particle-spin motion in storage rings. As to be seen in Chapter 8 this method clarifies and considerably extends the current theory of [BEH]. In fact it generalizes the concepts of invariant frame field, spin tune, spin-orbit resonances, invariant polarization field and invariant spin field to an arbitrary subgroup H of $SO(3)$ by using the concept of H-normal form and invariant (E, l)-field. This leads us to the Normal Form Theorem and various theorems which generalize some standard theorems that are also presented in this work. For short versions of the present work, see [HBEV1, HBEV2].

In [BEH] we assumed the particle motion to be independent of the spin, i.e., we neglected the Stern-Gerlach force. Also, the particle motion was described by an integrable Hamiltonian system in action-angle variables, $J, \phi \in \mathbb{R}^d$. We further assumed that the electric and magnetic fields were of class C^1, i.e., continuously differentiable) both in ϕ and θ. Thus the T–BMT equation became a linear system of ordinary differential equations (ODE) for the particle-spin-vector motion with smooth coefficients depending quasiperiodically on θ. This quasiperiodic structure led us to a generalization of the Floquet theorem and a new approach to the spin tune.

Although accelerator physicists tend to concentrate on studying specific models of particle-spin motion in real storage rings, many of the issues surrounding the spin tune and the invariant spin field depend just on the *structure* of the equations of particle-spin motion and can be treated in general terms. This is the strategy to be adopted here and it clears the way for the focus on purely mathematical matters and in particular for the exploitation of methods from Dynamical-Systems Theory and the theory of bundles.

In storage-ring physics there are two main approaches for dealing with the independent variable in the equations of motion (EOM), namely use of the flow formalism or the map formalism. In the flow formalism the EOM is an ODE, whence the independent variable is the continuous variable $\theta \in \mathbb{R}$ describing the distance around the ring. In the map formalism the independent variable in the EOM is the discrete variable $n \in \mathbb{Z}$ labelling thef turn number where $\mathbb{Z}$ denotes the set of integers. In Dynamical-Systems Theory it is common practice to refer to the independent variable in the EOM, such as θ, the "time" and that is the convention that we will use here. Thus there is a continuous-time and a discrete-time formalism. In [BEH] we used the former, here the emphasis is on discrete time. Nevertheless it would be possible to present the machinery of this work in the continuous-time formalism.

The external electrodynamic fields inside an accelerator's vacuum chamber are smooth, i.e., of class C^∞. So the C^1 assumption adopted in [BEH] appears to be perfectly reasonable. On the other hand, practical numerical spin–orbit tracking simulations are usually carried out with fields which cut off sharply at the ends of magnets and/or with thin-lens approximations. Thus in [BEH] our formalism involved class C^1 in the time variable θ although numerical calculations cited there in Sec. X had been obtained using hard-edged and thin-lens fields. However, hard-edged and thin-lens ring elements fit naturally into the discrete-time formalism. In particular, for this, we merely require that the fields are continuous (i.e., of class C^0) in the orbital phases and we allow jump discontinuities in θ. Of course, this still allows study of systems with fields smooth in θ and/or the orbital phases. The way that the discrete-time formalism derives from the continuous-time formalism is explained in Section 2.1.

This work is designed so that it can be read independently of [BEH]. However, we wish to avoid repeating the copious contextual material contained in [BEH]. We therefore invite the reader to consult the Introduction and the Summary and Conclusion in [BEH] in order to acquire a better appreciation of the context. In this work, as in [BEH], the particle motion is integrable and we allow the number of angle variables, d, to be arbitrary (but ≥ 1) although for particle-spin motion in storage rings, the case $d = 3$ is the most important. We use the symbols $\phi = (\phi_1, ..., \phi_d)^t$, $J = (J_1, ..., J_d)^t$ and $\omega(J) = (\omega_1(J), ..., \omega_d(J))^t$ respectively for the lists of orbital angles, orbital actions and orbital tunes where ^{t} denotes the transpose and where with continuous time $d\phi/d\theta = \omega(J)$. In the continuous-time formalism, the T–BMT equation is written as $d\mathbf{S}/d\theta = \mathbf{\Omega}(\theta, J, \phi(\theta)) \times \mathbf{S}$ where the 3-vector $\mathbf{S}$ is the spin expectation value ("the spin vector") in the rest frame of a particle and $\mathbf{\Omega}$ is the precession vector obtained as indicated in [BEH] from the electric and magnetic fields on the particle trajectory. For the purposes of this work we don't need to consider the whole (J, ϕ) phase space since it will suffice to confine ourselves to a fixed J-value, i.e., to particle motion on a single torus. Thus the actions J are just parameters. However it is likely that our work can be easily generalized to arbitrary particle motion if one maintains our condition that the particle motion is unaffected by the spin motion.

This work, in which we aim to present particle-spin motions in terms of Dynamical-Systems Theory, is structured as follows. In Section 2.1 we discuss the continuous-time formalism which will motivate, in Section 2.3, the discrete-time concept of the "spin-orbit system" (j, A) which characterizes a given setup by its 1-turn particle map j on the torus $\mathbb{T}^d$. While j characterizes the integrable particle dynamics, A is the 1-turn spin transfer matrix function, the latter being a continuous function from $\mathbb{T}^d$ to $SO(3)$. In the special case of the torus translation we have $j = \mathcal{P}_\omega$ where ω is the orbital tune and $\mathcal{P}_\omega$ is the corresponding translation on the torus after one turn. Thus in Section 2.1 we derive the discrete-time Poincaré map formalism from the continuous-time formalism and in Section 2.2 we introduce the torus $\mathbb{T}^d$ as a topological space. For the torus the angle variable ϕ is replaced by the angle variable z. Then in Section 2.3 we define the set $\mathcal{SOS}(d, j)$ of spin-orbit systems (j, A) to be considered in this work. and in Section 2.4 we introduce three important tools: the topological group, the group action, and the cocycle. These will carry us through the whole work and will reveal a host of well-known and less well-known structures underlying spin-orbit systems. In Chapter 3 we define polarization field trajectories and these lead to the definition of the invariant spin field (ISF). A transformation rule, $(j, A) \mapsto (j, A')$, is introduced in Chapter 4. This partitions $\mathcal{SOS}(d, j)$ into equivalence classes and spin-orbit systems belonging to the same equivalence class have similar properties. For the notions of partition and equivalence class, see Appendix A.2. It also leads us to structure-preserving transformations of particle-spin-vector trajectories and to structure-preserving transformations of polarization-field trajectories. In Chapter 5 the partition of $\mathcal{SOS}(d, j)$ leads us to several important subsets of $\mathcal{SOS}(d, j)$ which are denoted by $\mathcal{CB}_H(d, j)$. Each of these subsets of $\mathcal{SOS}(d, j)$ is defined in terms of a simple form of A. In particular a (j, A) in $\mathcal{SOS}(d, j)$ belongs to $\mathcal{CB}_H(d, j)$ iff it can be transformed to a (j, A') such that A' is H-valued where H is a subgroup of $SO(3)$. Then (j, A') is said to be an "H-normal form" of (j, A). The concept of H-normal form is also the driving force which leads us to the general theory of Chapter 8. In Chapter 5 we also formulate and prove a standard theorem, which connects the notions of ISF and invariant frame field (IFF) and which will turn out in Chapter 8 as the special case $H = SO(2)$ of the Normal Form Theorem.

In Chapter 6 the partition of $\mathcal{SOS}(d, j)$ leads us to the important subset $\mathcal{ACB}(d, j)$ of $\mathcal{SOS}(d, j)$. This subset $\mathcal{ACB}(d, j)$ of $\mathcal{SOS}(d, j)$ is defined in terms of another simple form of A. In particular a (j, A) in $\mathcal{SOS}(d, j)$ belongs to $\mathcal{ACB}(d, j)$ iff it can be transformed to a (j, A') such that A' is constant. On the other hand spin tunes describe constant rates of precession in appropriate reference frames so that one needs special spin-orbit systems which can be reached by transforming from the original spin-orbit systems to such frames. Indeed, it is $\mathcal{ACB}(d, j)$ which leads in Section 6.2 to the notion of spin tune and to the notion of spin-orbit resonance. Chapter 7 covers the topic of polarization. In particular in Section 7.1 we derive various formulas which estimate the bunch polarization with special emphasis on the situation where only two ISF's exist. In Section 7.2 we state and prove an important and well-known theorem which provides conditions under which only two ISF's exist. Then, in Chapter 8 we revisit and generalize the studies of the previous chapters using an approach that we call the "Technique of Association"(ToA) by which the $SO(3)$-spaces (E, l) label the different "contexts", covering all the different spin variables. The basic features of the ToA are defined in Section 8.2 and finer details in Sections 8.3 and 8.7 whereas applications are considered in Sections 8.4-8.6 and the bundle-theoretic origins of the ToA in Section 8.8. With the ToA we will see that the particle-spin-vector motion, i.e., the particle-spin-vector trajectories and the polarization-field trajectories introduced in Chapters 2-7 turn out to be tied to the special context, $(E, l) = (\mathbb{R}^3, l_v)$, of the ToA where the $\mathbb{R}^3$-valued spin variable is the spin vector S and where $l_v(r, S) = rS$. In [BEH] we didn't go beyond $(\mathbb{R}^3, l_v)$ but in this work we do. For example we will study (E_t, l_t) (see, e.g., Section 8.4) which encompasses the behavior of the spin tensor needed for spin-1 particles, and we will study other important (E, l) as well, in particular those needed for density matrices. With Chapter 8 it also becomes clear which of the concepts of Chapters 2-7 are (E, l)-dependent and which not. For example, the concepts of spin-orbit system, particle 1-turn map, spin transfer matrix function, spin tune, spin-orbit resonance, invariant frame field, H-normal form are (E, l)-independent since they only depend on (j, A). Clearly (E, l)-independent concepts are very general. In contrast the concepts of invariant field and the two ToA transformation rules are (E, l)-dependent. While the main dynamical themes of Chapter 8 are the Normal Form Theorem and various invariant-field theorems, a host of other results will be found along the way as well. In Appendix A we introduce the basic analytic notions like continuous functions and partitions. Appendix B contains some of our proofs.

Although many of our concepts, and in particular the ToA of Sections 8.1-8.7, have their origin in bundle theory as outlined in Section 8.8 we do not explicitly use bundle theory in those sections. Thus it is appropriate to briefly mention that the bundle machinery has many similarities with the so-called geometrical approach to Yang-Mills Theory. The hallmark of most bundle approaches is a carefully chosen principal bundle which allows one to store all data in the associated bundles of that principal bundle. Of course, one of the associated bundles is the principal bundle itself. In our application the underlying principal bundle carries the data from the particle motion and of the spin transfer matrix functions. Moreover the associated bundles are labeled by the $SO(3)$-spaces (E, l), i.e., they correspond to the above-mentioned "contexts". Thus each associated bundle carries a specific spin variable x, e.g., the spin vector S for the T–BMT spin motion, or a matrix M for the spin tensor motion needed for spin-1 particles. The specific design of our underlying principal bundle takes advantage of the fact that in polarized beam physics one neglects the Stern-Gerlach force, thereby allowing us to use techniques which were developed by R.J.Zimmer, R. Feres and others since the 1980's to study so-called rigidity problems in Dynamical-Systems Theory (see [Fe, Zi2, Zi3] and Chapter 9 in [HK1]). In contrast, in the geometrical approach to Yang-Mills Theory one picks a principal bundle which carries the data from the space-time and from the gauge potentials and gauge fields while the matter fields (leptons, quarks, Higgs particles, magnetic monopoles etc.) reside in specific associated bundles. The advantage of the use of bundles is its great flexibility and its ability to store and reveal data and structures. For example in our application we take advantage of the cocycle structure of the spin transfer matrix functions, of a SO(3)-gauge transformation structure connecting different spin-orbit systems, and of the duality of particle-spin and field motion. The duality provides the practically important ability to track polarization field trajectories, tensor field trajectories etc. in terms of the accelerator's particle trajectories. In the special case $(E, l) = (\mathbb{R}^3, l_v)$ the above duality is the duality between particle-spin-vector trajectories and polarization-field trajectories.

2 Spin-orbit systems

A central aim of this work is a study of the 1-turn particle-spin-vector map $\mathcal{P}[j, A]$ given by (2.23), i.e.,

$$\mathcal{P}[j, A](z, S) := \begin{pmatrix} j(z) \\ A(z)S \end{pmatrix} ,$$

where z is the angle variable on the torus and where j represents the 1-turn particle map whereas, in the case of real spin motion, a spin vector S would be mapped to $A(z)S$ after one turn according to the 1-turn spin transfer matrix function A derived from the T-BMT equation. These objects will be defined in detail in this section and the above map will be generalized in Chapter 8, from spin vectors to other objects related to spin. In Section 2.2 we discuss the torus as a topological space. In Section 2.3 we discuss the basic properties of $\mathcal{P}[j, A]$ and in Section 2.4 we define some group theoretical notions underlying $\mathcal{P}[j, A]$.

2.1 Deriving the discrete-time particle-spin-vector motion from the continuous-time particle-spin-vector motion

We begin our study by deriving our discrete-time particle-spin-vector motion from a continous-time initial value problem (IVP) which takes the form

$$\frac{d\phi}{d\theta} = \omega , \qquad \phi(0) = \phi_0 \in \mathbb{R}^d , \tag{2.1}$$

$$\frac{dS}{d\theta} = \mathcal{A}(\theta, \phi)S \ , \qquad S(0) = S_0 \in \mathbb{R}^3 \ , \tag{2.2}$$

where $\omega \in \mathbb{R}^d$ and where the matrix-valued function $\mathcal{A} : \mathbb{R}^{d+1} \to \mathbb{R}^{3\times3}$ is continuous in ϕ and piecewise continuous in θ. More precisely, $\mathcal{A}$ is either continuous or has not more than finitely many jump discontinuities at θ values $\theta_1, ..., \theta_N$ such that $\mathcal{A}$ is continuous on $(\mathbb{R} \setminus \{\theta_1, ..., \theta_N\}) \times \mathbb{R}^d$ and such that $\mathcal{A}(\theta_1; \cdot), ..., \mathcal{A}(\theta_N; \cdot)$ are continuous. For the $\cdot$ notation see Appendix A.1. Moreover we assume that $\mathcal{A}$ is 2π-periodic in each of its $d+1$ arguments and that it is skew–symmetric, i.e., $\mathcal{A}^t(\theta, \phi) = -\mathcal{A}(\theta, \phi)$. Without loss of generality and for simplicity of notation we choose $\theta = 0$ as the initial time. We denote the set of $\mathcal{A}$, where $\mathcal{A}$ satisfies the above conditions, by $\mathcal{BMT}(d)$.

As is clear from the above and the Introduction, the above IVP and the assumptions on $\mathcal{A}$ are motivated by our underlying interest in particle-spin-vector motion in storage rings. In the application to particle-spin-vector motion in storage rings, S is a column vector of components of the spin $\mathbf{S}$ and $\mathcal{A}(\theta, \phi) \equiv \mathcal{A}_J(\theta, \phi)$ represents the rotation rate vector $\mathbf{\Omega}(\theta, J, \phi)$ of the T–BMT equation [BEH]. Here J, ϕ are the action-angle variables of an integrable particle motion. Note that $\mathcal{A}(\theta, \phi)$ is 2π-periodic in θ because we deal with storage rings and $\mathcal{A}(\theta, \phi)$ is 2π-periodic in the d components of ϕ since the latter are angle variables. Moreover $\mathcal{A}$ is skew-symmetric by its origin in the T–BMT equation, thus preserving the norm of S. We suppress the J, except for a few occasions where we need it, since we work mainly on a fixed-J torus. The set $\mathcal{BMT}(d)$ includes standard particle-spin-vector motion but need not, and is only restricted by the above mentioned constraints on $\mathcal{A}$, in keeping with our wish to investigate the properties of any system defined by (2.1) and (2.2).

Since the system (2.1),(2.2) is periodic in θ it is convenient to study the behavior of solutions in terms of the Poincaré map (PM) [AP, HK2]. We now derive a convenient representation for the PM. Solving (2.1) gives

$$\phi(\theta) = \phi_0 + \omega\theta \ , \tag{2.3}$$

whence (2.2) reads as

$$\frac{dS}{d\theta} = \mathcal{A}(\theta, \phi_0 + \omega\theta)S \ , \qquad S(0) = S_0 \in \mathbb{R}^3 \ . \tag{2.4}$$

Since $\mathcal{A}(\theta; \phi)$ is piecewise continuous in θ it can be shown [Cr] that the IVP (2.4) has a unique solution S in the sense that

$$S(\theta) = S_0 + \int_0^\theta \mathcal{A}(t, \phi_0 + \omega t)S(t)dt \ . \tag{2.5}$$

It follows that $S(\theta)$ is continuous in θ. The proof in Cronin [Cr] does not include the parameter ϕ_0 but it is easily added.

Since (2.4) is linear in S the general solution of (2.5) can be written as

$$S(\theta) = \Phi_{CT}[\omega, \mathcal{A}](\theta; \phi_0)S_0 \ , \tag{2.6}$$

where, with (2.5), the function $\Phi_{CT}[\omega, \mathcal{A}] : \mathbb{R} \times \mathbb{R}^d \to SO(3)$ satisfies

$$\Phi_{CT}[\omega, \mathcal{A}](\theta; \phi_0) = I_{3\times3} + \int_0^\theta \mathcal{A}(t, \phi_0 + \omega t)\Phi_{CT}[\omega, \mathcal{A}](t; \phi_0)dt \ , \tag{2.7}$$

and where $I_{3\times3}$ is the 3×3 unit matrix and where the subscript "CT" indicates "continuous time". Since the values of $\mathcal{A}$ are real skew–symmetric 3×3 matrices, $\Phi_{CT}[\omega,\mathcal{A}]$ is $SO(3)$-valued where $SO(3)$ is the set of real 3×3–matrices R for which $R^tR = I_{3\times3}$ and $\det(R) = 1$. By adding the parameters ϕ_0 and ω in Cronin's proof, and using the fact that $\mathcal{A}(\theta;\phi)$ is continuous in ϕ, we conclude from (2.7) that $\Phi_{CT}[\omega,\mathcal{A}] \in \mathcal{C}(\mathbb{R}^{d+1}, SO(3))$ where $\mathcal{C}(\mathbb{R}^{d+1}, SO(3))$ is the set of continuous functions from $\mathbb{R}^{d+1}$ into $SO(3)$. See Appendix A.4 too. Furthermore $\Phi_{CT}[\omega,\mathcal{A}](\theta,\phi)$ is 2π-periodic in the components of ϕ. Using (2.3) and (2.6), the solution of the IVP (2.1),(2.2) can now be written

$$\begin{pmatrix} \phi(\theta) \\ S(\theta) \end{pmatrix} = \varphi(\theta;\phi_0,S_0)\;, \tag{2.8}$$

where the function $\varphi \in \mathcal{C}(\mathbb{R}^{d+4},\mathbb{R}^{d+3})$, is defined by

$$\varphi(\theta;\phi,S) := \begin{pmatrix} \phi+\omega\theta \\ \Phi_{CT}[\omega,\mathcal{A}](\theta,\phi)S \end{pmatrix}\;. \tag{2.9}$$

The PM on $\mathbb{R}^{d+3}$ is defined by $\varphi(2\pi;\cdot)$ and it reads as

$$\varphi(2\pi;\phi,S) = \begin{pmatrix} \phi+2\pi\omega \\ \Phi_{CT}[\omega,\mathcal{A}](2\pi;\phi)S \end{pmatrix}\;. \tag{2.10}$$

With (2.10) the Poincaré map $\varphi(2\pi;\cdot)$ is determined by the parameters ω and $\mathcal{A}$. With this the study of the non-autonomous continuous-time Dynamical System (DS) of (2.1),(2.2) has now been replaced by a study of an autonomous discrete-time DS given by the PM (2.10). In the following section we will see how the Poincaré map (2.10), which is expressed in terms of the angle variable ϕ, will be expressed in terms of the angle variable z on the torus, leading us to the Poincaré map $\mathcal{P}_{CT}[\omega,\mathcal{A}]$ in (2.19) below.

2.2 The torus $\mathbb{T}^d$ as the arena of the particle motion

Since ϕ is an angle variable and since we will have to prove many analytic properties later, it is very convenient to replace $\mathbb{R}^d$ by the torus $\mathbb{T}^d$. One often defines a torus as the space obtained by the function from $\phi\in\mathbb{R}^d$ to $\phi \bmod 2\pi$. However continuity plays a significant role in our study and so we give a definition suited for defining continuity in terms of a topology, i.e., in terms of open sets. For the basic topological notions, see Appendix A.3. A reader familiar with the torus can imagine (2.10) on the torus and safely move to Section 2.3. We define

$$\mathbb{T}^d := \{\phi+\tilde{\mathbb{Z}}^d : \phi\in\mathbb{R}^d\}\;, \tag{2.11}$$

where

$$\phi+\tilde{\mathbb{Z}}^d := \{\phi+\tilde{\phi} : \tilde{\phi}\in\tilde{\mathbb{Z}}^d\} = \{\phi+2\pi n : n\in\mathbb{Z}^d\}\;, \tag{2.12}$$

and where $\tilde{\mathbb{Z}}^d := \{2\pi n : n\in\mathbb{Z}^d\}$. So on $\mathbb{T}^d$, a chosen ϕ is accompanied by a countable infinity of points separated by 2π in each component of ϕ. In other words the elements of $\mathbb{T}^d$

are countably infinite subsets of $\mathbb{R}^d$. These subsets form a partition of $\mathbb{R}^d$. For the definition of partition see Appendix A.2. Each element of $\mathbb{T}^d$ can be represented by a unique element of $[0, 2\pi)^d$ and so $[0, 2\pi)^d$ is a "representing set" of the partition $\mathbb{T}^d$ of $\mathbb{R}^d$. For the definition of representing set of partition see Appendix A.2. Clearly $\phi' + \tilde{\mathbb{Z}}^d = \phi + \tilde{\mathbb{Z}}^d$ iff there exists an $m \in \mathbb{Z}^d$ such that $\phi' - \phi = 2\pi m$. The topology on $\mathbb{T}^d$ is defined in a standard way using the topology of $\mathbb{R}^d$.

As we will now show, with this definition of $\mathbb{T}^d$, the PM (2.10) can be written as in (2.19) as

$$\mathcal{P}_{CT}[\omega, \mathcal{A}](z, S) := \begin{pmatrix} \mathcal{P}_\omega(z) \\ A_{CT}[\omega, \mathcal{A}](z)S \end{pmatrix} ,$$

using the definitions in (2.17) and (2.18). The reader who is satisfied with this result might wish to jump to the summary of this subsection on a first reading. In any case we now continue with a rigorous justification and proof of the continuity of (2.19).

For that we now consider the onto function (surjection) $\pi_d : \mathbb{R}^d \to \mathbb{T}^d$ where

$$\pi_d(\phi) := \phi + \tilde{\mathbb{Z}}^d = \{\phi + 2\pi n : n \in \mathbb{Z}^d\} , \tag{2.13}$$

then a subset $B \subset \mathbb{T}^d$ is said to be open iff the inverse image, $\pi_d^{-1}(B) \subset \mathbb{R}^d$, of B under π_d is open (for the notion of inverse image see also Appendix A.1). Thus the function π_d and the natural topology on $\mathbb{R}^d$ are used to define a topology on $\mathbb{T}^d$. It is common to say that the topology on $\mathbb{T}^d$ is "co-induced by π_d" [wiki]. See Appendix A.6 too. Of course, π_d is continuous, i.e., $\pi_d \in \mathcal{C}(\mathbb{R}^d, \mathbb{T}^d)$. Furthermore the topology on $\mathbb{T}^d$ is the largest for which π_d is continuous. We will see many more co-induced topologies (on sets different from $\mathbb{T}^d$) in this work. In an older terminology the above topology on $\mathbb{T}^d$ is called the "identification topology" w.r.t. π_d and π_d is called an "identification map" [Du, Hu].

Remark:

(1) Another common definition of the torus is a cartesian product of circles, i.e., a subset $\hat{\mathbb{T}}^d$ of $\mathbb{R}^{2d}$. Here the topology is defined in terms of the Euclidean norm on $\mathbb{R}^{2d}$ giving the Euclidean metric. The two topological spaces $\hat{\mathbb{T}}^d$ and $\mathbb{T}^d$ are homeomorphic whence the topology of $\mathbb{T}^d$ has an underlying metric and thus continuity could be defined in terms of the standard "$\epsilon - \delta$" approach. For the notion of "homeomorphic" see Appendix A.4. However for the purposes of this work it is easier to work with the equivalent open-set definition of continuity. □

Clearly, by (2.11) and (2.13) and the remarks after (2.12),

$$\mathbb{T}^d = \{\pi_d(\phi) : \phi \in [0, 2\pi)^d\} = \{\pi_d(\phi) : \phi \in \mathbb{R}^d\} . \tag{2.14}$$

We can now prepare for the definition of the Poincaré map $\mathcal{P}_{CT}[\omega, \mathcal{A}]$ and the demonstration of its continuity. As a first step we use the above topology on $\mathbb{T}^d$ to obtain the following lemma:

Lemma 2.1 *(Torus Lemma)*
Let Y be a topological space and $F : \mathbb{R}^d \to Y$ be 2π-periodic in each of its d arguments. Then a unique function $f : \mathbb{T}^d \to Y$ exists such that

$$F = f \circ \pi_d \ . \tag{2.15}$$

Moreover if F is continuous then f is continuous, i.e., $f \in \mathcal{C}(\mathbb{T}^d, Y)$.

The situation in (2.15) is illustrated by the commutative diagram in Fig. 1.

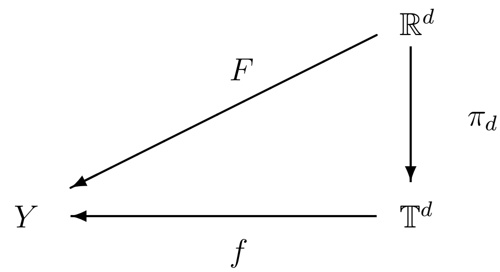

Figure 1: Commutative diagram for Lemma 2.1

Proof of Lemma 2.1: Given the function F we define the function $f : \mathbb{T}^d \to Y$ by

$$f(z) := F(\phi), \quad \phi \in z \ , \tag{2.16}$$

where the elements of the set z are defined by (2.13) so that, by (2.14), $\phi \in z$ iff $\pi_d(\phi) = z$. The function f is well defined since $F(\phi)$ has, by periodicity, the same value for every choice of $\phi \in z$. Clearly f satisfies (2.15). Furthermore, since π_d is a surjection, f is the only function which satisfies (2.15). Let F be continuous. To see that f is continuous we need to show that the inverse image $f^{-1}(V)$ is open for all open subsets V of Y. In fact, by (2.15), we compute, for the inverse images, $\pi_d^{-1}(f^{-1}(V)) = (f \circ \pi_d)^{-1}(V) = F^{-1}(V)$. Thus if V is open, $F^{-1}(V) = \pi_d^{-1}(f^{-1}(V))$ is open since F is continuous. Thus indeed f is continuous. Note that the second part of the proof also follows from the Continuity Lemma in Appendix A.6. □

Remark:

(2) We have the following simple corollary to the Torus Lemma.

Let $g \in \mathcal{C}(\mathbb{T}^d, Y)$ and let $F \in \mathcal{C}(\mathbb{R}^d, Y)$ be defined by $F := g \circ \pi_d$. Then a function $f : \mathbb{T}^d \to Y$ exists such that (2.15) holds and $f = g$.

Proof: Clearly F is continuous and 2π-periodic in all of its arguments. Thus we can apply Lemma 2.1 to F giving us a unique function $f : \mathbb{T}^d \to Y$ which satisfies (2.15). Thus and since $F = g \circ \pi_d$ we have $f = g$. □

Since π_d is continuous and 2π-periodic in all of its arguments, a trivial application of Lemma 2.1 is where $F = \pi_d$ and $f = id_{\mathbb{T}^d}$ where $id_{\mathbb{T}^d}$ is the identity function on $\mathbb{T}^d$ (for the latter see also Appendix A.1). More importantly, with Lemma 2.1 we can now rewrite the PM (2.10) in terms of $\mathbb{T}^d$. First we define the function $\mathcal{P}_\omega : \mathbb{T}^d \to \mathbb{T}^d$ by

$$\mathcal{P}_\omega(z) := (\phi + 2\pi\omega) + \tilde{\mathbb{Z}}^d \;, \quad \phi \in z \;. \tag{2.17}$$

This represents the particle dynamics on $\mathbb{T}^d$ and it simply is a linear translation on the torus which shifts the set $z = \phi + \tilde{\mathbb{Z}}^d$ to the set $(\phi + 2\pi\omega) + \tilde{\mathbb{Z}}^d$. Secondly, we define the function $A_{CT}[\omega, \mathcal{A}] : \mathbb{T}^d \to SO(3)$ by

$$A_{CT}[\omega, \mathcal{A}](z) := \Phi_{CT}[\omega, \mathcal{A}](2\pi; \phi) \;, \quad \phi \in z \;. \tag{2.18}$$

Here the functions $A_{CT}[\omega, \mathcal{A}]$ resp. $\Phi_{CT}[\omega, \mathcal{A}](2\pi; \cdot)$ correspond to f resp. F of Lemma 2.1 and thus $A_{CT}[\omega, \mathcal{A}]$ is continuous. Thus the PM (2.10) will be rewritten as the function $\mathcal{P}_{CT}[\omega, \mathcal{A}] : \mathbb{T}^d \times \mathbb{R}^3 \to \mathbb{T}^d \times \mathbb{R}^3$ defined by

$$\mathcal{P}_{CT}[\omega, \mathcal{A}](z, S) := \begin{pmatrix} \mathcal{P}_\omega(z) \\ A_{CT}[\omega, \mathcal{A}](z) S \end{pmatrix} \;. \tag{2.19}$$

We now argue that $\mathcal{P}_{CT}[\omega, \mathcal{A}]$ is continuous. It is easy to show, by (2.17), that $F(\phi) := (\mathcal{P}_\omega \circ \pi_d)(\phi) = \pi_d(\phi + 2\pi\omega)$ whence, and since π_d is continuous and 2π-periodic in its arguments, F belongs to $\mathcal{C}(\mathbb{R}^d, \mathbb{T}^d)$ and is 2π-periodic in all of its arguments so that, by Lemma 2.1, a unique function $f : \mathbb{T}^d \to \mathbb{T}^d$ exists such that $F = f \circ \pi_d$ and f is continuous. Of course since $F = \mathcal{P}_\omega \circ \pi_d$ we have $f = \mathcal{P}_\omega$ whence $\mathcal{P}_\omega \in \mathcal{C}(\mathbb{T}^d, \mathbb{T}^d)$. Since $\mathcal{P}_{-\omega}$ is the inverse of $\mathcal{P}_\omega$ we can write, due to the discussion after (2.19), $\mathcal{P}_\omega \in \text{Homeo}(\mathbb{T}^d)$. Here Homeo($Y$) denotes the set of homeomorphisms on the topological space Y (see also Appendix A.4). Since $A_{CT}[\omega, \mathcal{A}] \in \mathcal{C}(\mathbb{T}^d, SO(3))$, (2.19) implies that $\mathcal{P}_{CT}[\omega, \mathcal{A}] \in \mathcal{C}(\mathbb{T}^d \times \mathbb{R}^3, \mathbb{T}^d \times \mathbb{R}^3)$.

In summary, we have reduced the study of the continuous-time non-autonomous DS (2.1),(2.2) to the study of the discrete-time autonomous DS given by the map of (2.19). This map is determined by ω and $A_{CT}[\omega, \mathcal{A}]$. Thus we define

$$\mathcal{SOS}_{CT}(d, \omega) := \{(\mathcal{P}_\omega, A_{CT}[\omega, \mathcal{A}]) : \mathcal{A} \in \mathcal{BMT}(d)\} \;. \tag{2.20}$$

In the next section we will generalize (2.19) to the maps that we will consider in this work.

2.3 Introducing the set $\mathcal{SOS}(d, j)$ of spin-orbit systems

We now generalize $\mathcal{SOS}_{CT}(d, \omega)$ to $\mathcal{SOS}(d, j)$ by generalizing $\mathcal{P}_\omega$ and $A_{CT}[\omega, \mathcal{A}]$ to j and A giving us

$$\mathcal{SOS}(d, j) := \{(j, A) : A \in \mathcal{C}(\mathbb{T}^d, SO(3))\} \;, \tag{2.21}$$

where $j \in \text{Homeo}(\mathbb{T}^d)$ and where the matrix function A is arbitrary in $\mathcal{C}(\mathbb{T}^d, SO(3))$ and thus is not necessarily derived from the $\mathcal{A}$ of (2.1),(2.2).

Since $\mathcal{P}_\omega \in \text{Homeo}(\mathbb{T}^d)$, and since the function $A_{CT}[\omega, \mathcal{A}]$ belongs to $\mathcal{C}(\mathbb{T}^d, SO(3))$, we see from (2.20) and (2.21) that

$$\mathcal{SOS}_{CT}(d, \omega) \subset \mathcal{SOS}(d, \mathcal{P}_\omega) \;, \tag{2.22}$$

and it will be shown below that the inclusion in (2.22) is proper, i.e., that $\mathcal{SOS}_{CT}(d,\omega) \neq \mathcal{SOS}(d,\mathcal{P}_\omega)$.

We call every pair (j,A) in $\mathcal{SOS}(d,j)$ a "spin-orbit system". We call A the "1-turn spin transfer matrix function" of a spin-orbit system (j,A). We call ω the "orbital tune vector" of a spin-orbit system $(\mathcal{P}_\omega,A)$. We denote the union of the $\mathcal{SOS}(d,j)$ over j by $\mathcal{SOS}(d)$.

Motivated by (2.19), we define, for every (j,A) in $\mathcal{SOS}(d,j)$, the function $\mathcal{P}[j,A]: \mathbb{T}^d\times\mathbb{R}^3\to\mathbb{T}^d\times\mathbb{R}^3$ by

$$\mathcal{P}[j,A](z,S) := \begin{pmatrix} j(z) \\ A(z)S \end{pmatrix} , \tag{2.23}$$

and we call $\mathcal{P}[j,A]$ the "1-turn particle-spin-vector map of (j,A)". The map is invertible with inverse

$$\mathcal{P}[j,A]^{-1}(z,S) := \begin{pmatrix} j^{-1}(z) \\ A^t(j^{-1}(z))S \end{pmatrix} . \tag{2.24}$$

Clearly $\mathcal{P}[j,A]$ and $\mathcal{P}[j,A]^{-1}$ belong to $\mathcal{C}(\mathbb{T}^d\times\mathbb{R}^3,\mathbb{T}^d\times\mathbb{R}^3)$ whence $\mathcal{P}[j,A]$ is a homeomorphism and we write $\mathcal{P}[j,A]\in \mathrm{Homeo}(\mathbb{T}^d\times\mathbb{R}^3)$. In the special case where the spin-orbit system (j,A) belongs to $\mathcal{SOS}_{CT}(d,\omega)$ the 1-turn particle-spin-vector map of (j,A) carries the data of the PM, i.e., $j=\mathcal{P}_\omega$ and

$$\mathcal{P}_{CT}[\omega,\mathcal{A}] = \mathcal{P}[\mathcal{P}_\omega, A_{CT}[\omega,\mathcal{A}]] . \tag{2.25}$$

See also (2.18) and (2.19). In particular $\mathcal{P}_{CT}[\omega,\mathcal{A}]\in \mathrm{Homeo}(\mathbb{T}^d\times\mathbb{R}^3)$.

All physical applications we have in mind have $j=\mathcal{P}_\omega$ and so in this case j is just a shorthand. However, since for most notions and results of this work a general j is perfectly applicable, we do not confine ourselves to $j=\mathcal{P}_\omega$.

A central aim of this paper is a study of the DS defined by (2.23). We find it convenient to work in the more general setting of $\mathcal{SOS}(d,j)$ and (2.23) rather than the special setting of $\mathcal{SOS}_{CT}(d,\omega)$. However the main physical interest is in a small subset of $\mathcal{SOS}_{CT}(d,\omega)$. There is a natural question: given $(\mathcal{P}_\omega,A)$ in $\mathcal{SOS}(d,\mathcal{P}_\omega)$, does it belong to $\mathcal{SOS}_{CT}(d,\omega)$? This is an analogue of the following question from beam dynamics: given a symplectic map, can it be generated as the 1-turn map of a Hamiltonian system? We do not deal with this question. However, to show that the inclusion (2.22) is proper consider $\omega\in\mathbb{R}$ and $(\mathcal{P}_\omega,A)\in\mathcal{SOS}(1,\mathcal{P}_\omega)$ with $A\in\mathcal{C}(\mathbb{T}^1,SO(3))$ where m is an integer and

$$A(z) := \begin{pmatrix} \cos m\phi & -\sin m\phi & 0 \\ \sin m\phi & \cos m\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad \phi\in z . \tag{2.26}$$

It can be shown with Lemma 2.1 that A in (2.26) is well defined and continuous as we did for $A_{CT}[\omega,\mathcal{A}]$ after (2.18), thus $(\mathcal{P}_\omega,A)\in\mathcal{SOS}(1,\mathcal{P}_\omega)$. It was shown in [He2, Section 7.2], by using simple arguments from Homotopy Theory, that $(\mathcal{P}_\omega,A)\in\mathcal{SOS}_{CT}(1,\mathcal{P}_\omega)$ iff m is even. Thus for m odd we have an example showing that the inclusion (2.22) is proper. Note also that $(\mathcal{P}_\omega,A^2)\in\mathcal{SOS}_{CT}(1,\mathcal{P}_\omega)$ for every integer m.

We now discuss the DS defined by (2.23). It is a special case of a DS defined by a homeomorphism $f \in \mathcal{C}(Y,Y)$ on a topological space Y. The iterates are given by

$$y(n+1) = f(y(n)) \ , \quad y(0) = y_0 \ , \quad n \in \mathbb{Z} \ , \tag{2.27}$$

thus $y(1) = f(y_0), y(-1) = f^{-1}(y_0), y(2) = (f \circ f)(y_0), y(-2) = (f^{-1} \circ f^{-1})(y_0)$, etc. The solution of (2.27) can be written as

$$y(n) = \psi(n; y_0) \ , \qquad \psi(0; y_0) = y_0 \ , \tag{2.28}$$

where the function $\psi : \mathbb{Z} \times Y \to Y$ satisfies

$$\psi(n+m; y) = \psi(n; \psi(m; y)) \ , \qquad \psi(0; y_0) = y \ . \tag{2.29}$$

Let f^n be the n-fold composition of f with itself. Then $\psi(n; y_0) = f^n(y_0)$ and we call f^n the n-th iterate of f. We use the standard topology on $\mathbb{Z}$ (see Section 2.4) in which case a function on $\mathbb{Z} \times Y$ is continuous iff it is continuous in the second argument. Thus $\psi \in \mathcal{C}(\mathbb{Z} \times Y, Y)$. One proves (2.29) by noting that both $H_1(n) = \psi(n+m; y)$ and $H_2(n) = \psi(n; \psi(m; y))$ satisfy (2.27) with $H_1(0) = H_2(0)$. Thus by uniqueness they are equal for all n.

For our case we have $Y = \mathbb{T}^d \times \mathbb{R}^3$, $f = \mathcal{P}[j, A]$ and

$$y(n) = \begin{pmatrix} z(n) \\ S(n) \end{pmatrix} \tag{2.30}$$

whence, by (2.23),

$$\begin{pmatrix} z(n+1) \\ S(n+1) \end{pmatrix} = \begin{pmatrix} j(z(n)) \\ A(z(n))S(n) \end{pmatrix} \tag{2.31}$$

with $z(0) = z_0$ and $S(0) = S_0$ given. We call $z(\cdot)$ a "particle trajectory of (j, A)" and $S(\cdot)$ a "spin-vector trajectory of (j, A)". Moreover we call a function $(z(\cdot), S(\cdot))$ a "particle-spin-vector trajectory of (j, A)". The notion of particle-spin-vector trajectory will be generalized in Chapter 8 where we generalize the spin vector to an arbitrary variable related to spin.

We now derive a convenient representation for ψ in our case where $f = \mathcal{P}[j, A]$. We follow the procedure in Section 2.1 going from (2.1) and (2.2) to (2.9). Clearly $z(n) = j^n(z_0)$. Define $L[j] : \mathbb{Z} \times \mathbb{T}^d \to \mathbb{T}^d$ via

$$L[j](n; z) := j^n(z) \ , \tag{2.32}$$

then

$$L[j](n+m, z) = L[j](n; L[j](m; z)) \ , \tag{2.33}$$

and $S(n+1) = A(L[j](n; z_0))S(n)$ and $S(-n) = A^t(L[j](-n; z_0))S(-n+1)$ so that

$$\begin{aligned} S(n) &= A(L[j](n-1; z)) \cdots A(L[j](1; z))A(z)S_0 \ , \qquad (n = 1, 2, ...) \\ S(n) &= A^t(L[j](n; z)) \cdots A^t(L[j](-1; z))S_0 \ , \qquad (n = -1, -2, ...) \ , \end{aligned} \tag{2.34}$$

where we used the fact that $A^t(z)A(z) = I_{3\times 3}$. Thus

$$S(n) := \Psi[j, A](n; z_0)S_0 \ , \tag{2.35}$$

where

$$\begin{aligned}
&\Psi[j, A](0; z) = I_{3\times 3} \ , \\
&\Psi[j, A](n; z) = A(L[j](n-1; z)) \cdots A(L[j](1; z))A(z) \ , \qquad (n = 1, 2, ...) \ , \\
&\Psi[j, A](n; z) = A^t(L[j](n; z)) \cdots A^t(L[j](-1; z)) \ , \quad (n = -1, -2, ...) \ .
\end{aligned} \tag{2.36}$$

We now have the desired representation for ψ given by the function $L[j, A] : \mathbb{Z} \times \mathbb{T}^d \times \mathbb{R}^3 \to \mathbb{T}^d \times \mathbb{R}^3$ defined by the nth iteration of $\mathcal{P}[j, A]$:

$$L[j, A](n; z, S) := \mathcal{P}[j, A]^n(z, S) = \begin{pmatrix} L[j](n; z) \\ \Psi[j, A](n; z)S \end{pmatrix} \ . \tag{2.37}$$

With (2.28) or (2.37) the solution of (2.31) is

$$\begin{pmatrix} z(n) \\ S(n) \end{pmatrix} = L[j, A](n, z_0, S_0) \ . \tag{2.38}$$

Also from (2.29) or (2.37) we get

$$L[j, A](n + m, z, S) = L[j, A](n; L[j, A](m; z, S)) \ , \quad L[j, A](0; z, S) = \begin{pmatrix} z \\ S \end{pmatrix} \ . \tag{2.39}$$

Inserting (2.37) into (2.39) gives

$$\Psi[j, A](n + m; z) = \Psi[j, A](n; L[j](m; z)) \ \Psi[j, A](m; z) \ . \tag{2.40}$$

We now introduce some additional terminology which will be useful in the following. We call the n-th iterate $\mathcal{P}[j, A]^n = L[j, A](n; \cdot)$ the "n-turn particle-spin-vector map of (j, A)", we call $\Psi[j, A]$ the "spin transfer matrix function" of (j, A) and we call $\Psi[j, A](n; \cdot)$ the "n-turn spin transfer matrix function" of (j, A). Since $\Psi[j, A](n; \cdot)$ is continuous, every spin transfer matrix function is a continuous function due to the standard topology on $\mathbb{Z}$. Clearly

$$\Psi[j, A](1; z) = A(z) \ , \tag{2.41}$$

which justifies calling A the 1-turn spin transfer matrix function.

The behavior of the spin-vector trajectories in (2.35) depends on the values of A reached by the particle motion $L[j](n; z_0)$ in its argument, which in turn depends on j. In the case $j = \mathcal{P}_\omega$ the argument $z(n)$ of A in (2.31) will remain in a confined subset of the torus for some values of ω and for other values it will cover the torus densely. To be more precise we define resonance. We say $\chi \in \mathbb{R}^n$ is *resonant* if there exists a non-zero integer vector $k \in \mathbb{Z}^n$ such that $k \cdot \chi = 0$ and nonresonant if not resonant. If $j = \mathcal{P}_\omega$ and $(1, \omega)$ is nonresonant then the argument $z(n)$ of A in (2.31) covers the torus densely and since A is continuous all values of A affect the spin-vector trajectory whereas if $(1, \omega)$ is resonant the values of A are only sampled by its values on a sub-torus. The spin-orbit system $(\mathcal{P}_\omega, A)$ is said to be "off orbital resonance" if $(1,\omega)$ is nonresonant and "on orbital resonance" if $(1,\omega)$ is resonant. Thus spin-vector trajectories may exhibit significantly different qualitative behaviors on and off orbital resonance. We will now generalize the notion "off orbital resonance". One says that $j \in \text{Homeo}(\mathbb{T}^d)$ is "topologically transitive" if a $z_0 \in \mathbb{T}^d$ exists such that the set $B := \{j^n(z_0) : n \in \mathbb{Z}\}$ is dense in $\mathbb{T}^d$, i.e., $\overline{B} = \mathbb{T}^d$ where $\overline{B}$ denotes the topological closure of B, see Appendix A.3. An important special case is when $j = \mathcal{P}_\omega$: then j is topologically transitive iff $(1,\omega)$ is nonresonant.

It is interesting to relate again the motion defined by (2.1),(2.2) to the motion defined by (2.23). Let $\phi_0 \in \mathbb{R}^d$ and $\omega \in \mathbb{R}^d, \mathcal{A} \in \mathcal{BMT}(d)$ and let S be a solution of the IVP (2.4). Defining the function $\hat{S} : \mathbb{Z} \to \mathbb{R}^3$ by $\hat{S}(n) := S(2\pi n)$ we observe that $\hat{S}(\cdot)$ is a spin-vector trajectory of $(\mathcal{P}_\omega, A_{CT}[\omega, \mathcal{A}])$, i.e., in addition to (2.25) we get

$$\Psi[\mathcal{P}_\omega, A_{CT}[\omega, \mathcal{A}]](n; \phi_0 + \tilde{\mathbb{Z}}^d) = \Phi_{CT}[\omega, \mathcal{A}](2\pi n; \phi_0) \; . \tag{2.42}$$

2.4 Group actions and cocycles

We now continue with the DS defined by (2.23) and we will define some group theoretical notions underlying $\Psi[j, A], L[j]$ and $L[j, A]$ which will be crucial for the remainder of this work.

Definition 2.2 *(Group)*
A "group" is a pair $(G, *)$ where G is a set and $*$ is a binary operation such that

(G0) (Binary operation)	$\forall_{g_1, g_2 \in G} \; (g_1 * g_2) \in G \; ,$
(G1) (Associativity)	$\forall_{g_1, g_2, g_3 \in G} \; (g_1 * g_2) * g_3 = g_1 * (g_2 * g_3) \; ,$
(G2) (Identity element e_G)	$\exists_{e_G \in G} \; \forall_{g \in G} \; e_G = e_G * g = g * e_G \; ,$
(G3) (Inverse elements)	$\forall_{g_1 \in G} \; \exists_{g_2 \in G} \; e_G = g_1 * g_2 = g_2 * g_1 \; .$

We will abbreviate $(G, *)$ as G when the operation $*$ is clear from the context and we often write $g_1 * g_2$ as $g_1 g_2$ when the operation $*$ is clear from the context. The inverse element of a $g \in G$ is denoted by g^{-1}. If H is a subset of G and if $g, g' \in G$ then we define $gHg' := \{ghg' : h \in H\}$.

A subset G' of G is called a "subgroup of G" if it is a group w.r.t. to the restriction of $*$ to G'. Two elements g', g'' of a group G are called "conjugate" if $g \in G$ exists such that $g'' = gg'g^{-1}$. Two subgroups G', G'' of a group G are called "conjugate" if $g \in G$ exists such that $G'' = gG'g^{-1}$.

A group $(G, *)$ is called "Abelian" if

(G4) (Commutativity)	$\forall_{g_1, g_2 \in G} \; g_1 * g_2 = g_2 * g_1 \; ,$

in which case $*$ is often replaced by $+$. □

Important examples of groups in Section 2.3 are $(\mathbb{Z}, +)$, $(SO(3), *)$ (in the latter case the binary operation is matrix multiplication).

Definition 2.3 *(G-action, G-set, isotropy group)*
Consider a group G and a set E. Then a function $l : G \times E \to E$ is called a "G-action on E" if, for $g_1, g_2 \in G, x \in E$,

$$l(e_G; x) = x \tag{2.43}$$
$$l(g_1 g_2; x) = l(g_1; l(g_2; x)) \ . \tag{2.44}$$

If l is a G-action on E then the pair (E, l) is called a "G-set".

Let $x \in E$. Then we denote by $Iso(E, l; x)$ the set of those $g \in G$ for which x is a fixed point of $l(g; \cdot)$ i.e.,

$$Iso(E, l; x) := \{g \in G : l(g; x) = x\} \ . \tag{2.45}$$

Using (2.43) and (2.44) it is a simple exercise to show that $Iso(E, l; x)$ is a subgroup of G, and it is called the "isotropy group" (or "stabilizer group") of (E, l) at x. □

If (E, l) is a G-set then, for each $g \in G$, the function $l(g; \cdot) : E \to E$ is onto since, for every $y \in E$, the equality $l(g; x) = y$ is solved by $l(g^{-1}; y) = x$. Moreover $l(g; \cdot)$ is one-one since the equality $l(g; x) = l(g; y)$ implies that $x = l(g^{-1}; l(g; x)) = l(g^{-1}; l(g; y)) = y$. Thus $l(g; \cdot)$ is a bijection with inverse $l(g^{-1}; \cdot)$. For the definition of "bijection", see Appendix A.1.

It is clear by (2.33) that $L[j]$ is a $\mathbb{Z}$-action on $\mathbb{T}^d$ whence $(\mathbb{T}^d, L[j])$ is a $\mathbb{Z}$-set. Analogously it follows from (2.39) that $L[j, A]$ is a $\mathbb{Z}$-action on $\mathbb{T}^d \times \mathbb{R}^3$ and that $(\mathbb{T}^d \times \mathbb{R}^3, L[j, A])$ is a $\mathbb{Z}$-set. Apart from this $\mathbb{Z}$-set we will see many more $\mathbb{Z}$-sets in this work which are tied with (j, A). In particular in Chapter 8 we will define an infinite collection of $\mathbb{Z}$-sets tied with (j, A).

Definition 2.4 *((E, l)-orbit)*
Let (E, l) be a G-set. If $x \in E$ then the set $\{l(g; x) : g \in G\}$ is called the "(E, l)-orbit of x". We denote the set of (E, l)-orbits by E/l and define the function $l(G; \cdot) : E \to E/l$ by $l(G; x) := \{l(g; x) : g \in G\}$. Thus $l(G; x)$ is the (E, l)-orbit of x.

A G-set is called "transitive" if it has only one orbit, i.e., if E is the (E, l)-orbit of every x in E. □

The E/l in Definition 2.4 is a partition of E (see Appendix A.2) and thus $l(G; \cdot)$ is well defined and a surjection. Isotropy groups are important tools for dealing with G-sets since they allow one to conveniently deal with the (E, l)-orbits. In fact the isotropy groups of $SO(3)$-sets will be used in this capacity in Chapter 8 - in particular they will give us more insights into particle-spin-vector motions and polarization-field motions (the latter being defined in Chapter 3).

Definition 2.5 *(Topological group)*
A "topological group" is a group $(G, *)$ where G is a topological space, where the binary operation $*$ is continuous and where the function $g \mapsto g^{-1}$ on G is also continuous. □

The above-mentioned groups $\mathbb{Z}$ and $SO(3)$ in Section 2.3 are topological as we consider them to be equipped with their standard topologies. Thus the topology of $\mathbb{Z}$ is discrete, i.e., every subset of $\mathbb{Z}$ is open, and $SO(3)$ is equipped with the subspace topology from $\mathbb{R}^{3\times 3}$ (for the latter notion, see Appendix A.3). In this work we are often interested in G-sets where G and E have a topology and l is continuous. This is formalized in the following definition.

Definition 2.6 *(G-space)*
Let (E, l) be a G-set, let E be a topological space, G be a topological group, and let $l \in \mathcal{C}(G \times E, E)$ where $G \times E$ carries the product topology. For the latter notion, see Appendix A.5. Then the pair (E, l) is called a "G-space".

The definitions of "transitive", "isotropy group", (E, l)-orbit and E/l are the same as for G-sets. Also, we equip each (E, l)-orbit with the subspace topology from E. □

If (E, l) is a G-space then each $l(g; \cdot)$ is a homeomorphism. Moreover in the important subcase when the topology of G is discrete (e.g., when $G = \mathbb{Z}$) the condition that l is continuous is equivalent to $l(g; \cdot)$ being continuous for all $g \in G$.

Since, by (2.32), $L[j](n; \cdot)$ is continuous it is clear that the $\mathbb{Z}$-set $(\mathbb{T}^d, L[j])$ is a $\mathbb{Z}$-space and $L[j](n; \cdot) \in \text{Homeo}(\mathbb{T}^d)$. Recalling that $\Psi[j, A]$ is continuous, it is equally clear by (2.37) that $L[j, A](n; \cdot)$ is continuous whence the $\mathbb{Z}$-set $(\mathbb{T}^d \times \mathbb{R}^3, L[j, A])$ is also a $\mathbb{Z}$-space and $L[j, A](n; \cdot) \in \text{Homeo}(\mathbb{T}^d \times \mathbb{R}^3)$.

There are many books which cover G-spaces. A useful book, dedicated exclusively to G-sets and G-spaces, is [Ka]. Clearly G-sets and G-spaces are 2-tuples (E, l). The use of the terms set and space in this context simply arise out of the need for simple names and the fact that E is either a set or a topological space. The term G-set is synonymous with the term "transformation group" often used in the physics literature.

The spin transfer matrix function is an example of a cocycle and (2.40) is the cocycle condition. We thus define:

Definition 2.7 *(Cocycle)*
Let (E, l) be a G-space and K be a topological group. Then a function $f \in \mathcal{C}(G \times E, K)$ is called a "K cocycle over the G-space (E, l)" if, for $g, g' \in G, x \in E$,

$$f(gg'; x) = f(g; l(g'; x)) f(g'; x) \ . \tag{2.46}$$

Here $G \times E$ carries the product topology. □

For literature on cocycles, see, e.g., [KR, Zi1] and Chapter 1 in [HK1]. The reader will easily appreciate the similarity between the structures of (2.40) and (2.46) and the correspondence between the functions $\Psi[j, A] \in \mathcal{C}(\mathbb{Z} \times \mathbb{T}^d, SO(3))$ and $f \in \mathcal{C}(G \times E, K)$. Since $(\mathbb{T}^d, L[j])$ is a $\mathbb{Z}$-space and $SO(3)$ is a topological group, the set of $SO(3)$ cocycles over $(\mathbb{T}^d, L[j])$ is well defined. In fact since $\Psi[j, A] \in \mathcal{C}(\mathbb{Z} \times \mathbb{T}^d, SO(3))$ it follows from (2.40) that, for every $(j, A) \in \mathcal{SOS}(d, j)$, $\Psi[j, A]$ is a $SO(3)$ cocycle over $(\mathbb{T}^d, L[j])$. Conversely, every $SO(3)$ cocycle Ψ over $(\mathbb{T}^d, L[j])$ is the spin transfer matrix function of a spin-orbit system since, by defining $A := \Psi(1; \cdot)$, we have $\Psi[j, A] = \Psi$ so that Ψ is the spin transfer matrix function of (j, A). Clearly the cocycle property of the spin transfer matrix function $\Psi[j, A]$ is an important structure of the particle-spin-vector motion of spin-orbit systems and in Section 8.2.2 their importance will be extended to more general spin variables. Furthermore cocycles are of key importance in the underlying bundle theory (see Section 8.8).

3 Polarization-field trajectories and invariant polarization fields

In this chapter we introduce the notions of polarization field, invariant polarization field, spin field and invariant spin field and we present their most basic properties.

3.1 Generalities

High precision measurements of the spin-dependent properties of the interactions of colliding particles in storage rings depend on the equilibrium spin polarization being maximized. This, in turn, is facilitated by an understanding of the meaning of the term equilibrium, both as it applies to the value of the polarization and to its direction at each point in phase space. We will return to these matters in Section 7.1 but continue now with a definition and an exploration of the effects of maps.

Suppose therefore that $(j, A) \in \mathcal{SOS}(d, j)$ and that, at time $n = 0$, a spin vector S_{z_0} has been assigned to every point $z_0 = \phi_0 + \tilde{\mathbb{Z}}^d \in \mathbb{T}^d$ of the "particle torus" and consider their evolution according to (2.35). Let $S_{z_0}(\cdot)$ denote the spin-vector trajectory with the initial value $S_0 = S_{z_0}(0)$. We define the field trajectory $\mathcal{S} = \mathcal{S}(n, z)$ by $\mathcal{S}(n, j^n(z)) = S_z(n)$ where n and z vary over $\mathbb{Z}$ and $\mathbb{T}^d$ respectively. Clearly $\mathcal{S}(n, \cdot)$ is the distribution of spins which started at $n = 0$ with the assignments S_{z_0} and evolved under the dynamics of (2.35). Since (2.35) gives us $S_z(n + 1) = A(j^n(z))S_z(n)$, we have

$$\mathcal{S}(n + 1, z) = A\Big(j^{-1}(z)\Big)\mathcal{S}\Big(n, j^{-1}(z)\Big) \ . \tag{3.1}$$

Definition 3.1 *(Polarization-field trajectory, invariant polarization field, ISF)*
Let $(j, A) \in \mathcal{SOS}(d, j)$. We call a function $\mathcal{S} \in \mathcal{C}(\mathbb{Z} \times \mathbb{T}^d, \mathbb{R}^3)$ a "polarization-field trajectory of (j, A)", if it satisfies the evolution equation (3.1). Clearly $\mathcal{S}(n, \cdot) \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ and we call $\mathcal{S}(0, \cdot)$ the "initial value of $\mathcal{S}$". A function $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ is called an "invariant polarization field of (j, A)" if it satisfies

$$f \circ j = Af \ . \tag{3.2}$$

A polarization-field trajectory $\mathcal{S}$ is also called a "spin-field trajectory" if $|\mathcal{S}| = 1$. An invariant polarization field f is called an "invariant spin field (ISF)" if $|f| = 1$. We denote the set of invariant spin fields of (j, A) by $\mathcal{ISF}(j, A)$. □

At (2.23) we defined the function $\mathcal{P}[j, A](z, S)$ for transporting particles and their spin vectors.

The IRT, Theorem 8.15 in Section 8.7.1, will show that the ISF is a rather deep concept. We now define the function $\tilde{\mathcal{P}}[j, A] : \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3) \to \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ for the field evolution by

$$\tilde{\mathcal{P}}[j, A](f) := (Af) \circ j^{-1} \ , \tag{3.3}$$

i.e., $(\tilde{\mathcal{P}}[j, A](f))(z) := A(j^{-1}(z))f(j^{-1}(z))$ where $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$. It is an easy exercise to show that, for $(j, A) \in \mathcal{SOS}(d, j)$ and $(j', A') \in \mathcal{SOS}(d, j')$,

$$\tilde{\mathcal{P}}[j', A'] \circ \tilde{\mathcal{P}}[j, A] = \tilde{\mathcal{P}}[j' \circ j, A''] \ , \tag{3.4}$$

where $A'' \in \mathcal{C}(\mathbb{T}^d, SO(3))$ is defined by $A'' := (A' \circ j)A$, whence

$$\tilde{\mathcal{P}}[j, A] = \tilde{\mathcal{P}}[j, A_{d,0}] \circ \tilde{\mathcal{P}}[id_{\mathbb{T}^d}, A] \ , \tag{3.5}$$

where $A_{d,0} \in \mathcal{C}(\mathbb{T}^d, SO(3))$ is defined by $A_{d,0}(z) := I_{3\times 3}$. Then the inverse, $\tilde{\mathcal{P}}[j, A]^{-1}$, of $\tilde{\mathcal{P}}[j, A]$ is given by

$$\tilde{\mathcal{P}}[j, A]^{-1} = \tilde{\mathcal{P}}[id_{\mathbb{T}^d}, A^t] \circ \tilde{\mathcal{P}}[j^{-1}, A_{d,0}] = \tilde{\mathcal{P}}[j^{-1}, A^t \circ j^{-1}] \ . \tag{3.6}$$

Thus $\tilde{\mathcal{P}}[j, A]$ is a bijection so that the function $\tilde{L}[j, A] : \mathbb{Z} \times \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3) \to \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$, defined by

$$\tilde{L}[j, A](n; \cdot) := \tilde{\mathcal{P}}[j, A]^n \ , \tag{3.7}$$

is a $\mathbb{Z}$-action on $\mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ where $\tilde{\mathcal{P}}[j, A]^n$ denotes the n-th iteration of $\tilde{\mathcal{P}}[j, A]$. Clearly $(\mathcal{C}(\mathbb{T}^d, \mathbb{R}^3), \tilde{L}[j, A])$ is a $\mathbb{Z}$-set. Note that, by (2.36), (3.3) and (3.7),

$$\tilde{L}[j, A](n; f) = \Big(\Psi[j, A](n; \cdot) f\Big) \circ L[j](-n; \cdot) \ , \tag{3.8}$$

i.e., $(\tilde{L}[j, A](n; f))(z) = \Psi[j, A](n; L[j](-n; z)) f(L[j](-n; z))$. Of course, with (3.3) the evolution equation (3.1) can be written as $\mathcal{S}(n + 1, \cdot) = \tilde{\mathcal{P}}[j, A](\mathcal{S}(n, \cdot))$ whence, by (3.7), for every polarization-field trajectory $\mathcal{S}$

$$\mathcal{S}(n, \cdot) = \tilde{L}[j, A](n; \mathcal{S}(0, \cdot)) \ . \tag{3.9}$$

3.2 Invariant polarization fields

In this section we take a closer look at invariant polarization fields.

We first recall from (3.1) that if $\mathcal{S}$ is a polarization-field trajectory of (j, A) then

$$\mathcal{S}(n + 1, j(z)) = A(z)\mathcal{S}(n, z) \ , \tag{3.10}$$

whence if $\mathcal{S}$ is also time-independent then

$$\mathcal{S}(n, j(z)) = \mathcal{S}(n + 1, j(z)) = A(z)\mathcal{S}(n, z) \ . \tag{3.11}$$

It follows from (3.10) and (3.11) and Definition 3.1 and by induction in n that if $\mathcal{S}$ is a polarization-field trajectory of (j, A) then $\mathcal{S}$ is time-independent iff its initial value $\mathcal{S}(0, \cdot)$ is an invariant polarization field of (j, A).

Invariant polarization fields play an important role in polarized beam physics since they can be used to estimate the maximum attainable polarization of a bunch as we explain in Section 7.1, and since they are closely tied to the notions of spin tune and spin-orbit resonance (see Chapter 6). In fact as indicated in the Introduction invariant polarization fields are central to this work. This becomes especially clear when we generalize the notions of invariant polarization field to the notion of invariant (E, l)-field in Chapter 8 whereby (3.2) will turn out to be a so-called stationarity equation.

We now make some comments on the question of the existence of the ISF for spin-orbit systems in $\mathcal{SOS}(d,j)$. It should be clear that the constraints involved in the definition of the ISF are nontrivial. However, if a spin-orbit system (j,A) has an ISF f then $-f$ is also an ISF of (j,A). So since $f \neq -f$, if (j,A) has a finite number of ISF's, then this number is even. The important subcase where (j,A) has exactly two ISF's is dealt with in Chapter 7.

It is also known [BV1] and examined in Section 8.5.2 that spin-orbit systems exist which are on orbital resonance and which have no continuous ISF of the kind that we treat here. At the same time there are some indications, mainly from numerical computations on ISF's, that practically relevant spin-orbit systems which have no ISF are "rare". Thus we state the following conjecture, which we call the "ISF-conjecture": If (j,A) is a spin-orbit system such that j is topologically transitive then (j,A) has an ISF. Note that a special case of this conjecture is: If a spin-orbit system $(\mathcal{P}_\omega, A)$ is off orbital resonance, then it has an ISF.

The ISF-conjecture is, at least to our knowledge, unresolved. The question of the existence of the ISF is widely considered important both as a theoretical matter and as it relates to the practical matter of deciding whether a beam can have stable, non-vanishing polarization. Chapter 8 presents a new framework for discussing it.

Since the ISF-conjecture deals with topologically transitive j we state the following theorem which considers this situation.

Theorem 3.2 *Let $(j,A) \in \mathcal{SOS}(d,j)$ where j is topologically transitive. If f is an invariant polarization field of (j,A) then $|f|$ is constant, i.e., $|f(z)|$ is independent of z. Also (j,A) has an ISF iff it has an invariant polarization field which is not identically zero.*

Proof of Theorem 3.2: Let f be an invariant polarization field of (j,A). Then, by Definition 3.1, $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ and

$$|f(j(z))| = |f(z)| \; . \tag{3.12}$$

We pick a $z_0 \in \mathbb{T}^d$ such that the set $B := \{j^n(z_0) : n \in \mathbb{Z}\}$ is dense in $\mathbb{T}^d$, i.e., $\overline{B} = \mathbb{T}^d$ and we define $S_0 := f(z_0)$. Since f is an invariant polarization field we have $B \subset C := \{z \in \mathbb{T}^d : |f(z)| = |S_0|\}$. On the other hand, the sphere of radius $|S_0|$, i.e., the set $\{S \in \mathbb{R}^3 : |S| = |S_0|\}$ is a closed subset of $\mathbb{R}^3$ whence, because f is continuous, C is a closed subset of $\mathbb{T}^d$. Therefore $\mathbb{T}^d = \overline{B} \subset \overline{C} = C$ so that $\mathbb{T}^d = C$. Thus, by the definition of C, we conclude that $|f(z)| = |S_0|$ for all $z \in \mathbb{T}^d$.

To prove the second claim, let f be an invariant polarization field of (j,A) which is not identically zero. Clearly by the first claim $|f|$ is constant and takes a nonzero value because $|f|$ is not identically zero. Thus we define $g \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ by $g := f/|f|$ whence, by Definition 3.1, g is an ISF of (j,A). Conversely every ISF of (j,A) is an invariant polarization field of (j,A) which is not identically zero. □

In the special case when $j = \mathcal{P}_\omega$ with $(1,\omega)$ nonresonant one can prove Theorem 3.2 alternatively by some simple Fourier Analysis of f [He2]. With Theorem 3.2, the ISF conjecture is equivalent to the following statement: If j is topologically transitive then (j,A) has an invariant polarization field which is not identically zero. Note also that Theorem 3.2 will be generalized by Lemma 8.4.

A less formal picture surrounding Theorem 3.2 is as follows. When j is topologically transitive, the whole of $\mathbb{T}^d$ can effectively be reached from any starting position z_0 by repeated application of j. Moreover, by a corresponding repeated application of A, $f(z_0)$ generates $f(z)$ at effectively all points on $\mathbb{T}^d$. So the $f(z)$ on $\mathbb{T}^d$ are all "connected". Also, since A is $SO(3)$-valued all the $|f(z)|$ are the same. On the other hand, if j is not transitive, the $f(z)$ at different z need not be connected. For example the $f(z)$ at adjacent z could have opposite signs. We will encounter a related situation in Section 8.5.2 for $j = \mathcal{P}_\omega$ with $\omega = 1/2$ and in Remark 12 of Chapter 8.

4 Transforming spin-orbit systems

In this chapter we introduce the transformation of any $(j, A) \in \mathcal{SOS}(d, j)$ under any $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ and we show how this is accompanied by a transformation of $\mathcal{P}[j, A]$ and $\tilde{\mathcal{P}}[j, A]$ as well as by a transformation of particle-spin-vector trajectories and polarization-field trajectories.

4.1 The transformation rule of spin-orbit systems

We now show how to partition $\mathcal{SOS}(d, j)$ into subsets within which the dynamics is similar. Consider $(j, A) \in \mathcal{SOS}(d, j)$ and let $(z(\cdot), S(\cdot))$ be a particle-spin-vector trajectory of (j, A), i.e., let (2.31) hold so that $S(\cdot)$ is a spin-vector trajectory of (j, A) and thus $S(n+1) = A(z(n))S(n)$. For arbitrary $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$, the function $S' : \mathbb{Z} \to \mathbb{R}^3$ defined by $S'(n) := T^t(z(n))S(n)$ satisfies $S'(n+1) = T^t(z(n+1))A(z(n))T(z(n))S'(n)$. So $S'(\cdot)$ is a spin-vector trajectory of a new spin-orbit system, namely of $(j, A') \in \mathcal{SOS}(d, j)$ which is defined by

$$A'(z) := T^t(j(z))A(z)T(z) . \tag{4.1}$$

Note that (4.1) implies $A(z) = T(j(z))A'(z)T^t(z)$. Thus $(z(\cdot), S'(\cdot))$ is a particle-spin-vector trajectory of (j, A'). Recalling from (2.23) that $\mathcal{P}[j, A](z, S) = \begin{pmatrix} j(z) \\ A(z)S \end{pmatrix}$ and $\mathcal{P}[j, A'](z, S) = \begin{pmatrix} j(z) \\ A'(z)S \end{pmatrix}$ it is easy to show that (4.1) holds iff

$$\mathcal{P}[id_{\mathbb{T}^d}, T]^{-1} \circ \mathcal{P}[j, A] \circ \mathcal{P}[id_{\mathbb{T}^d}, T] = \mathcal{P}[j, A'] . \tag{4.2}$$

Eq. (4.1) gives rise to a partition of $\mathcal{SOS}(d, j)$ as we formalize in the next two definitions.

Definition 4.1 *(Transformation rule of spin-orbit systems)*
Let (j, A) and (j, A') be in $\mathcal{SOS}(d, j)$. Then a T in $\mathcal{C}(\mathbb{T}^d, SO(3))$ is called a "transfer field from (j, A) to (j, A')" iff (4.1) holds. We also say that "(j, A') is the transform of (j, A) under T". We denote the collection of all transfer fields from (j, A) to (j, A') by $\mathcal{TF}(A, A'; d, j)$. Note that if $T \in \mathcal{TF}(A, A'; d, j)$ then $T^t \in \mathcal{TF}(A', A; d, j)$, i.e., (j, A) is the transform of (j, A') under T^t. □

Clearly $\mathcal{TF}(A, A'; d, j) \neq \emptyset$ iff (j, A') is a transform of (j, A) as in (4.1). Note that in general we don't have transfer fields, i.e., $\mathcal{TF}(A, A'; d, j) = \emptyset$ (see, e.g., Remark 9 in Chapter 6).

Following Appendix A.2 we make the definition:

Definition 4.2 *Let (j, A) and (j, A') be in $\mathcal{SOS}(d, j)$. Then we write $(j, A) \sim (j, A')$ and say that (j, A) and (j, A') are "equivalent" iff (j, A') is a transform of (j, A) under some $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$. Clearly the relation $\sim$ is symmetric, reflexive, and transitive and thus is an equivalence relation on $\mathcal{SOS}(d, j)$. Let $\overline{(j, A)} := \{(j, A'') : (j, A'') \sim (j, A)\}$, i.e., the equivalence class of (j, A) under $\sim$. As outlined in Appendix A.2, the sets $\overline{(j, A)}$ partition $\mathcal{SOS}(d, j)$.* □

Two spin-orbit systems which are equivalent share many important properties, e.g., the existence or nonexistence of an ISF (see Remark 3 below). We will see other properties shared by equivalent spin-orbit systems throughout this work. For checking those shared properties it can be convenient to check them for a "simple" element of $\overline{(j, A)}$ (see especially Chapters 5 and 6).

The transformation rule $(j, A) \longrightarrow (j, A')$ also gives the following transformation rule of spin transfer matrix functions:

$$\Psi[j, A] \longrightarrow \Psi[j, A'] \ . \tag{4.3}$$

It follows from (2.40) and (4.1) and by induction in n that, if $T \in \mathcal{TF}(A, A'; d, j)$, with $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ then the spin transfer matrix functions of $(j, A) \in \mathcal{SOS}(d, j)$ and $(j, A') \in \mathcal{SOS}(d, j)$ are related by

$$\Psi[j, A'](n; z) = T^t(L[j](n; z))\Psi[j, A](n; z)T(z) \ . \tag{4.4}$$

Recall from Section 2.4 that $\Psi[j, A]$ and $\Psi[j, A']$ are cocycles. Then (4.4) implies that the cocycles $\Psi[j, A]$ and $\Psi[j, A']$ are "cohomologous". For this notion, see, e.g., [He2, KR, Zi1] and Chapter 1 in [HK1].

4.2 Transforming particle-spin-vector trajectories and polarization-field trajectories. Topological G-maps of G-spaces

With Definition 4.1 we arrive at the following transformation rule of $\mathbb{Z}$-actions:

$$L[j, A] \longrightarrow L[j, A'] \ , \tag{4.5}$$
$$\tilde{L}[j, A] \longrightarrow \tilde{L}[j, A'] \ , \tag{4.6}$$

where A, A' are related by (4.1) with $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ and $(j, A) \in \mathcal{SOS}(d, j)$. It is easy to see how the $\mathbb{Z}$-actions $L[j, A]$ and $L[j, A']$ in the transformation rule (4.5) are related. In fact it follows from (4.2) that

$$\mathcal{P}[id_{\mathbb{T}^d}, T]^{-1} \circ \mathcal{P}[j, A]^n \circ \mathcal{P}[id_{\mathbb{T}^d}, T] = \mathcal{P}[j, A']^n \ . \tag{4.7}$$

Therefore, by (2.37), $L[j, A'](n; \cdot) = \mathcal{P}[id_{\mathbb{T}^d}, T]^{-1} \circ L[j, A](n; \cdot) \circ \mathcal{P}[id_{\mathbb{T}^d}, T]$, so that

$$\mathcal{P}[id_{\mathbb{T}^d}, T]^{-1} \circ L[j, A](n; \cdot) = L[j, A'](n; \cdot) \circ \mathcal{P}[id_{\mathbb{T}^d}, T]^{-1} \ . \tag{4.8}$$

Moreover it is easy to see how the $\mathbb{Z}$-actions $\tilde{L}[j, A]$ and $\tilde{L}[j, A']$ in the transformation rule (4.6) are related. In fact we conclude from (3.4) that

$$\tilde{\mathcal{P}}[id_{\mathbb{T}^d}, T^t] = \tilde{\mathcal{P}}[id_{\mathbb{T}^d}, T]^{-1} \ , \tag{4.9}$$

$$\tilde{\mathcal{P}}[id_{\mathbb{T}^d}, T]^{-1} \circ \tilde{\mathcal{P}}[j, A] \circ \tilde{\mathcal{P}}[id_{\mathbb{T}^d}, T] = \tilde{\mathcal{P}}[j, A'] \ , \tag{4.10}$$

whence, by (3.7),

$$\tilde{\mathcal{P}}[id_{\mathbb{T}^d}, T]^{-1} \circ \tilde{L}[j, A](n; \cdot) = \tilde{L}[j, A'](n; \cdot) \circ \tilde{\mathcal{P}}[id_{\mathbb{T}^d}, T]^{-1} \ . \tag{4.11}$$

The following definition provides a simple classification of the relations (4.8) and (4.11). Recall that G-sets and G-spaces are defined in Definitions 2.3 and 2.6.

Definition 4.3 *(G-maps of G-sets, topological G-maps of G-spaces)*
a) Consider G-sets (E_1, l_1) and (E_2, l_2). A function $f : E_1 \to E_2$ is called a "G-map from (E_1, l_1) to (E_2, l_2)" if, for every $g \in G$, $f \circ l_1(g; \cdot) = l_2(g; f(\cdot))$, i.e., if for every $g \in G, x \in E_1$,

$$f(l_1(g; x)) = l_2(g; f(x)) \ . \tag{4.12}$$

b) Consider the G-spaces (E_1, l_1) and (E_2, l_2) and let $f \in \mathcal{C}(E_1, E_2)$. If f satisfies (4.12) then f is called a "topological G-map from (E_1, l_1) to (E_2, l_2)". □

If f is a G-map from the G-set (E_1, l_1) to the G-set (E_2, l_2) and if f is a bijection, then f^{-1} is a G-map from (E_2, l_2) to (E_1, l_1) and (E_2, l_2) and (E_1, l_1) are said to be "isomorphic". We then also say that l_2 and l_1 are "isomorphic" and that f is an "isomorphism" from (E_1, l_1) to (E_2, l_2). Note that isomorphic G-sets share many properties.

Analogously, when f is a topological G-map from the G-space (E_1, l_1) to the G-space (E_2, l_2) and if f is a homeomorphism then (E_2, l_2) and (E_1, l_1) are said to be "isomorphic". We then also say that f is an "isomorphism" from (E_1, l_1) to (E_2, l_2). Clearly then f^{-1} is an isomorphism from (E_2, l_2) to (E_1, l_1). Note that isomorphic G-spaces share many properties.

It is clear how the relations (4.8) and (4.11) can be formulated in terms of Definition 4.3. First, since $\mathcal{P}[id_{\mathbb{T}^d}, T]^{-1} \in \text{Homeo}(\mathbb{T}^d \times \mathbb{R}^3)$ it follows from (4.8) and Definitions 4.1 and 4.3 that if T is a transfer field from (j, A) to (j, A') then $\mathcal{P}[id_{\mathbb{T}^d}, T]^{-1}$ is an isomorphism from the $\mathbb{Z}$-space $(\mathbb{T}^d \times \mathbb{R}^3, L[j, A])$ to the $\mathbb{Z}$-space $(\mathbb{T}^d \times \mathbb{R}^3, L[j, A'])$. In particular, the transformation rule (4.5) relates isomorphic $\mathbb{Z}$-spaces. Secondly, since $\tilde{\mathcal{P}}[id_{\mathbb{T}^d}, T]^{-1}$ is a bijection it follows from (4.11) and Definition 4.3 that $\tilde{\mathcal{P}}[id_{\mathbb{T}^d}, T]^{-1}$ is an isomorphism from the $\mathbb{Z}$-set $(\mathcal{C}(\mathbb{T}^d, \mathbb{R}^3), \tilde{L}[j, A])$ to the $\mathbb{Z}$-set $(\mathcal{C}(\mathbb{T}^d, \mathbb{R}^3), \tilde{L}[j, A'])$.

The transformation rules (4.5) and (4.6) will be generalized in Section 8.2.3. In particular, as we point out in Section 8.8, they derive from an $SO(3)$-gauge transformation rule.

The transformation rules (4.5) and (4.6) transform $\mathbb{Z}$-actions, i.e., they transform dynamics. We now supplement (4.5) and (4.6) by transformation rules of the underlying histories, i.e., transformation rules of particle-spin-vector trajectories and polarization-field trajectories. First of all, as mentioned at the beginning of this section, we arrive at the transformation rule of particle-spin-vector trajectories:

$$(z(\cdot), S(\cdot)) \longrightarrow (z(\cdot), S'(\cdot)) \ , \quad S'(n) := T^t(z(n))S(n) \ . \tag{4.13}$$

Clearly if $(z(\cdot), S(\cdot))$ is a particle-spin-vector trajectory of (j, A) then $(z(\cdot), S'(\cdot))$ is a particle-spin-vector trajectory of (j, A').

In parallel to (4.13) one can also transform polarization-field trajectories. In fact if f is the initial value of a polarization-field trajectory $\mathcal{S}$ of (j, A) then we can relate it to the polarization-field trajectory $\mathcal{S}'$ of (j, A') whose initial value is $\tilde{\mathcal{P}}[id_{\mathbb{T}^d}, T]^{-1}(f) = T^t f$. Thus, by (3.9),

$$\mathcal{S}(n, \cdot) = \tilde{L}[j, A](n; f) \ , \quad \mathcal{S}'(n, \cdot) = \tilde{L}[j, A'](n; T^t f) \ , \tag{4.14}$$

whence, by (3.7) and (4.10),

$$\begin{aligned}
\mathcal{S}'(n, \cdot) &= \tilde{L}[j, A'](n; \mathcal{S}'(0, \cdot)) \\
&= \tilde{\mathcal{P}}[j, A']^n(\mathcal{S}'(0, \cdot)) = (\tilde{\mathcal{P}}[id_{\mathbb{T}^d}, T]^{-1} \circ \tilde{\mathcal{P}}[j, A]^n \circ \tilde{\mathcal{P}}[id_{\mathbb{T}^d}, T])(\mathcal{S}'(0, \cdot)) \\
&= (\tilde{\mathcal{P}}[id_{\mathbb{T}^d}, T]^{-1} \circ \tilde{\mathcal{P}}[j, A]^n \circ \tilde{\mathcal{P}}[id_{\mathbb{T}^d}, T] \circ \tilde{\mathcal{P}}[id_{\mathbb{T}^d}, T]^{-1})(f) = \tilde{\mathcal{P}}[id_{\mathbb{T}^d}, T]^{-1}(\tilde{\mathcal{P}}[j, A]^n(f)) \\
&= \tilde{\mathcal{P}}[id_{\mathbb{T}^d}, T]^{-1}(\tilde{L}[j, A](n, f)) = \tilde{\mathcal{P}}[id_{\mathbb{T}^d}, T]^{-1}(\mathcal{S}(n, \cdot)) \ ,
\end{aligned} \tag{4.15}$$

i.e.,

$$\mathcal{S}'(n, \cdot) = \tilde{\mathcal{P}}[id_{\mathbb{T}^d}, T]^{-1}(\mathcal{S}(n, \cdot)) \ . \tag{4.16}$$

We conclude from (4.16) that if $\mathcal{S}$ is a polarization-field trajectory of (j, A) then $\mathcal{S}'$, defined by (4.16), is a polarization-field trajectory of (j, A'). Thus with (4.16) we have a natural transformation rule of polarization-field trajectories. Note that (3.3) and (4.16) give us

$$\mathcal{S} \longrightarrow \mathcal{S}' \ , \quad \mathcal{S}'(n, z) := T^t(z)\mathcal{S}(n, z) \ . \tag{4.17}$$

With (4.17) and by the remarks after (3.11) we have the following transformation rule of invariant polarization fields:

$$f \longrightarrow f' \ , \quad f'(z) := T^t(z) f(z) \ . \tag{4.18}$$

In fact if f is an invariant polarization field of (j, A) then f', defined by (4.18), is an invariant polarization field of (j, A'). In Section 8.2 we will generalize the notions of particle-spin-vector trajectory, polarization-field trajectory, and invariant polarization field. Then the transformation rules (4.13), (4.17) and (4.18) will be generalized accordingly.

Remarks:

(1) The transformation rules (4.5), (4.6), (4.13), (4.17) and (4.18) are no strangers to the polarized-beam community. In fact when researchers deal with the topics of spin tune, spin frequency, spin resonances, resonance strengths etc. then they often appeal more or less directly to these transformation rules. In those applications the aim, typically, is to transform (j, A) to a "simple" (j, A').

(2) The transformation rule (4.13) could be generalized to

$$(z(\cdot), S(\cdot)) \longrightarrow (z(\cdot), S'(\cdot)) \ , \quad S'(n) := R^t(n, z(\cdot))S(n) \ , \tag{4.19}$$

where $R(n, z(\cdot))$ generalizes $T(z(n))$ by allowing an arbitrary dependence on the particle trajectory $z(\cdot)$. However, as can be easily shown [He2], if $(\mathcal{P}_\omega, A) \in \mathcal{SOS}(d, \mathcal{P}_\omega)$ then, in general the $(z(\cdot), S'(\cdot))$ in (4.19) is not the particle-spin-vector trajectory of *any* $(\mathcal{P}_\omega, A') \in \mathcal{SOS}(d, \mathcal{P}_\omega)$!

(3) It is clear that (4.17) maps the polarization-field trajectories of (j, A) bijectively onto the set of polarization-field trajectories of (j, A'). It is equally clear that (4.18) maps $\mathcal{ISF}(j, A)$ bijectively onto $\mathcal{ISF}(j, A')$. In particular equivalent spin-orbit systems have the same number of ISF's.

(4) When transforming (j, A) to a "simple" (j, A') the polarization-field trajectories of the latter are "simple" too whence (4.17) allows one to compute polarization-field trajectories by transforming "simple" polarization-field trajectories. For this philosophy see Chapters 5 and 6 too. □

4.3 Remarks on conjugate 1-turn particle-spin-vector maps and structure preserving homeomorphisms

Note that Homeo($\mathbb{T}^d \times \mathbb{R}^3$) forms a group, where the group multiplication is understood to be the composition of functions. Thus, since $\mathcal{P}[j, A] \in$ Homeo($\mathbb{T}^d \times \mathbb{R}^3$), it follows from (4.2) and Definitions 2.2 and 4.2 that if $(j, A) \sim (j, A')$ then $\mathcal{P}[j, A]$ and $\mathcal{P}[j, A']$ are conjugate elements of the group Homeo($\mathbb{T}^d \times \mathbb{R}^3$), i.e., a $\mathcal{T} \in$ Homeo($\mathbb{T}^d \times \mathbb{R}^3$) exists such that $\mathcal{P}[j, A'] = \mathcal{T}^{-1} \circ \mathcal{P}[j, A] \circ \mathcal{T}$. In fact $\mathcal{T} = \mathcal{P}[id_{\mathbb{T}^d}, T]$ with $T \in \mathcal{TF}(A, A'; d, j)$ is an example. We call a $\mathcal{T} \in$ Homeo($\mathbb{T}^d \times \mathbb{R}^3$) "structure preserving for a $(j, A) \in \mathcal{SOS}(d, j)$" if the homeomorphism $\mathcal{T}^{-1} \circ \mathcal{P}[j, A] \circ \mathcal{T}$ in Homeo($\mathbb{T}^d \times \mathbb{R}^3$) is of the form $\mathcal{P}[j', A']$ for some $(j', A') \in \mathcal{SOS}(d, j')$. We call a $\mathcal{T} \in$ Homeo($\mathbb{T}^d \times \mathbb{R}^3$) "structure preserving" if it is structure preserving for all $(j, A) \in \mathcal{SOS}(d, j)$. As we discovered in Section 4.1, every $\mathcal{P}[id_{\mathbb{T}^d}, T]$ with $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ is structure preserving.

Thus three natural questions arise. First, what are the structure-preserving $\mathcal{T} \in$ Homeo($\mathbb{T}^d \times \mathbb{R}^3$) of a given $(j, A) \in \mathcal{SOS}(d, j)$? Secondly, which $\mathcal{T} \in$ Homeo($\mathbb{T}^d \times \mathbb{R}^3$) are structure preserving? Thirdly, which $\mathcal{TF}(A, A'; d, j)$ are nonempty? While these questions from Dynamical-Systems Theory will not be fully addressed in this work we now give a brief glimpse. Let $(j, A) \in \mathcal{SOS}(d, j)$ and $(j, A') \in \mathcal{SOS}(d, j)$ and let $\mathcal{T} \in$ Homeo($\mathbb{T}^d \times \mathbb{R}^3$). Writing $\mathcal{T}$ in terms of components $\mathcal{T} = (\mathcal{T}_{part}, \mathcal{T}_v)$ we compute

$$
\begin{aligned}
&(\mathcal{T} \circ \mathcal{P}[j, A'])(z, S) = \mathcal{T}(j(z), A'(z)S) = (\mathcal{T}_{part}(j(z), A'(z)S), \mathcal{T}_v(j(z), A'(z)S)) \,, \\
&(\mathcal{P}[j, A] \circ \mathcal{T})(z, S) = \mathcal{P}[j, A](\mathcal{T}_{part}(z, S), \mathcal{T}_v(z, S)) \\
&\quad = (j(\mathcal{T}_{part}(z, S)), A(\mathcal{T}_{part}(z, S))\mathcal{T}_v(z, S)) \,,
\end{aligned}
$$

whence $\mathcal{P}[j, A'] = \mathcal{T}^{-1} \circ \mathcal{P}[j, A] \circ \mathcal{T}$ iff

$$
\begin{aligned}
\mathcal{T}_{part}(j(z), A'(z)S) &= j(\mathcal{T}_{part}(z, S)) \,, \\
\mathcal{T}_v(j(z), A'(z)S) &= A(\mathcal{T}_{part}(z, S))\mathcal{T}_v(z, S) \,.
\end{aligned}
\tag{4.20}
$$

The system of equations (4.20) plays a central role when one addresses the aforementioned questions. Of course in the special case $\mathcal{T} = \mathcal{P}[id_{\mathbb{T}^d}, T]$ with $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ we see that $\mathcal{T}_{part} = id_{\mathbb{T}^d}$ and $\mathcal{T}_v(z, S)) = T(z)S$ so that in that case we recover the fact from Section 4.1 that $\mathcal{P}[j, A'] = \mathcal{T}^{-1} \circ \mathcal{P}[j, A] \circ \mathcal{T}$ iff $T \in \mathcal{TF}(A, A'; d, j)$. We finally mention that there are structure preserving $\mathcal{T} \in$ Homeo($\mathbb{T}^d \times \mathbb{R}^3$) which are different from any $\mathcal{P}[id_{\mathbb{T}^d}, T]$. For example, defining $\mathcal{T} \in \text{Homeo}(\mathbb{T}^d \times \mathbb{R}^3)$ by $\mathcal{T} = (\mathcal{T}_{part}, \mathcal{T}_v)$ where $\mathcal{T}_{part}(z) = z, \mathcal{T}_v(z, S) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} S$ one easily sees that $\mathcal{T}$ is structure preserving and is different from any $\mathcal{P}[id_{\mathbb{T}^d}, T]$. The latter follows from the fact that $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ has determinant -1.

5 H-normal forms and the subsets $\mathcal{CB}_H(d, j)$ of $\mathcal{SOS}(d, j)$

As in Section 4.3, we wish to know which $\mathcal{TF}(A, A'; d, j)$ are nonempty, i.e., which spin-orbit systems in $\mathcal{SOS}(d, j)$ are equivalent. In fact, by Remark 9 in Chapter 6, every $\mathcal{SOS}(d, j)$ is partitioned into uncountably many equivalence classes if j is of the form $\mathcal{P}_\omega$. We have already remarked on the advantages of transforming to a "simple" (j, A'). Now Remark 9 in Chapter 6 suggests that to gain insight into the partition of $\mathcal{SOS}(d, j)$ it is fruitful to find "simple" elements (j, A') in an equivalence class $\overline{(j, A)}$ and to compare different equivalence classes in terms of their "simple" elements. In this chapter we apply this philosophy by focusing on those "simple" elements (j, A') in $\overline{(j, A)}$ for which A' is H-valued where H is a fixed subgroup of $SO(3)$. Then (j, A') is called an H-normal form of (j, A). Note that the notion of H-normal form is different from the usual definition of normal form for spin [Yo2] but it is inspired by the $SO(2)$-normal forms studied in [Yo1].

We will proceed as follows. In Section 5.1 we will define the notion of H-normal form. Then in Section 5.2 we will consider the important case $H = SO(2)$ where $SO(2)$ is defined by (5.5) and we will see that the notion of $SO(2)$-normal form is not new and is connected with the notion of the ISF via the IFF Theorem, Theorem 5.4b.

5.1 Generalities

Definition 5.1 *(H-normal form, $\mathcal{CB}_H(d, j)$)*
Consider a subgroup, H, of $SO(3)$ and let (j, A) be in $\mathcal{SOS}(d, j)$. Then we call a (j, A') in $\mathcal{SOS}(d, j)$ an "H-normal form of (j, A)" if A' is H-valued and $(j, A) \sim (j, A')$, i.e., $(j, A') \in \overline{(j, A)}$. We denote by $\mathcal{CB}_H(d, j)$ the set of all spin-orbit systems in $\mathcal{SOS}(d, j)$ which have an H-normal form. Thus $(j, A) \in \mathcal{CB}_H(d, j)$ iff $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ exists such that

$$T^t(j(z))A(z)T(z) \in H \ , \tag{5.1}$$

holds for every $z \in \mathbb{T}^d$. The acronym $\mathcal{CB}$ will be explained in Remark 6 of Chapter 6. We also define

$$\mathcal{TF}_H(j, A) := \Big\{ T \in \mathcal{C}(\mathbb{T}^d, SO(3)) : (\forall \ z \in \mathbb{T}^d) \ T^t(j(z))A(z)T(z) \in H \Big\} \ . \tag{5.2}$$

Thus $(j, A) \in \mathcal{CB}_H(d, j)$ iff $\mathcal{TF}_H(j, A)$ is nonempty. Note that the elements of $\mathcal{TF}_H(j, A)$ are the transfer fields from (j, A) to those (j, A') for which A' is H-valued. □

In Chapter 8 we will take a deeper look into H-normal forms for arbitrary subgroups H of $SO(3)$. See for example the Normal Form Theorem, Theorem 8.1, in Section 8.2.4.

We now make some remarks on Definition 5.1.

Remarks:

(1) Definition 5.1 gives us another property shared by equivalent spin-orbit systems since it implies that if (j, A) belongs to $\mathcal{CB}_H(d, j)$ then every spin-orbit system in $\overline{(j, A)}$ belongs to $\mathcal{CB}_H(d, j)$.

(2) Let (j, A) be in $\mathcal{SOS}(d, j)$ and let H' and H be subgroups of $SO(3)$ such that $H \subset H'$. Then, by Definition 5.1, $\mathcal{TF}_H(j, A) \subset \mathcal{TF}_{H'}(j, A)$. On the other hand, by Definition 5.1, if $(j, A) \in \mathcal{CB}_H(d, j)$ then $\mathcal{TF}_H$ is nonempty whence $\mathcal{TF}_{H'}$ is nonempty so that, by Definition 5.1, $(j, A) \in \mathcal{CB}_{H'}(d, j)$. Thus

$$\mathcal{CB}_H(d, j) \subset \mathcal{CB}_{H'}(d, j) \ . \tag{5.3}$$

This fact is even true under more general conditions than $H \subset H'$ as explained after Definition 5.2. This fact also implies that the "larger H" the more likely it is that a given (j, A) has an H-normal form. For more details on this aspect see the remarks after the NFT in Section 8.2.4.

(3) Let (j, A) be in $\mathcal{SOS}(d, j)$, let H be a subgroup of $SO(3)$ and $r \in SO(3)$. Then it is an easy exercise to show, by Definition 5.1, that $\mathcal{TF}_{rHr^t}(j, A) = \{Tr^t : T \in \mathcal{TF}_H(j, A)\}$. $\square$

To relate H-normal forms for different H we make the following definition:

Definition 5.2 *Let H and H' be subsets of $SO(3)$. We write $H \trianglelefteq H'$ if an $r \in SO(3)$ exists such that $rHr^t \subset H'$. For the notation rHr^t see Definition 2.2. Recalling Appendix A.2, $\trianglelefteq$ is a relation on the set of subsets of $SO(3)$ and it is easy to show that $\trianglelefteq$ is reflexive and transitive but not symmetric.* $\square$

Note that $H \trianglelefteq H'$ if $H \subset H'$. If H, H' are subgroups of $SO(3)$ then $H \trianglelefteq H'$ iff H is conjugate to a subgroup of H'. It is an easy exercise to show, by Remarks 2 and 3, that (5.3) holds if $H \trianglelefteq H'$ (this strengthens Remark 2). Thus via $\trianglelefteq$ one can order spin-orbit tori in terms of their normal forms. It is also a simple exercise to show that if H and H' are conjugate subgroups of $SO(3)$ then $H' \trianglelefteq H$ and $H \trianglelefteq H'$ whence, by (5.3),

$$\mathcal{CB}_{H'}(d, j) = \mathcal{CB}_H(d, j) \ . \tag{5.4}$$

The relation $\trianglelefteq$ is well-known in Mathematics and will become an important tool in Chapter 8.

5.2 $SO(2)$-normal forms. The IFF Theorem

In this section we consider H-normal forms in the special case $H = SO(2)$ where the subgroup $SO(2)$ of $SO(3)$ is defined by

$$SO(2) := \left\{ \begin{pmatrix} \cos(x) & -\sin(x) & 0 \\ \sin(x) & \cos(x) & 0 \\ 0 & 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$$

$$= \{\exp(x\mathcal{J}) : x \in \mathbb{R}\} = \{\exp(x\mathcal{J}) : x \in [0, 2\pi)\} \ , \tag{5.5}$$

with

$$\mathcal{J} := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ . \tag{5.6}$$

For reasons that will become clear below, we now come to:

Definition 5.3 *(Invariant frame field)*
Let $(j, A) \in \mathcal{SOS}(d, j)$. We call every element of $\mathcal{TF}_{SO(2)}(j, A)$ an "Invariant Frame Field (IFF) of (j, A)". Clearly, by Definition 5.1, $\mathcal{TF}_{SO(2)}(j, A)$ is nonempty iff $(j, A) \in \mathcal{CB}_{SO(2)}(d, j)$. □

Moreover, for any subgroup $H \neq SO(2)$ of $SO(3)$, we will view the elements of $\mathcal{TF}_H(j, A)$ as generalized IFF's of (j, A). Definition 5.3 sets the stage for

Theorem 5.4 *a) ($SO(2)$-Lemma) A matrix r in $SO(3)$ belongs to $SO(2)$ iff $r(0, 0, 1)^t = (0, 0, 1)^t$.*

b) (IFF Theorem) Let $(j, A) \in \mathcal{SOS}(d, j)$. Then T is an IFF of (j, A) iff $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ and the third column of T is an ISF of (j, A). In other words, a $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ belongs to $\mathcal{TF}_{SO(2)}(j, A)$ iff $f(z) := T(z)(0, 0, 1)^t$ satisfies (3.2).

Proof of Theorem 5.4a: The claim follows immediately from (5.5). □
Proof of Theorem 5.4b: "⇒": Let $T \in \mathcal{TF}_{SO(2)}(j, A)$. Then, by Definition 5.1, $T^t(j(z))A(z)T(z) \in SO(2)$ whence, by Theorem 5.4a, $T^t(j(z))A(z)T(z)(0, 0, 1)^t = (0, 0, 1)^t$ so that $A(z)T(z)(0, 0, 1)^t = T(j(z))(0, 0, 1)^t$ whence, by Definition 3.1, $T(0, 0, 1)^t$ is an ISF of (j, A).

"⇐": Let $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ and let $T(0, 0, 1)^t$ be an ISF of (j, A) whence, by Definition 3.1, $A(z)T(z)(0, 0, 1)^t = T(j(z))(0, 0, 1)^t$ so that $T^t(j(z))A(z)T(z)(0, 0, 1)^t = (0, 0, 1)^t$. Thus, by Theorem 5.4a, $T^t(j(z))A(z)T(z) \in SO(2)$. It now follows from Definition 5.1 that $T \in \mathcal{TF}_{SO(2)}(j, A)$. □

By Theorem 5.4b, IFF's are those continuous T's whose third columns are ISF's. In fact this is to be expected given the definition of the IFF in the continuous-time formalism in [BEH]. There, we begin with the ISF at each point in phase space, and then construct the IFF by attaching two unit vectors to the ISF at each point so as to form a local orthonormal coordinate system for spin at each point in phase space. Spin vector motion within the IFF is then a simple precession around the ISF. Here, in constrast, we come from the opposite direction by noting that by definition spin vector motion w.r.t. an element of $T \in \mathcal{TF}_{SO(2)}(j, A)$ as obtained by a transformation of the kind in (4.13) (say), is a simple precession around the third axis. We then discover that the third axis must be an ISF. In this way we prepare to state and prove two general theorems in Chapter 8. In fact the Normal Form Theorem, Theorem 8.1 in Section 8.2.4, will generalize Theorem 5.4b. Most importantly Theorem 5.4b connects the concepts of normal form and invariant field and the Normal Form Theorem will generalize this connection from $SO(2)$ to an arbitrary subgroup

H of $SO(3)$ using a generalization of the notion of invariant polarization field. In fact it will turn out that $SO(2)$ is an isotropy group closely related to ISF's and IFF's. The second general theorem to be mentioned is the Cross Section Theorem, Theorem 8.19 in Section 8.7.2 which will show that the IFF is a rather deep concept. See Remark 30 of Chapter 8 too.

Remarks:

(4) A question closely related to Theorem 5.4b is: if $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ with $|f| = 1$, is there a $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ such that f is the third column of T? In fact it is well-known, as pointed out in Section 8.7.2, that in general such a T does not exist. The above question will also be generalized in Chapter 8 - see Section 8.2.4.

(5) One can show, e.g., as in Appendix C in [He2], that if $A \in \mathcal{C}(\mathbb{T}^d, SO(3))$ is $SO(2)$-valued then a constant $N \in \mathbb{Z}^d$ and an $a \in \mathcal{C}(\mathbb{T}^d, \mathbb{R})$ exist such that

$$A(z) = \exp(\mathcal{J}[N \cdot \phi + 2\pi a(z)]) \; , \tag{5.7}$$

where $\phi \in z$. The fact that a is continuous is the only nontrivial detail of (5.7).

(6) If (j, A) belongs to $\mathcal{CB}_{SO(2)}(d, j)$ then $(j, A) \sim (j, A')$ where A' is $SO(2)$-valued whence, by Remark 5, a constant $N' \in \mathbb{Z}^d$ and an $a' \in \mathcal{C}(\mathbb{T}^d, \mathbb{R})$ exist such that

$$A'(z) = \exp(\mathcal{J}[N' \cdot \phi + 2\pi a'(z)]) \; , \tag{5.8}$$

where $\phi \in z$. It is noteworthy that the constant N' in (5.8) carries interesting information about A'. For example, as shown in Section 7.2 of [He2] by using simple arguments from Homotopy Theory, for $(\mathcal{P}_\omega, A')$ to belong to $\mathcal{SOS}_{CT}(d, \omega)$ it is necessary that all d components of N' are even integers.

If $(z(\cdot), S'(\cdot))$ is a particle-spin-vector trajectory of (j, A') then, by (2.31) and (5.8), S' evolves simply as:

$$S'(n+1) = \exp\Big(\mathcal{J}[N' \cdot \phi_1 + 2\pi a'(L[j](n; z(0)))]\Big) S'(n) \; , \tag{5.9}$$

where $\phi_1 \in L[j](n; z(0))$. Note that the spin vector motion in (5.9) is planar, i.e., the points $S'(n)$ lie in a plane parallel to the 1-2-plane.

If $T \in \mathcal{TF}(A, A'; d, j)$ and if $(z(\cdot), S(\cdot))$ is a particle-spin-vector trajectory of (j, A) then, by the transformation rule (4.13), $(z(\cdot), S(\cdot))$ transforms into the particle-spin-vector trajectory $(z(\cdot), S'(\cdot))$ of (j, A') where $S'(n) := T^t(z(n))S(n)$. Thus $S'(\cdot)$ obeys (5.9). □

6 $\mathcal{ACB}(d, j)$ and the notions of spin tune and spin-orbit resonance

As mentioned at the beginning of Chapter 5, it is natural to ask which (j, A) in $\mathcal{SOS}(d, j)$ are equivalent and it is fruitful to find "simple" elements (j, A') in an equivalence class $\overline{(j, A)}$

and to compare different equivalence classes in terms of their "simple" elements. In this chapter we apply this philosophy by focusing on those "simple" elements (j, A') in $\overline{(j, A)}$ for which A' is constant, i.e., for which $A'(z)$ is independent of z. This leads us in Section 6.1 to the definition and main basic properties of the subset $\mathcal{ACB}(d, j)$ of $\mathcal{SOS}(d, j)$.

In Section 6.2 this approach will enable us to associate tunes in addition to ω, namely spin tunes, with our spin-orbit systems. As in other dynamical systems, tunes can lead to the recognition of resonances and consequent instabilities. Here, spin tunes will lead to recognition of spin-orbit resonances. In the case of real spin vector motion, where spins are subject to the electric and magnetic fields on synchro-betatron trajectories, the definition of spin-orbit resonance allows us to predict at which orbital tunes spin vector motion might be particularly unstable. The definition of spin tune is also closely related with the concept of H-normal form as Theorem 6.2 will reveal.

6.1 The subset $\mathcal{ACB}(d, j)$ of $\mathcal{SOS}(d, j)$

We first define:

Definition 6.1 *($\mathcal{ACB}(d, j)$)*
We denote by $\mathcal{ACB}(d, j)$ the set of those $(j, A) \in \mathcal{SOS}(d, j)$ for which $\overline{(j, A)}$ contains a (j, A') such that A' is constant, i.e., such that $A'(z)$ is independent of z. □

The set $\mathcal{ACB}(d, j)$ contains the most important spin-orbit systems in $\mathcal{SOS}(d, j)$ when it comes to applications. See the remarks after Definition 6.3 too. However, as explained in Section 7.6 in [He2], it is easy to artificially construct $(j, A) \in \mathcal{SOS}(d, j)$ which are not in $\mathcal{ACB}(d, j)$ (for an example, see Section 8.5.2). The problem of deciding whether a given spin-orbit system is in $\mathcal{ACB}(d, j)$, is fruitful both theoretically and practically.

Remarks:

(1) Definition 6.1 gives us another property shared by equivalent spin-orbit systems since it implies that if (j, A) belongs to $\mathcal{ACB}(d, j)$ then every spin-orbit system in $\overline{(j, A)}$ belongs to $\mathcal{ACB}(d, j)$.

(2) Let $(j, A) \in \mathcal{SOS}(d, j)$ such that A is constant. Then, by (2.36),

$$\Psi[j, A](n; z) = A^n \, , \tag{6.1}$$

whence every n-turn spin transfer matrix function of (j, A) is a constant function so that, by Definition 6.1, a spin-orbit system belongs to some $\mathcal{ACB}(d, j)$ iff it is equivalent to a spin-orbit system for which every n-turn spin transfer matrix function is a constant function. This motivates our acronym $\mathcal{ACB}$ in Definition 6.1 since the spin transfer matrix functions of the spin-orbit systems in any $\mathcal{ACB}(d, j)$ are so-called "almost coboundaries" (see, e.g., [KR]).

(3) If $(\mathcal{P}_\omega, A) \in \mathcal{SOS}(d, \mathcal{P}_\omega)$ such that A is constant, then $(\mathcal{P}_\omega, A) \in \mathcal{SOS}_{CT}(d, \omega)$ since one easily shows that a function $\mathcal{A} : \mathbb{R}^{d+1} \to \mathbb{R}^{3\times 3}$ exists which is constant and whose

constant value is a skew-symmetric matrix and such that $A = \exp(2\pi\mathcal{A})$. Then we see, by (2.18), that

$$A_{CT}[\omega, \mathcal{A}] = A \ , \tag{6.2}$$

whence, by (2.20), indeed $(\mathcal{P}_\omega, A) \in \mathcal{SOS}_{CT}(d, \omega)$. □

The following theorem gives us insights into $\mathcal{ACB}(d, j)$ and for that purpose we need some notation. We begin by defining, for every $\nu \in [0, 1)$ and every positive integer d, the constant function $A_{d,\nu} \in \mathcal{C}(\mathbb{T}^d, SO(3))$ as

$$A_{d,\nu}(z) := \exp(2\pi\nu\mathcal{J}) = \begin{pmatrix} \cos(2\pi\nu) & -\sin(2\pi\nu) & 0 \\ \sin(2\pi\nu) & \cos(2\pi\nu) & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{6.3}$$

Clearly, for every $j \in \mathrm{Homeo}(\mathbb{T}^d)$, the spin-orbit system $(j, A_{d,\nu})$ belongs to $\mathcal{ACB}(d, j)$ since $A_{d,\nu}$ is a constant function. Secondly, for every $\nu \in [0, 1)$ we define

$$G_\nu := \{\exp(2\pi n\nu\mathcal{J}) : n \in \mathbb{Z}\} = \{\exp(2\pi(n\nu + m)\mathcal{J}) : m, n \in \mathbb{Z}\} \ , \tag{6.4}$$

where the, trivial, second equality highlights the fact that G_ν consists of matrices $\exp(2\pi\mu\mathcal{J})$ where $\mu \in [0, 1)$. It is clear by (5.5) that G_ν is a subgroup of $SO(2)$. Due to (6.3) every $A_{d,\nu}$ is G_ν-valued and $G_0 = \{I_{3\times3}\}$ is the trivial subgroup of $SO(3)$. Finally, for every $(j, A) \in \mathcal{SOS}(d, j)$ we define the set

$$\Xi(j, A) := \{\nu \in [0, 1) : (j, A_{d,\nu}) \in \overline{(j, A)}\} \ . \tag{6.5}$$

The following theorem shows some relationships between these concepts. Note that the main purpose of part c) is to help prove part d).

Theorem 6.2 *a) Let $(j, A) \in \mathcal{SOS}(d, j)$. Then $(j, A) \in \mathcal{ACB}(d, j)$ iff a $\nu \in [0, 1)$ exists such that $(j, A_{d,\nu})$ belongs to $\overline{(j, A)}$.*

b) Let $(j, A) \in \mathcal{SOS}(d, j)$. Then $(j, A) \in \mathcal{ACB}(d, j)$ iff $\Xi(j, A)$ is nonempty.

c) Let $\nu \in [0, 1)$ and $A \in \mathcal{C}(\mathbb{T}^d, SO(3))$ be G_ν-valued. Then A is a constant function.

d) Let $j \in \mathrm{Homeo}(\mathbb{T}^d)$. Then

$$\mathcal{ACB}(d, j) = \bigcup_{\nu\in[0,1)} \mathcal{CB}_{G_\nu}(d, j) \ . \tag{6.6}$$

e) Let $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ and let $(j, A') \in \mathcal{SOS}(d, j)$ be the transform of $(j, A) \in \mathcal{SOS}(d, j)$ under T, i.e., $T \in \mathcal{TF}(A, A'; d, j)$. Then T belongs to $\bigcup_{\nu\in[0,1)} \mathcal{TF}_{G_\nu}(j, A)$ iff T is an IFF of (j, A) and A' is a constant function.

f) Let $(\mathcal{P}_\omega, A) \in \mathcal{SOS}(d, \mathcal{P}_\omega)$. If $\nu \in \Xi(\mathcal{P}_\omega, A)$ then

$$\Xi(\mathcal{P}_\omega, A) = [0, 1) \cap \left\{\varepsilon\nu + m \cdot \omega + n : \varepsilon \in \{1, -1\}, m \in \mathbb{Z}^d, n \in \mathbb{Z}\right\} . \tag{6.7}$$

Proof of Theorem 6.2a: If $\nu \in [0,1)$ exists such that $(j, A_{d,\nu})$ belongs to $\overline{(j,A)}$ then, by Definition 6.1, $(j,A) \in \mathcal{ACB}(d,j)$ since $A_{d,\nu}$ is constant. To prove the converse, let $(j,A) \in \mathcal{ACB}(d,j)$. Then, by Definition 6.1, $\overline{(j,A)}$ contains a (j,A') such that A' is constant with constant value, say r. By some simple Linear Algebra, a $\nu \in [0,1)$ and a $W \in SO(3)$ can be found such that

$$r = W \exp(2\pi\nu\mathcal{J})W^t \ . \tag{6.8}$$

See, e.g., Lemma 2.1 of [BEH]. Thus, defining the constant function $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ by $T(z) := W$ we observe by (6.8) and Definition 4.1 that $T \in \mathcal{TF}(A', A_{d,\nu}, d, j)$ whence $(j,A') \sim (j, A_{d,\nu})$ so that $(j,A) \sim (j, A_{d,\nu})$ which implies that $(j, A_{d,\nu})$ belongs to $\overline{(j,A)}$. □
Proof of Theorem 6.2b: The claim is a simple consequence of (6.5) and Theorem 6.2a. □
Proof of Theorem 6.2c: Since A is G_ν-valued it follows from (6.4) that a function $\tilde{n} : \mathbb{T}^d \to \mathbb{Z}$ exists such that $A(z) = \exp(\mathcal{J}2\pi\nu\tilde{n}(z))$ whence

$$A(z) = \exp(\mathcal{J}2\pi\nu\tilde{n}(z)) \ . \tag{6.9}$$

Clearly A is $SO(2)$-valued whence, by Remark 5 in Chapter 5, a constant $N \in \mathbb{Z}^d$ and an $a \in \mathcal{C}(\mathbb{T}^d, \mathbb{R})$ exist such that

$$\exp(\mathcal{J}2\pi\nu\tilde{n}(z)) = A(z) = \exp(\mathcal{J}[N \cdot \phi + 2\pi a(z)]) \ , \tag{6.10}$$

where $\phi \in z$ and where in the first equality we used (6.9). It follows from (5.6) and (6.10) that a function $k : \mathbb{T}^d \to \mathbb{Z}$ exists such that

$$2\pi\nu\tilde{n}(\phi + \tilde{\mathbb{Z}}^d) + 2\pi k(\phi + \tilde{\mathbb{Z}}^d) = N \cdot \phi + 2\pi a(\phi + \tilde{\mathbb{Z}}^d) \ . \tag{6.11}$$

Since $\tilde{n}(\phi + \tilde{\mathbb{Z}}^d)$, $k(\phi + \tilde{\mathbb{Z}}^d)$ and $a(\phi + \tilde{\mathbb{Z}}^d)$ are 2π-periodic in the components of ϕ it follows from (6.11) that $N \cdot \phi$ is 2π-periodic in the components of ϕ whence $N = 0$ so that, by (6.11), for all $z \in \mathbb{T}^d$

$$\nu\tilde{n}(z) + k(z) = a(z) \ . \tag{6.12}$$

Since $\tilde{n}$ and k are $\mathbb{Z}$-valued, the function $\nu\tilde{n} + k$ can take at most countably many values whence, by (6.12), the function a can take at most countably many values. On the other hand since a is continuous and since its domain, $\mathbb{T}^d$, is a path-connected topological space, the range of a is a path-connected subset of $\mathbb{R}$, i.e., an interval, say I (for the notion of range see also Appendix A.1). However since a takes at most countably many values, I contains at most only countably many points whence, being an interval, I contains just one point which implies that a is constant. Since a is constant and $N = 0$ it follows from (6.10) that A is constant. □
Proof of Theorem 6.2d: "$\subset$": Let $\underline{(j,A)} \in \mathcal{ACB}(d,j)$. Then, by Theorem 6.2a, a $\nu \in [0,1)$ exists such that $(j, A_{d,\nu})$ belongs to $\overline{(j,A)}$. By a remark after (6.4), $A_{d,\nu}$ is G_ν-valued whence, by Definition 5.1, $(j,A) \in \mathcal{CB}_{G_\nu}(d,j)$.
"$\supset$": Let $\nu \in [0,1)$ and $(j,A) \in \mathcal{CB}_{G_\nu}(d,j)$ whence, by Definition 5.1, $\mathcal{TF}_{G_\nu}(j,A)$ is nonempty. So pick a $T \in \mathcal{TF}_{G_\nu}(j,A)$. Then, by Definitions 4.1 and 5.1, $T \in \mathcal{TF}(A, A'; d, j)$

where A' is G_ν-valued. Since A' is G_ν-valued it follows from Theorem 6.2c that A' is constant which implies, by Definition 6.1, that $(j, A) \in \mathcal{ACB}(d, j)$. □

Proof of Theorem 6.2e: "⇒": Let $T \in \mathcal{TF}_{G_\nu}(j, A)$. Since G_ν is a subgroup of $SO(2)$ we conclude from Remark 2 in Chapter 5 that $T \in \mathcal{TF}_{SO(2)}(j, A)$ so that T is an IFF of (j, A). Also, A' is G_ν-valued whence, by Theorem 6.2c, A' is constant.

"⇐": Let T be an IFF of (j, A) and let A' be constant. Clearly, by Definition 5.3, A' is $SO(2)$-valued whence $\nu \in [0, 1)$ exists such that $A' = A_{d,\nu}$ which implies that A' is G_ν-valued so that $T \in \mathcal{TF}_{G_\nu}(j, A)$. □

Proof of Theorem 6.2f: The claim is proved in Chapter 8 of [He2] by using the tool of quasiperiodic functions [He2]. □

Theorem 6.2d provides the insight that every $\mathcal{ACB}(d, j)$ can be understood in terms of H-normal forms, a fact that is not obvious by Definition 6.1. The purpose of Theorem 6.2e is to lead us to the definition of the uniform IFF in Remark 5 below. As shown in Remark 9 below, Theorem 6.2f gives an insight into how the $\mathcal{SOS}(d, \mathcal{P}_\omega)$ are partitioned w.r.t. to the equivalence relation $\sim$ and it will also play a role for the notion of spin-orbit resonance in Section 6.2. Theorem 6.2a will be used in Remark 5 and was used for proving Theorems 6.2b and 6.2d while Theorem 6.2b will be used in Remark 5 and in Section 6.2.

Remarks:

(4) Let $j \in \mathrm{Homeo}(\mathbb{T}^d)$ and $\nu \in [0, 1)$. By a remark after (6.4) we have $G_\nu \subset SO(2)$ whence, by (5.3),

$$\mathcal{CB}_{G_\nu}(d, j) \subset \mathcal{CB}_{SO(2)}(d, j) \ . \tag{6.13}$$

It follows from (6.13) and Theorem 6.2d that

$$\mathcal{ACB}(d, j) \subset \mathcal{CB}_{SO(2)}(d, j) \ . \tag{6.14}$$

Then, by Definition 5.3, $(j, A) \in \mathcal{ACB}(d, j)$ has an IFF whence, by Theorem 5.4b, (j, A) has an ISF, say f. In fact, recalling Section 3.2, (j, A) has at least the two ISF's f and $-f$. For example, the constant functions on $\mathbb{T}^d$ with values $(0, 0, 1)^t$ and $(0, 0, -1)^t$ are ISF's of every spin-orbit system of the form $(j, A_{d,\nu})$.

(5) Let $(j, A) \in \mathcal{SOS}(d, j)$. A $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ is called a "uniform IFF of (j, A)" iff it is an IFF of (j, A) and A', defined by (4.1), is a constant function. It is clear that the uniform IFF's are the discrete-time analogues of the so-called uniform invariant frame fields introduced in the continuous-time formalism of [BEH]. Also, by Theorem 6.2e, the elements of $\bigcup_{\nu\in[0,1)} \mathcal{TF}_{G_\nu}(j, A)$ are the uniform IFF's of (j, A). This implies, by Theorem 6.2c, that A', defined by (4.1), is of the form $A_{d,\nu}$. It thus follows that the set of uniform IFF's of (j, A) reads as

$$\bigcup_{\nu\in[0,1)} \mathcal{TF}_{G_\nu}(j, A) = \bigcup_{\nu\in[0,1)} \mathcal{TF}(A, A_{d,\nu}; d, j) = \bigcup_{\nu\in\Xi(j,A)} \mathcal{TF}(A, A_{d,\nu}; d, j) \ , \tag{6.15}$$

where in the second equality we used (6.5). Using (6.15) and Theorem 6.2b we also observe that a (j, A) has a uniform IFF iff $(j, A) \in \mathcal{ACB}(d, j)$.

Note that if (j, A) has a uniform IFF, say T, then $TA_{d,\mu}$ is also a uniform IFF of (j, A) where $\mu \in [0, 1)$ whence every $(j, A) \in \mathcal{ACB}(d, j)$ has uncountably many uniform IFF's. Note also, by (6.15), that (j, A) has at least as many uniform IFF's as there are elements in $\Xi(j, A)$. This is especially evident in the case when $j = \mathcal{P}_\omega$ and $(j, A) \in \mathcal{ACB}(d, j)$ since then, by (6.7), $\Xi(j, A)$ has at most countably many elements so here the uniform IFF's considerably outnumber the elements of $\Xi(j, A)$. On the other hand, if $(\mathcal{P}_\omega, A) \in \mathcal{ACB}(d, j)$ it rarely happens that $\Xi(\mathcal{P}_\omega, A)$ has finitely many elements since one can show [He1] that this only happens iff all d components of ω are rational numbers.

(6) Let $(j, A) \in \mathcal{CB}_{G_0}(d, j)$. Thus, by Definition 5.1, $\mathcal{TF}_{G_0}(j, A)$ is nonempty and every T in $\mathcal{TF}_{G_0}(j, A)$ satisfies $(T^t \circ j)AT = A_{d,0}$ so that, by (4.4),

$$\Psi[j, A](n; z) = T(L[j](n; z))T^t(z) \ . \tag{6.16}$$

Eq. (6.16) motivates the acronym $\mathcal{CB}$ in Definition 5.1 since the spin transfer matrix function $\Psi[j, A]$ in (6.16) belongs to that class of cocycles which are called “coboundaries” (see [HK2] and Chapter 1 in [HK1]).

(7) Let $(\mathcal{P}_\omega, A) \in \mathcal{SOS}(d, \mathcal{P}_\omega)$. It can be easily shown, by using (6.5), that $(\mathcal{P}_\omega, A) \in \mathcal{CB}_{G_0}(d, \mathcal{P}_\omega)$ iff $0 \in \Xi(\mathcal{P}_\omega, A)$.

(8) Let $(j, A) \in \mathcal{SOS}(d, j)$. It is easy to show, by (6.3), (6.4) and (6.5) and for every $\nu \in [0, 1)$, that either $\mathcal{TF}_{G_\nu}(j, A)$ is empty or there exist integers m, n such that $(m\nu + n) \in \Xi(j, A)$ (note that n ensures that $(m\nu + n) \in [0, 1)$). This implies, by Theorem 6.2f, that if j is of the form $\mathcal{P}_\omega$ then a subset B of $[0, 1)$ exists which has at most countably many elements and such that $\bigcup_{\nu\in[0,1)} \mathcal{TF}_{G_\nu}(\mathcal{P}_\omega, A) = \bigcup_{\nu\in B} \mathcal{TF}_{G_\nu}(\mathcal{P}_\omega, A)$.

(9) As mentioned in Section 4.3, in this work we do not fully address how the $\mathcal{SOS}(d, j)$ are partitioned w.r.t. to the equivalence relation $\sim$. Thus it may come as a surprise that Theorem 6.2f sheds light on this issue. In fact if j is of the form $\mathcal{P}_\omega$ then $\mathcal{SOS}(d, j)$ contains uncountably many equivalence classes as follows.

To prove this claim we first of all note that $\mathcal{SOS}(d, \mathcal{P}_\omega)$ has uncountably many elements since ν is a continuous parameter whence there are uncountably many $A_{d,\nu}$, i.e., the spin-orbit systems $(\mathcal{P}_\omega, A_{d,\nu})$ form an uncountable subset, say B, of $\mathcal{SOS}(d, \mathcal{P}_\omega)$ (note that ω is fixed but ν varies over $[0, 1)$). Note also that both B and $\overline{(\mathcal{P}_\omega, A_{d,\nu})}$ have uncountably many elements but, as will be shown below, $B \cap \overline{(\mathcal{P}_\omega, A_{d,\nu})}$ has at most countably many elements. In fact in our proof the sets $B \cap \overline{(\mathcal{P}_\omega, A_{d,\nu})}$ for each ν will play a key role and we already note here that they form a partition of B since the $\overline{(\mathcal{P}_\omega, A_{d,\nu})}$, being equivalence classes, are mutually disjoint. In particular, if $\overline{(\mathcal{P}_\omega, A_{d,\nu})}$ and $\overline{(\mathcal{P}_\omega, A_{d,\mu})}$ are different then they are disjoint and belong to different equivalence classes of the equivalence relation $\sim$. The crucial question now is: how many of the sets $B \cap \overline{(\mathcal{P}_\omega, A_{d,\nu})}$ are different? In other words how common is it that two spin-orbit systems in B are equivalent? This is where Theorem 6.2 engages. In fact, by (6.7), each set $\Xi(\mathcal{P}_\omega, A_{d,\nu})$ contains at most countably many elements. On the other hand if $\nu, \mu \in [0, 1)$ then, by (6.5), $(\mathcal{P}_\omega, A_{d,\mu}) \in \overline{(\mathcal{P}_\omega, A_{d,\nu})}$ iff $\mu \in \Xi(\mathcal{P}_\omega, A_{d,\nu})$. Thus every set of the form $B \cap \overline{(\mathcal{P}_\omega, A_{d,\nu})}$ contains at most countably many elements of B. Thus we need uncountably many of the sets $B \cap \overline{(\mathcal{P}_\omega, A_{d,\nu})}$ to overlap B whence the $B \cap \overline{(\mathcal{P}_\omega, A_{d,\nu})}$ form an uncountable partition of B. Since different $B \cap \overline{(\mathcal{P}_\omega, A_{d,\nu})}$ are contained in different equivalence classes we thus have shown that there are uncountably many equivalence classes of the form $\overline{(\mathcal{P}_\omega, A_{d,\nu})}$. Thus, as was to be shown, $\mathcal{SOS}(d, \mathcal{P}_\omega)$ is partitioned into uncountably many equivalence classes w.r.t. to the equivalence relation $\sim$. □

6.2 Spin tunes and spin-orbit resonances

Definition 6.1 and Theorem 6.2 lead us naturally to the notions of spin tune and spin-orbit resonance. A $\nu \in [0, 1)$ is said to be a spin tune for $(j, A) \in \mathcal{SOS}(d, j)$ if (j, A) is equivalent to (j, A') with $A'(z) = \exp(2\pi\nu\mathcal{J})$, i.e., if $(j, A_{d,\nu})$ belongs to $\overline{(j, A)}$. We thus arrive at the following definition:

Definition 6.3 *(Spin tune, spin-orbit resonance)*
We call the elements of $\Xi(j, A)$ the spin tunes of (j, A). We say that $(\mathcal{P}_\omega, A)$ is "on spin-orbit resonance (SOR)" if $(\mathcal{P}_\omega, A) \in \mathcal{ACB}(d, \mathcal{P}_\omega)$ and if for every $\nu \in \Xi(\mathcal{P}_\omega, A)$ we can find $m \in \mathbb{Z}^d, n \in \mathbb{Z}$ such that

$$\nu = m \cdot \omega + n \; . \tag{6.17}$$

We say that $(\mathcal{P}_\omega, A)$ is "off spin-orbit resonance" iff $(\mathcal{P}_\omega, A) \in \mathcal{ACB}(d, \mathcal{P}_\omega)$ and if $(\mathcal{P}_\omega, A)$ is not on spin-orbit resonance. Note that a spin-orbit system which has no spin tunes is neither on nor off spin-orbit resonance. This happens in particular when j is not a torus translation. □

It follows from Definition 6.3 and Theorem 6.2b that a $(j, A) \in \mathcal{SOS}(d, j)$ has spin tunes iff $(j, A) \in \mathcal{ACB}(d, j)$. This has the implication that, by Theorem 6.2d, spin tunes can be understood in terms of normal forms. Furthermore it has the implication, by Remark 5, that (j, A) has a spin tune iff it has a uniform IFF.

In [BEH] spin-orbit systems with spin tunes belong to the class of "well tuned" systems and most of the systems with no spin tunes are said to be "ill-tuned".

In [He2] the spin tune and spin-orbit resonances defined here are called spin tune of the first kind and spin-orbit resonances of the first kind respectively since [He2] finds it convenient to distinguish between two kind of spin tune. That distinction is not needed here.

If one considers a family $(j_J, A_J)_{J \in \Lambda}$ of spin-orbit systems (see the Introduction and Chapter 7) and if every (j_J, A_J) has a spin tune, say ν_J, then ν_J is called an amplitude dependent spin tune (ADST). Recall from Remark 5 that if T_J is a uniform IFF of (j_J, A_J) then $T_J^t(j(z))A_J(z)T_J(z) = A_{d,\nu_J}(z) = \exp(2\pi\nu_J\mathcal{J})$.

As stated at the beginning of this chapter spin-orbit resonance can lead to a large angular spread of the ISF and that can lead to unacceptably low equilibrium polarization as explained in Chapter 7. The large angular spread also means that if a particle beam occupies a large volume of phase space at injection while the spins all point in roughly the same direction, the polarization of the beam can be very unstable while the spin precess around their individual ISF's. See [Ho] for an example of this. See [Ho],[Ma],[Vo],[Yo1] for formalisms and calculations which have demonstrated the potential for a large spread of the ISF near spin-orbit resonances. For detailed further comments see Section X in [BEH].

Moreover, since the ADST can vary with orbital amplitude J, particles at one amplitude can be close to spin-orbit resonance while particles as nearby amplitudes need not be. Manifestations of this are beautifully demonstrated in [Ho, Vo, BHV00, HV] where the value of a rigorous definition of spin tune is made crystal clear. Note that as shown in those works, spin-orbit resonances tend to be rather repelling than attractive. The rigorous definition of spin tune and of spin-orbit resonance also will lead us in Chapter 7 to the Uniqueness Theorem for the ISF [Yo1, DK73]. In summary, a rigorous definition, as in Definition 6.3, is very important for a detailed understanding of real spin vector motion.

As explained in Section X of [BEH] and in [BV1], as well as in other literature, a real spin-orbit system $(\mathcal{P}_\omega, A)$ on orbital resonance normally has no spin tune. One exception is the so-called single resonance model underlying the model with two Siberian snakes in Section 8.5.2. Nevertheless, such a system can, but need not, have an ISF of the continuous kind defined here. An example of a spin-orbit system on orbital resonance which has no ISF, and thus no spin tune, is studied in Section 8.5.2 . If the d components of ω are rational numbers then it is easy to calculate an ISF f by finding the real eigenvector $f(z)$ of the matrix $\Psi[\mathcal{P}_\omega, A](n; z)$ for the number of turns n for which the particle returns to its starting position z. The discontinuous "ISF" of [BV1] can also be calculated in this way (and this is also done in our example in Section 8.5.2). Recall also from the ISF conjecture in Chapter 3 that we expect an ISF to exist off orbital resonance.

The ISF and the ADST for real spin vector motion off orbital resonance in storage rings can be computed in a number of ways [Be],[Fo],[HH],[Ho],[Ma],[Vo],[Yo3]. Here we describe two of them and we start with a method of computing the ADST, implemented in the computer code SPRINT [He2, Ho, Vo] (as an alternative method, SPRINT offers an implementation of the SODOM-2 algorithm). The calculations proceed in two steps [BEH00, BHV98, Ho, Vo]. For simplicity we consider a fixed but arbitrary action value J and assume that the spin-orbit system belongs to $\mathcal{ACB}(d, \mathcal{P}_\omega)$ and is off orbital resonance and off spin-orbit resonance. As we will see in Chapter 7, by the Uniqueness Theorem, Theorem 7.1b, the given spin-orbit system $(\mathcal{P}_\omega, A)$ has only two ISF's, say f and $-f$. Of course f and $-f$ in general are unknown and in fact one only attempts to compute a discretization of them. In the first step, f is computed at some point z on the torus at some point θ on a ring using stroboscopic averaging [EH, HH] giving us $f(z)$. By Remark 4 an IFF, say T, exists and, due to Theorem 5.4b, the third column of T is either f or $-f$ and here $T(z)$ is constructed by a simple orthonormalization procedure in which $f(z)$ is the third column is $T(z)$. The axis represented by the second column of $T(z)$ could, for example, be chosen so as to have no component along the direction of the beam. In the next step the spin value $f(z)$ is tracked forwards turn by turn, according to (3.2), resulting in the discretization $f(z), f(\mathcal{P}_\omega(z)), f(\mathcal{P}_\omega^2(z)), ..., f(\mathcal{P}_\omega^N(z))$ of f for some large integer N. Accordingly $T(z), T(\mathcal{P}_\omega(z)), T(\mathcal{P}_\omega^2(z)), ..., T(\mathcal{P}_\omega^N(z))$ are constructed at the end of each turn according to the chosen prescription. Then, the average spin precession angle around the ISF w.r.t. this IFF is computed for a very large number of turns N. In fact since T is an

IFF, by Remark 6 in Chapter 5, an $N \in \mathbb{Z}^d$ and an $a \in \mathcal{C}(\mathbb{T}^d, \mathbb{R})$ exist such that

$$T^t\left(\left(\phi + 2\pi(n+1)\omega\right) + \tilde{\mathbb{Z}}^d\right) A\left(\left(\phi + 2\pi n\omega\right) + \tilde{\mathbb{Z}}^d\right) T\left(\left(\phi + 2\pi n\omega\right) + \tilde{\mathbb{Z}}^d\right)$$
$$= \exp\left(\mathcal{J}[N \cdot (\phi + 2\pi n\omega) + 2\pi a\left((\phi + 2\pi n\omega) + \tilde{\mathbb{Z}}^d\right)]\right) , \tag{6.18}$$

where $\phi \in z$ and $n = 0, ..., N$. One can show [He2, Vo] that the average $< a >$ of a, given by

$$< a > := \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} a(\pi_d(\phi)) d\phi , \tag{6.19}$$

is a spin tune of $(\mathcal{P}_\omega, A)$. On the other hand, (6.18), gives us $a(z), a(\mathcal{P}_\omega(z)), a(\mathcal{P}_\omega^2(z)), ..., a(\mathcal{P}_\omega^N(z))$ which allows one to approximate the average of a. This delivers an ADST for the given J but the member of the set $\Xi(\mathcal{P}_\omega, A)$ that emerges will depend on the convention used to choose the first and second axes of T.

Another practical way to compute spin tunes is by using the spectrum of the spin vector motion as follows. For simplicity we consider a fixed but arbitrary action value J and assume that the spin-orbit system belongs to $\mathcal{ACB}(d, \mathcal{P}_\omega)$. Then let $(\mathcal{P}_\omega, A)$ have a spin-vector trajectory $S(\cdot)$. The discrete Fourier transform (DFT) of $S(0), ..., S(N)$ is defined by $\hat{S}$ where

$$\hat{S}(k) := \frac{1}{N+1} \sum_{n=0}^{N} S(n) \exp(-2\pi i n k/(N+1)) , \tag{6.20}$$

and where $k = 0, ..., N$. We define, for $\lambda \in [0, 1)$ and nonnegative integer N,

$$a_N(S, \lambda) := (N+1)^{-1} \sum_{n=0}^{N} S(n) \exp(-2\pi i n \lambda) . \tag{6.21}$$

It can be easily shown [He2] that $a_N(S, \lambda)$ converges as $N \to \infty$ and we denote the limit of $a_N(S, \lambda)$ by $a(S, \lambda)$ and we define the "spectrum $\Lambda(S)$ of S" by $\Lambda(S) := \{\lambda \in [0, 1) : a(S, \lambda) \neq 0\}$. From (6.20) and (6.21) it is clear that $a(S, \lambda)$ can be approximated by using standard DFT software. Then, as can be easily shown [He2], spin tunes are contained in the spectrum since

$$\Lambda(S) \subset \Xi(\mathcal{P}_\omega, A) \cup \{l \cdot \omega + n : l \in \mathbb{Z}^d, n \in \mathbb{Z}\} . \tag{6.22}$$

Moreover, the spectrum can contain many of the spin tunes in $\Xi(\mathcal{P}_\omega, A)$. Theorem 9.1c in the continuous-time formalism of [BEH] reaches the same conclusions. With this we have a direct relationship between the set $\Xi(\mathcal{P}_\omega, A)$ appearing in Theorem 6.2 and a "measureable" quantity, namely the spectrum. This way of getting ADST's has been essential for interpreting spin vector motion near to resonance with oscillating external magnetic fields [Ba].

Remark:

(10) By (6.7) and Definition 6.3 an $(\mathcal{P}_\omega, A) \in \mathcal{ACB}(d, \mathcal{P}_\omega)$ is on spin-orbit resonance iff (6.17) holds for just one choice of $m \in \mathbb{Z}^d, n \in \mathbb{Z}, \nu \in \Xi(\mathcal{P}_\omega, A)$. Thus a single spin tune ν of $(\mathcal{P}_\omega, A)$ is sufficient to determine if $(\mathcal{P}_\omega, A)$ is on spin-orbit resonance. Note also, by (6.7) and Definition 6.3, that a spin-orbit system $(\mathcal{P}_\omega, A) \in \mathcal{SOS}(d, \mathcal{P}_\omega)$ is on spin-orbit resonance iff $0 \in \Xi(\mathcal{P}_\omega, A)$. Thus, by Remark 7, $(\mathcal{P}_\omega, A)$ is on SOR iff $(\mathcal{P}_\omega, A) \in \mathcal{CB}_{G_0}(d, \mathcal{P}_\omega)$. □

In analogy with Theorem 5.4b we now state:

Theorem 6.4 *a) Let $(j, A) \in \mathcal{SOS}(d, j)$ and $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$. Then T satisfies*

$$T \circ j = AT \ , \tag{6.23}$$

iff it belongs to $\mathcal{TF}_{G_0}(j, A)$.

b) (SOR Theorem) Let $(\mathcal{P}_\omega, A) \in \mathcal{SOS}(d, \mathcal{P}_\omega)$. Then $(\mathcal{P}_\omega, A)$ is on SOR iff $\mathcal{TF}_{G_0}(\mathcal{P}_\omega, A)$ is nonempty, i.e., iff $(\mathcal{P}_\omega, A) \in \mathcal{CB}_{G_0}(d, \mathcal{P}_\omega)$.

Proof of Theorem 6.4a: By Definition 5.1, $T \in \mathcal{TF}_{G_0}(j, A)$ iff $T^t(j(z))A(z)T(z) \in G_0$ whence, by (6.4), $T \in \mathcal{TF}_{G_0}(j, A)$ iff $T^t(j(z))A(z)T(z) = I_{3\times 3}$ which proves the claim. □
Proof of Theorem 6.4b: By Remark 10, $(\mathcal{P}_\omega, A)$ is on SOR iff $0 \in \Xi(\mathcal{P}_\omega, A)$ iff $(\mathcal{P}_\omega, A) \in \mathcal{CB}_{G_0}(d, \mathcal{P}_\omega)$. The claim now follows from Definition 5.1. □

We will use Theorem 6.4 in the proof of the Uniqueness Theorem, Theorem 7.1b. Moreover the Normal Form Theorem, Theorem 8.1 in Section 8.2.4, will generalize Theorem 6.4a from G_0 to an arbitrary subgroup H of $SO(3)$. It will thereby turn out that (6.23) is an example of a so-called stationarity equation. The second general theorem to be mentioned is the Cross Section Theorem, Theorem 8.19 in Section 8.7.2 which will show that the SOR is a rather deep concept. Spin-orbit resonances will be further studied in the following chapter 7.

7 Polarization

In this chapter we tie together the concepts of polarization field and polarization.

7.1 Estimating the polarization

Consider a family $(j_J, A_J)_{J\in\Lambda}$ of spin-orbit systems where $(j_J, A_J) \in \mathcal{SOS}(d, j_J)$ and $\Lambda \subset \mathbb{R}^d$ is the set of action values.

We note (see also [BH, BV1]) that, for every $J \in \Lambda$, we have a so-called "local polarization", say $\mathcal{S}_{loc,J}$, which by definition is a polarization-field trajectory of (j_J, A_J) satisfying

$$|\mathcal{S}_{loc,J}| \leq 1 \ . \tag{7.1}$$

The associated polarization on the torus J at time n is then given by

$$P_J(n) := (\frac{1}{2\pi})^d \Big| \int_{[0,2\pi]^d} d\phi \mathcal{S}_{loc,J}(n, \pi_d(\phi)) \Big| \ . \tag{7.2}$$

We will see below how P_J can be estimated by (7.5) which makes P_J a convenient tool for analyzing the bunch polarization. In the so-called "spin equilibrium" the polarization-field trajectory $\mathcal{S}_{loc,J}$ is, by the definition of the spin equilibrium, time-independent for every J whence its initial value, $\mathcal{S}_{loc,J}(0,\cdot)$ is an invariant polarization field of (j_J, A_J). Thus for the spin equilibrium we get

$$P_J(n) = P_J(0) = (\frac{1}{2\pi})^d \Big| \int_{[0,2\pi]^d} d\phi \mathcal{S}_{loc,J}(0,\pi_d(\phi)) \Big| \,. \tag{7.3}$$

Let j_J be topologically transitive. Then, by Theorem 3.2, $|\mathcal{S}_{loc,J}(0,z)|$ is independent of z and, if $\mathcal{S}_{loc,J}(0,\cdot)$ is not the zero function, then $|\mathcal{S}_{loc,J}(0,z)| > 0$ and $\mathcal{S}_{loc,J}(0,\cdot)/|\mathcal{S}_{loc,J}(0,\cdot)|$ is an ISF of (j_J, A_J) whence, by (7.1),(7.3),

$$\begin{aligned} P_J(n) = P_J(0) &= (\frac{1}{2\pi})^d \Big| \int_{[0,2\pi]^d} d\phi |\mathcal{S}_{loc,J}(0,\pi_d(\phi))| \, \frac{\mathcal{S}_{loc,J}(0,\pi_d(\phi))}{|\mathcal{S}_{loc,J}(0,\pi_d(\phi))|} \Big| \\ &\leq (\frac{1}{2\pi})^d \Big| \int_{[0,2\pi]^d} d\phi \frac{\mathcal{S}_{loc,J}(0,\pi_d(\phi))}{|\mathcal{S}_{loc,J}(0,\pi_d(\phi))|} \Big| \,, \end{aligned} \tag{7.4}$$

so that

$$P_J(n) = P_J(0) \leq P_{J,max} \,, \tag{7.5}$$

where

$$P_{J,max} := (\frac{1}{2\pi})^d \sup \Big\{ \Big| \int_{[0,2\pi]^d} d\phi f(\pi_d(\phi)) \Big| : f \in \mathcal{ISF}(j_J, A_J) \Big\} \,. \tag{7.6}$$

Of course (7.5) also holds if $\mathcal{S}_{loc,J}(0,\cdot)$ is the zero function because in that case $P_J(n) = P_J(0) = 0$. Thus (7.5) holds for the spin equilibrium if j_J is topologically transitive and (j_J, A_J) has an ISF. We conclude from (7.5) that the ISF's provide an upper bound for P_J and this is one reason why they are so important in practice. One can simplify (7.6) in the important case where the spin-orbit system (j_J, A_J) in (7.6) has exactly two ISF's, say $f_J, -f_J$. Then (7.6) simplifies to

$$P_{J,max} = (\frac{1}{2\pi})^d \Big| \int_{[0,2\pi]^d} d\phi f_J(\pi_d(\phi)) \Big| \,. \tag{7.7}$$

Clearly $P_{J,max}$ is small if the range of f_J is spread out. In Section 7.2 we will see how the Uniqueness Theorem leads to the situation underlying (7.7).

Of course P_J can also be used for an estimation of the bunch polarization which is given by

$$P(n) = (\frac{1}{2\pi})^d \Big| \int_\Lambda dJ \rho_{eq}(J) \int_{[0,2\pi]^d} d\phi \mathcal{S}_{loc,J}(n,\pi_d(\phi)) \Big| \,, \tag{7.8}$$

where $(\frac{1}{2\pi})^d \rho_{eq}$ is the equilibrium particle phase-space density (for more details underlying (7.8) see Section 8.6.1). Thus the bunch polarization for the combined beam equilibrium and spin equilibrium reads as

$$P(n) = P(0) = (\frac{1}{2\pi})^d \Big| \int_\Lambda dJ \rho_{eq}(J) \int_{[0,2\pi]^d} d\phi \mathcal{S}_{loc,J}(0,\pi_d(\phi)) \Big| \,. \tag{7.9}$$

Let the conditions underlying (7.5) hold for almost all J, i.e., let a set $M \subset \Lambda$ exist which has Lebesgue measure zero and such that, for every $J \in (\Lambda \setminus M)$, the spin-orbit system (j_J, A_J) has an ISF and j_J is topologically transitive. Then, by (7.3),(7.5),(7.9), we have for the spin equilibrium

$$P(n) = P(0) \leq (\frac{1}{2\pi})^d \int_\Lambda dJ \rho_{eq}(J) \Big| \int_{[0,2\pi]^d} d\phi \mathcal{S}_{loc,J}(0, \pi_d(\phi)) \Big|$$
$$= \int_\Lambda dJ \rho_{eq}(J) P_J(0) \leq \int_\Lambda dJ \rho_{eq}(J) P_{J,max} \ . \tag{7.10}$$

Note that we assume that $\rho_{eq}(J)$ and $P_{J,max}$ depend on J regularly enough to ensure that the integrals in (7.8), (7.9) and (7.10) are meaningful. Using (7.7) one can simplify (7.10) in the case where, for every $J \in (\Lambda \setminus M)$, the spin-orbit system (j_J, A_J) has two ISF's $f_J, -f_J$ and no others. Then (7.10) simplifies, thanks to (7.7), to

$$P(n) = P(0) \leq (\frac{1}{2\pi})^d \int_\Lambda dJ \rho_{eq}(J) \Big| \int_{[0,2\pi]^d} d\phi f_J(\pi_d(\phi)) \Big| \ , \tag{7.11}$$

where we also assume that the functional dependences of $\rho_{eq}(J)$ and f_J on J are regular enough to ensure that the integrals in (7.11) are meaningful. For more details on estimating the bunch polarization, also for non-equilibrium spin fields, see [Ho, Vo].

7.2 The Uniqueness Theorem of invariant spin fields

We saw in (7.5) and (7.7), how in a situation where only two ISF's exist, the invariant spin fields govern the estimation of P_J. In this section we will see that this situation is very common off spin-orbit resonance.

Let $(j, A) \in \mathcal{ACB}(d, j)$. Then, by Remark 4 in Chapter 6, (j, A) has an ISF and so it natural to ask about the impact of the set $\Xi(j, A)$ on $\mathcal{ISF}(j, A)$. In fact, if $j = \mathcal{P}_\omega$ and $(\mathcal{P}_\omega, A)$ is off orbital resonance, this question is partially answered by part b) of the following theorem.

Theorem 7.1 *a) Let $(j, A) \in \mathcal{SOS}(d, j)$ and let f and g be invariant polarization fields of (j, A). Then $h \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$, defined by $h(z) := f(z) \times g(z)$, is an invariant polarization field of (j, A) where $\times$ denotes the vector product.*

b) (The Uniqueness Theorem) Let $(\mathcal{P}_\omega, A) \in \mathcal{ACB}(d, \mathcal{P}_\omega)$ be off orbital resonance, i.e., let $(1, \omega)$ be nonresonant. Also, let $(\mathcal{P}_\omega, A)$ be off spin-orbit resonance. Then $(\mathcal{P}_\omega, A)$ has an ISF, say F, and F and $-F$ are the only ISF's of $(\mathcal{P}_\omega, A)$.

Proof of Theorem 7.1a: Since f and g are invariant polarization fields of (j, A) it follows from Definition 3.1 that $f \circ j = Af$ and $g \circ j = Ag$ whence

$$h(j(z)) = (f(j(z)) \times g(j(z))) = (A(z)f(z) \times A(z)g(z)) = A(z)(f(z) \times g(z)) = A(z)h(z) \ ,$$

so that, by Definition 3.1, h is an invariant polarization field of (j, A). □

Proof of Theorem 7.1b: Let $(\mathcal{P}_\omega, A) \in \mathcal{ACB}(d, \mathcal{P}_\omega)$ be off orbital resonance. The claim to be proved is equivalent to its contrapositive which is the following claim: If the total number of ISF's of $(\mathcal{P}_\omega, A)$ is not 2, then $(\mathcal{P}_\omega, A)$ is not off spin-orbit resonance. Now, we know from Remark 4 in Chapter 6 that $(\mathcal{P}_\omega, A)$ has at least two ISF's so that if the number of ISF's differs from 2, there are more than two ISF's. Also, since $(\mathcal{P}_\omega, A) \in \mathcal{ACB}(d, \mathcal{P}_\omega)$ we know from a remark after Definition 6.3 that $(\mathcal{P}_\omega, A)$ has spin tunes. Then if the system is not off spin-orbit resonance, it must be on spin-orbit resonance. Thus the above claim we have to prove is equivalent to the following claim: If the total number of ISF's of $(\mathcal{P}_\omega, A)$ is larger than two, then $(\mathcal{P}_\omega, A)$ is on spin-orbit resonance.

In fact we will now prove the latter claim. So let $(\mathcal{P}_\omega, A)$ have more than two ISF's. Recalling Section 3.2, we then conclude that $(\mathcal{P}_\omega, A)$ has ISF's, say f and g, such that $g \neq f$ and $g \neq -f$. Note that $f, -f$ and g are three different ISF's of $(\mathcal{P}_\omega, A)$. We define $h \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ by $h(z) := f(z) \times g(z)$ and observe, by Theorem 7.1a, that h is an invariant polarization field of $(\mathcal{P}_\omega, A)$. On the other hand, since $(\mathcal{P}_\omega, A)$ is off orbital resonance, $\mathcal{P}_\omega$ is topologically transitive whence, by Theorem 3.2, $|h|$ is constant, i.e., $|h(z)| =: \lambda$ is independent of z. We first consider the case where $\lambda = 0$, i.e., where $f \times g$ is the zero function. Then a function $\tilde{h} : \mathbb{T}^d \to \mathbb{R}$ exists such that $g = \tilde{h} f$ whence $g \cdot f = \tilde{h}|f|^2 = \tilde{h}$ which implies that $\tilde{h}$ is continuous. On the other hand $1 = |g| = |\tilde{h} f| = |\tilde{h}|$ whence $\tilde{h}$ can take values only in $\{1, -1\}$ whence, since $\tilde{h}$ is continuous and $\mathbb{T}^d$ is pathwise connected, $\tilde{h}$ is constant. Thus either $g = f$ or $g = -f$ which is a contradiction. So the case where $\lambda = 0$ cannot occur. Thus $\lambda > 0$. Since h is an invariant polarization field of $(\mathcal{P}_\omega, A)$ and since the real number λ is positive we define $k \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ by $k(z) := h(z)/\lambda = h(z)/|h(z)|$ and observe, by using Definition 3.1, that k is an invariant polarization field of $(\mathcal{P}_\omega, A)$. Of course $|k(z)| = |h(z)|/|h(z)| = 1$ whence k is an ISF of $(\mathcal{P}_\omega, A)$. We also define $l \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ by $l(z) := k(z) \times f(z)$ and observe, by Theorem 7.1a, that l is an invariant polarization field of $(\mathcal{P}_\omega, A)$. Of course $f(z) \cdot k(z) = (f(z) \cdot h(z))/|h(z)| = f(z) \cdot (f(z) \times g(z))/\lambda = 0$ whence, for every $z \in \mathbb{T}^d$,

$$0 = l(z) \cdot k(z) = l(z) \cdot f(z) = f(z) \cdot k(z) \, . \tag{7.12}$$

Clearly $|l(z)| = |k(z) \times f(z)| = \sqrt{|k(z)|^2 \, |f(z)|^2 - (k(z) \cdot f(z))^2} = \sqrt{1 - (k(z) \cdot f(z))^2} = 1$ which implies that l is an ISF of $(\mathcal{P}_\omega, A)$ and that

$$1 = |l(z)| = |k(z)| = |f(z)| \, . \tag{7.13}$$

It follows from (7.12) and (7.13) that

$$[l(z), k(z), f(z)]^t [l(z), k(z), f(z)] = I_{3\times 3} \, . \tag{7.14}$$

Moreover, by (7.13), $\det([l(z), k(z), f(z)]) = l(z) \cdot (k(z) \times f(z)) = |l(z)|^2 = 1$ whence, by (7.14), for every $z \in \mathbb{T}^d$, the 3×3-matrix $[l(z), k(z), f(z)]$ belongs to $SO(3)$. We thus define $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ by $T(z) := [l(z), k(z), f(z)]$. Since all three columns of T are invariant polarization fields of $(\mathcal{P}_\omega, A)$ we have, by Definition 3.1,

$$\begin{aligned} A(z)T(z) &= A(z)[l(z), k(z), f(z)] = [A(z)l(z), A(z)k(z), A(z)f(z)] \\ &= [l(\mathcal{P}_\omega(z)), k(\mathcal{P}_\omega(z)), f(\mathcal{P}_\omega(z))] = T(\mathcal{P}_\omega(z)) \, , \end{aligned}$$

whence $T \circ \mathcal{P}_\omega = AT$ so that, by Theorem 6.4a, T belongs to $\mathcal{TF}_{G_0}(j, A)$. Thus, by Theorem 6.4b, $(\mathcal{P}_\omega, A)$ is on spin-orbit resonance as was to be shown. □

The claim of Theorem 7.1b that $(\mathcal{P}_\omega, A)$ has an ISF is trivial because of Remark 4 in Chapter 6. Thus the essence of the claim of Theorem 7.1b is that $(\mathcal{P}_\omega, A)$ has only two ISF's. Recall also from Chapter 3 that the set of ISF's of a spin-orbit system is either infinite or contains an even number of elements. Note that in this work the term "finite number" includes the case of zero. Indeed if a spin-orbit system has no ISF then its number of ISF's is zero, an even number!

8 Unified treatment of spin-orbit systems by using the Technique of Association (ToA)

8.1 Orientation

We now come to our generalization of the notions of particle-spin-vector motion and polarization-field motion by introducing our "Technique of Association" (ToA). With this we will see that while the spin-orbit systems are still the same, they can be exploited further to generate and encompass new perspectives. We thereby see that while the spin-orbit systems do not change, their scope widens. By the ToA the well established notions of invariant polarization field and invariant spin field are generalized to invariant (E, l)-fields where (E, l) is an $SO(3)$-space. The origin of the ToA is an underlying principal bundle and its associated bundles, hence the terminology. However since the principal bundle is of product form, we can easily present the theory in a fashion which does not use bundle theory. For a short account of the bundle aspect see Section 8.8 where we also briefly mention the relation to Yang-Mills Theory. Several major theorems are presented, among them the Normal Form Theorem which ties invariant fields with the notion of normal form, the Decomposition Theorem, which allows one to compare different invariant fields, the Invariant Reduction Theorem, which gives new insights into the question of existence of invariant fields (and in particular invariant spin fields), and the Cross Section Theorem which supplements the Invariant Reduction Theorem. It thus turns out that the well established notions of invariant frame field, spin tune, and spin-orbit resonance are generalized by the normal form concept whereas the well established notions of invariant polarization field and invariant spin field are generalized to invariant (E, l)-fields. In particular we see that the $SO(3)$-space $(\mathbb{R}^3, l_v)$ has been implicitly used in Chapters 2-7. With the flexibility in the choice of (E, l) we also have a unified way to study the dynamics of spin-1/2 and spin-1 particles. Accordingly the special cases $(E, l) = (\mathbb{R}^3, l_v)$ (for spin vectors) and $(E, l) = (E_t, l_t)$ (for spin tensors) are discussed in some detail.

8.2 Defining the ToA

8.2.1 The maps

For given (E, l) each particle carries, in addition to its position z on the torus $\mathbb{T}^d$, an E-valued quantity x that we call spin. Depending on the choice of (E, l), x can be the spin vector S or another quantity related to spin motion. We consider the autonomous DS given by the 1-turn particle-spin map

$$\begin{pmatrix} z \\ x \end{pmatrix} \mapsto \begin{pmatrix} z' \\ x' \end{pmatrix} = \begin{pmatrix} j(z) \\ l(A(z); x) \end{pmatrix} =: \mathcal{P}[E, l, j, A](z, x) \ , \tag{8.1}$$

where $z, z' \in \mathbb{T}^d, x, x' \in E$. In our formalism, (8.1) is the most general description of particle-spin dynamics and the choice of (E, l) depends on the situation, e.g., $(E, l) = (\mathbb{R}^3, l_v)$ for spin-1/2 particles - see below (in that case x is the spin vector S). Note that the function $\mathcal{P}[E, l, j, A] : \mathbb{T}^d \times E \to \mathbb{T}^d \times E$, defined by (8.1), belongs to Homeo$(\mathbb{T}^d \times E)$ since its inverse is $\mathcal{P}[E, l, j^{-1}, A^t \circ j^{-1}]$.
For an E-valued field $f : \mathbb{T}^d \to E$ set $x = f(z)$ in (8.1) so that the particle motion moves z to $j(z)$ and the field value at $j(z)$ becomes $l(A(z); f(z))$. Thus the field f evolves into the field $f' : \mathbb{T}^d \to E$ where $f'(z) := l(A(j^{-1}(z)); f(j^{-1}(z)))$. Therefore we have obtained a map of fields, i.e., the autonomous discrete-time DS given by the 1-turn field map

$$f \mapsto f' := l\Big(A \circ j^{-1}; f \circ j^{-1}\Big) =: \tilde{\mathcal{P}}[E, l, j, A](f) \ . \tag{8.2}$$

We are only interested in continuous fields, i.e., $f \in \mathcal{C}(\mathbb{T}^d, E)$ so that $f' \in \mathcal{C}(\mathbb{T}^d, E))$ too. Note that the function $\tilde{\mathcal{P}}[E, l, j, A] : \mathcal{C}(\mathbb{T}^d, E) \to \mathcal{C}(\mathbb{T}^d, E)$, defined by (8.2), is a bijection since its inverse is $\tilde{\mathcal{P}}[E, l, j^{-1}, A^t \circ j^{-1}]$. We call an $f \in \mathcal{C}(\mathbb{T}^d, E)$ an "invariant (E, l)-field of (j, A)" if it is mapped by (8.2) into itself, i.e., if

$$f \circ j = l(A; f) \ . \tag{8.3}$$

We call (8.3) the "(E, l)-stationarity equation of (j, A)". Clearly an $f \in \mathcal{C}(\mathbb{T}^d, E)$ is an invariant (E, l)-field of (j, A) iff $\tilde{\mathcal{P}}[E, l, j, A](f) = f$. We call an $f \in \mathcal{C}(\mathbb{T}^d, E)$ an "invariant n-turn (E, l)-field of (j, A)" iff $\tilde{\mathcal{P}}[E, l, j, A]^n(f) = f$ where n is a positive integer. Of course the notions of invariant (E, l)-field and invariant 1-turn (E, l)-field are identical. Invariant n-turn (E, l)-fields with $n > 1$ play a role for spin-orbit systems $j = \mathcal{P}_\omega$ with $(1, \omega)$ nonresonant (see Remark 12 and Section 8.5.2).

Remark:

(1) Consider the special case $(E, l) = (\mathbb{R}^3, l_v)$ where we define the function $l_v : SO(3) \times \mathbb{R}^3 \to \mathbb{R}^3$ by

$$l_v(r, S) := rS \ . \tag{8.4}$$

Then the above particle-spin and field maps of (8.1) and (8.2) become the particle-spin-vector and polarization field maps we know from Chapters 2-7. In fact it is a simple exercise to show that $(\mathbb{R}^3, l_v)$ is an $SO(3)$-space and that, by (2.23), the map (8.1) becomes

$$\begin{pmatrix} z \\ S \end{pmatrix} \mapsto \begin{pmatrix} j(z) \\ l_v(A(z); S) \end{pmatrix} = \begin{pmatrix} j(z) \\ A(z)S \end{pmatrix} = \mathcal{P}[j, A](z, S) \ , \tag{8.5}$$

i.e., $\mathcal{P}[\mathbb{R}^3, l_v, j, A] = \mathcal{P}[j, A]$ and, by (3.3), the map (8.2) becomes

$$f \mapsto f' = l_v\Big(A \circ j^{-1}; f \circ j^{-1}\Big) = (Af) \circ j^{-1} = \tilde{\mathcal{P}}[j, A](f) \;, \tag{8.6}$$

i.e., $\tilde{\mathcal{P}}[\mathbb{R}^3, l_v, j, A] = \tilde{\mathcal{P}}[j, A]$.

With (8.5) and (8.6) we see that it is the special case $(E, l) = (\mathbb{R}^3, l_v)$ that underlies the particle-spin-vector and polarization field motion of Chapters 2-7. Thus the ToA is a generalization of the particle-spin-vector and polarization field motion of Chapters 2-7 to arbitrary (E, l).

We also recover the notion of invariant polarization field. In fact an $f \in \mathcal{C}(\mathbb{T}^d, E)$ is an invariant $(\mathbb{R}^3, l_v)$-field of (j, A) iff it satisfies the $(\mathbb{R}^3, l_v)$-stationarity equation of (j, A), $f \circ j = l_v(A; f)$, i.e., iff $f \circ j = Af$. Thus, by Definition 3.1, the notion of invariant polarization field is identical with the notion of "invariant $(\mathbb{R}^3, l_v)$-field". Since we use the terminology "field" so often, it is important to mention that the notions of IFF and uniform IFF are different from the notion of invariant (E, l)-field. □

The $SO(3)$-spaces (E, l) can take a variety of forms. For example in addition to $(\mathbb{R}^3, l_v)$ we will consider (E_t, l_t) where $E_t \subset \mathbb{R}^{3\times 3}$ is defined in Section 8.4.1. This $SO(3)$-space is used for studies of polarized beams of spin-1 particles like deuterons.

8.2.2 The trajectories

Iterating the particle-spin map (8.1), the particle-spin trajectories $(z(\cdot), x(\cdot))$ are defined by

$$\begin{pmatrix} z(n+1) \\ x(n+1) \end{pmatrix} = \begin{pmatrix} j(z(n)) \\ l(A(z(n)); x(n)) \end{pmatrix} \;, \tag{8.7}$$

with $z(0) = z_0, x(0) = x_0$ whence

$$\begin{pmatrix} z(n) \\ x(n) \end{pmatrix} = \begin{pmatrix} j^n(z_0) \\ l\Big(A(j^{n-1}(z_0))A(j^{n-2}(z_0)) \cdots A(z_0)); x_0\Big) \end{pmatrix} \;. \tag{8.8}$$

It is convenient to introduce the corresponding $\mathbb{Z}$-action which is the function $L[E, l, j, A] : \mathbb{Z} \times \mathbb{T}^d \times E \to \mathbb{T}^d \times E$ defined by

$$L[E, l, j, A](n; z, x) := \mathcal{P}[E, l, j, A]^n(z, x) = \begin{pmatrix} L[j](n; z) \\ l(\Psi[j, A](n; z); x) \end{pmatrix} \;. \tag{8.9}$$

It is easy to show that $(\mathbb{T}^d \times E, L[E, l, j, A])$ is a $\mathbb{Z}$-space. In the study of this $\mathbb{Z}$-space, which will not be fully addressed in this work, it is of key importance that $\Psi[j, A]$ is a cocycle (recall Definition 2.7). With (2.32) and $L[E, l, j, A]$ the solution (8.8) of (8.7) can be written as

$$\begin{pmatrix} z(n) \\ x(n) \end{pmatrix} = L[E, l, j, A](n, z_0, x_0) \;. \tag{8.10}$$

For the record, we call a function $(z(\cdot), x(\cdot)) : \mathbb{Z} \to \mathbb{T}^d \times E$ an (E, l)-trajectory (or, just "particle-spin trajectory") of (j, A) if (8.10) holds for all $n \in \mathbb{Z}$ (i.e., if (8.7) holds for all $n \in \mathbb{Z}$).

On iteration of the field map (8.2), the field trajectories F emerge in terms of the equation of motion

$$F(n+1, z) = l\Big(A(j^{-1}(z)); F(n, j^{-1}(z))\Big) , \tag{8.11}$$

whence

$$F(n, z) = l\Big(\Psi[j, A](n; L[j](-n; z)); F(0, L[j](-n; z))\Big) , \tag{8.12}$$

where, as always, $\Psi[j, A](n; \cdot)$ is the spin transfer matrix function of (j, A). For the record, we call a continuous function $F : \mathbb{Z} \times \mathbb{T}^d \to E$ an (E, l)-field trajectory of (j, A) if it satisfies (8.11) or, equivalently, (8.12)). The terminology "trajectory" is justified since the function $n \mapsto F(n, \cdot)$ is a "trajectory" of fields belonging to $\mathcal{C}(\mathbb{T}^d, E)$. Clearly an (E, l)-field trajectory is time-independent iff its initial value $F(0, \cdot) \in \mathcal{C}(\mathbb{T}^d, E)$ is an invariant (E, l)-field. The notion of (E, l)-field trajectory generalizes the notion of polarization-field trajectory as pointed out in Remark 2 below.

We define the $\mathbb{Z}$-action $\tilde{L}[E, l, j, A] : \mathbb{Z} \times \mathcal{C}(\mathbb{T}^d, E) \to \mathcal{C}(\mathbb{T}^d, E)$ by

$$\tilde{L}[E, l, j, A](n; f) := g \ , \quad g(z) := l\Big(\Psi[j, A]\Big(n; L[j](-n; z)\Big); f(L[j](-n; z))\Big) \ . \tag{8.13}$$

With (8.13) we can write (8.12) as

$$F(n, \cdot) = \tilde{L}[E, l, j, A](n, F(0, \cdot)) \ . \tag{8.14}$$

Clearly a continuous function $F : \mathbb{Z} \times \mathbb{T}^d \to E$ is an (E, l)-field trajectory of (j, A) iff it satisfies (8.14).

If F is an (E, l)-field trajectory and $z(\cdot)$ is an particle trajectory of (j, A) then by (8.12) the function $n \mapsto (z(n), F(n, z(n))$ is an (E, l)-trajectory of (j, A). Thus the particle-spin motion can be viewed as a characteristic motion of the field motion. Of course in the special case $(E, l) = (\mathbb{R}^3, l_v)$ this is well-known and is the basic fact underlying all spin vector tracking methods.

Remark:

(2) In the special case where $(E, l) = (\mathbb{R}^3, l_v)$ the above trajectories become trajectories from Chapters 2-7. In fact, by (2.37), we have $L[\mathbb{R}^3, l_v, j, A] = L[j, A]$. This implies that the notion of "$(\mathbb{R}^3, l_v)$-trajectory is identical to the notion of "particle-spin-vector trajectory".

Moreover, by (3.7), we have $\tilde{L}[\mathbb{R}^3, l_v, j, A] = \tilde{L}[j, A]$. This implies that the notion of "$(\mathbb{R}^3, l_v)$-field trajectory is identical to the notion of "polarization-field trajectory". □

Since we work in the framework of topological dynamical systems, A, j, l are continuous functions and we therefore require our fields to be continuous, in particular the invariant (E,l)-fields. Thus every (E,l)-field trajectory F fulfills two different conditions: the "dynamical" condition (8.11) and the "regularity" condition that $F(0,\cdot)$ is continuous. However, in contrast to the dynamical condition, the regularity condition is a matter of choice. While in this work, and in [He2], we choose continuity as the regularity property, this property can basically vary between the extremes "Borel measurable" and "of class C^∞".

8.2.3 The First ToA Transformation Rule

The First ToA Transformation Rule generalizes the transformation rule given by (4.2) and (4.10) to (8.21) and (8.27), i.e., it is a generalization from $(\mathbb{R}^3, l_v)$ to (E,l) whence it is closely related to the concept of the H normal form. We aim to understand the dependence of particle-spin and field motions on A in the general ToA setting just as we did in the $(E,l) = (\mathbb{R}^3, l_v)$ setting of Chapters 2-7. By the transformation rule (4.13), the map (8.5), i.e.,

$$\begin{pmatrix} z \\ S \end{pmatrix} \mapsto \begin{pmatrix} j(z) \\ A(z)S \end{pmatrix} , \tag{8.15}$$

is transformed into the map

$$\begin{pmatrix} \zeta \\ \sigma \end{pmatrix} \mapsto \begin{pmatrix} \zeta' \\ \sigma' \end{pmatrix} = \begin{pmatrix} j(\zeta) \\ A'(\zeta)\sigma \end{pmatrix} , \tag{8.16}$$

where

$$\begin{pmatrix} \zeta \\ \sigma \end{pmatrix} := \begin{pmatrix} z \\ T^t(z)S \end{pmatrix} , \tag{8.17}$$

with $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ and $A' \in \mathcal{C}(\mathbb{T}^d, SO(3))$ defined by (4.1), i.e., $A'(z) := T^t(j(z))A(z)T(z)$. It is easy to generalize the transformation rule (8.15) and (8.16) by replacing $(\mathbb{R}^3, l_v)$ with (E,l). Thus the map (8.1), i.e.,

$$\begin{pmatrix} z \\ x \end{pmatrix} \mapsto \begin{pmatrix} j(z) \\ l(A(z);x) \end{pmatrix} , \tag{8.18}$$

is transformed into the map

$$\begin{pmatrix} \zeta \\ \xi \end{pmatrix} \mapsto \begin{pmatrix} \zeta' \\ \xi' \end{pmatrix} = \begin{pmatrix} j(\zeta) \\ l(A'(\zeta);\xi) \end{pmatrix} , \tag{8.19}$$

where

$$\begin{pmatrix} \zeta \\ \xi \end{pmatrix} := \begin{pmatrix} z \\ l(T^t(z);x) \end{pmatrix} , \tag{8.20}$$

with $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ and $A' \in \mathcal{C}(\mathbb{T}^d, SO(3))$ defined by (4.1), i.e., $A'(z) := T^t(j(z))A(z)T(z)$.
Remark:

(3) Using the notation of (8.1) one sees that $\mathcal{P}[E,l,j,A]$ is the map (8.18) and that $\mathcal{P}[E,l,j,A']$ is the map (8.19) whence $\mathcal{P}[E,l,j,A]$ is transformed into $\mathcal{P}[E,l,j,A']$. Moreover it is a simple exercise to show that

$$\mathcal{P}[E,l,j,A'] = \mathcal{P}[E,l,id_{\mathbb{T}^d},T]^{-1} \circ \mathcal{P}[E,l,j,A] \circ \mathcal{P}[E,l,id_{\mathbb{T}^d},T] \ . \tag{8.21}$$

In fact by defining $\mathcal{T} := \mathcal{P}[E,l,id_{\mathbb{T}^d},T]^{-1}$ we first note that

$$\mathcal{T}^{-1}(z,x) = \begin{pmatrix} z \\ l(T(z);x) \end{pmatrix} , \tag{8.22}$$

whence, by (2.32),(4.1) and (8.1),

$$\begin{aligned}
\Big(\mathcal{T} \circ \mathcal{P}[E,l,j,A] \circ \mathcal{T}^{-1}\Big)(z,x) &= \Big(\mathcal{T} \circ \mathcal{P}[E,l,j,A]\Big) \begin{pmatrix} z \\ l(T(z);x) \end{pmatrix} \\
&= \mathcal{T} \begin{pmatrix} j(z) \\ l(A(z);l(T(z);x)) \end{pmatrix} = \mathcal{T} \begin{pmatrix} j(z) \\ l(A(z)T(z);x) \end{pmatrix} \\
&= \begin{pmatrix} j(z) \\ l\Big(T^t(j(z));l(A(z)T(z);x)\Big) \end{pmatrix} = \begin{pmatrix} j(z) \\ l\Big(T^t(j(z))A(z)T(z);x\Big) \end{pmatrix} \\
&= \mathcal{P}[E,l,j,A'](z,x) ,
\end{aligned}$$

as was to be shown. □

By iteration of the maps, the First ToA Transformation Rule, (8.18) and (8.19), delivers the following transformation rule of trajectories:

$$(z(\cdot),x(\cdot)) \longrightarrow (z(\cdot),x'(\cdot)) \ , \quad x'(n) := l(T^t(z(n));x(n)) \ , \tag{8.23}$$

and we observe that if $(z(\cdot),x(\cdot))$ is an (E,l)-trajectory of (j,A) then $(z(\cdot),x'(\cdot))$ is a (E,l)-trajectory of (j,A'). In the special case where $(E,l) = (\mathbb{R}^3,l_v)$ the transformation rule (8.23) becomes (4.13).

Remark:

(4) By iterating (8.21) it is an easy exercise to show that $\mathcal{P}[E,l,id_{\mathbb{T}^d},T]^{-1}$ is an isomorphism from the $\mathbb{Z}$-space $(\mathbb{T}^d \times E, L[E,l,j,A])$ to the $\mathbb{Z}$-space $(\mathbb{T}^d \times E, L[E,l,j,A'])$. Thus the particle-spin motion of (j,A') is redundant since it can be covered by the particle-spin motion of (j,A). □

With fields we proceed analogously. In fact the map (8.2), i.e.,

$$f \mapsto f' := l\Big(A \circ j^{-1}; f \circ j^{-1}\Big) , \tag{8.24}$$

is transformed into the map

$$g \mapsto g' := l\Big(A' \circ j^{-1}; g \circ j^{-1}\Big) , \tag{8.25}$$

where

$$g := l(T^t; f) \; , \tag{8.26}$$

with $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ and $A' \in \mathcal{C}(\mathbb{T}^d, SO(3))$ defined by (4.1).

Remark:

(5) Using the notation of (8.2) one sees that $\tilde{\mathcal{P}}[E, l, j, A]$ is the map (8.24) and that $\tilde{\mathcal{P}}[E, l, j, A']$ is the map (8.25) whence $\tilde{\mathcal{P}}[E, l, j, A]$ is transformed into $\tilde{\mathcal{P}}[E, l, j, A']$. Moreover it is a simple exercise to show that

$$\tilde{\mathcal{P}}[E, l, j, A'] = \tilde{\mathcal{P}}[E, l, id_{\mathbb{T}^d}, T]^{-1} \circ \tilde{\mathcal{P}}[E, l, j, A] \circ \tilde{\mathcal{P}}[E, l, id_{\mathbb{T}^d}, T] \; . \tag{8.27}$$

□

By iteration of maps the First ToA Transformation Rule, (8.24) and (8.25), delivers the following transformation rule of (E, l)-field trajectories:

$$F \longrightarrow F' \; , \quad F'(n, z) := l(T^t(z); F(n, z)) \; , \tag{8.28}$$

and we observe that if F is an (E, l)-field trajectory of (j, A) then F' is a (E, l)-field trajectory of (j, A') with $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ and $A' \in \mathcal{C}(\mathbb{T}^d, SO(3))$ defined by (4.1). In the special case where $(E, l) = (\mathbb{R}^3, l_v)$ the transformation rule (8.28) becomes (4.17).

Remark:

(6) By iterating (8.27) it is an easy exercise to show that $\tilde{\mathcal{P}}[E, l, id_{\mathbb{T}^d}, T]^{-1}$ is an isomorphism from the $\mathbb{Z}$-set $(\mathcal{C}(\mathbb{T}^d, E), \tilde{L}[E, l, j, A])$ to the $\mathbb{Z}$-set $(\mathcal{C}(\mathbb{T}^d, E), \tilde{L}[E, l, j, A'])$. So the field motion of (j, A') is redundant since it can be covered by the field motion of (j, A). □

While in the First ToA Transformation Rule (E, l) and j are held fixed and A is transformed, we will also introduce, in Section 8.2.6, the Second ToA Transformation Rule where j and A are held fixed and (E, l) is transformed.

8.2.4 The Normal Form Theorem (NFT)

In this section we generalize the IFF Theorem, Theorem 5.4b, and Theorem 6.4a into the NFT. With this we expect the latter to have implications for the concepts of ISF, IFF and SOR. In fact its importance goes even beyond this as will become evident later. Recall that Theorems 5.4b and 6.4a are linked to the groups $SO(2)$ and G_0 respectively. In fact we will show in the examples in Remarks 8 and 9 below that in Theorems 5.4b and Theorem 6.4a the groups $SO(2)$ and G_0 play the role of isotropy groups. The NFT links the notions of invariant (E, l)-field and H-normal form for arbitrary subgroups H of $SO(3)$.

Theorem 8.1 *(NFT) Let (E, l) be an $SO(3)$-space, fix $x \in E$ and define $H(x) := Iso(E, l; x) = \{r \in SO(3) : l(r; x) = x\}$. Moreover, let $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$, $(j, A) \in \mathcal{SOS}(d, j)$ and define the function $f \in \mathcal{C}(\mathbb{T}^d, E)$ by*

$$f(z) := l(T(z); x) \; . \tag{8.29}$$

Then, for all $z \in \mathbb{T}^d$*,*

$$T^t(j(z))A(z)T(z) \in H(x) \,, \tag{8.30}$$

iff, for all $z \in \mathbb{T}^d$*,*

$$f(j(z)) = l(A(z); f(z)) \,. \tag{8.31}$$

In other words, $T \in \mathcal{TF}_{H(x)}(j, A)$ *iff* f *is an invariant* (E, l)*-field of* (j, A)*.*

Proof of Theorem 8.1: "$\Rightarrow$": Let T satisfy (8.30), i.e., let $l(T^t(j(z))A(z)T(z); x) = x$. We compute

$$f(j(z)) = l(T(j(z)); x) = l\Big(T(j(z)); l(T^t(j(z))A(z)T(z); x)\Big)$$
$$= l(A(z)T(z); x) = l(A(z); l(T(z); x)) = l(A(z); f(z)) \,,$$

whence f is an invariant (E, l)-field of (j, A).
"$\Leftarrow$": Let f satisfy the (E, l)-stationarity equation (8.31) of (j, A). Thus, by (8.29),

$$l(T(j(z)); x) = f(j(z)) = l(A(z); f(z))$$
$$= l\Big(A(z); l(T(z); x)\Big) = l(A(z)T(z); x) \,,$$

whence $x = l\Big(T^t(j(z)); l(A(z)T(z); x)\Big) = l\Big(T^t(j(z))A(z)T(z); x\Big)$ which implies (8.30) by (2.45). $\square$

The moral of the NFT is that f and T are effectively equivalent, i.e., that one can view invariant fields from two different perspectives: the perspective of Definition 3.1 and the perspective of H-normal form. To shed further light on this we note that f in (8.29) takes values in only one (E, l)-orbit. In fact if (E, l) is an $SO(3)$-space and $x \in E$ and if $f \in \mathcal{C}(\mathbb{T}^d, E)$ takes values only in $l(SO(3); x)$ then we call $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ an "(E, l)-lift of f" if (8.29) holds. Thus the NFT says that if an invariant (E, l)-field has an (E, l)-lift, say T, then $T \in \mathcal{TF}_{H(x)}(j, A)$ where $H(x) := Iso(E, l; x)$ and $x \in E$ has the property that f takes only values in $l(SO(3); x)$. Thus the notion of (E, l)-lift will give us insight into invariant fields via isotropy groups. Note that, by the IFF Theorem, Theorem 5.4b, for an ISF the notions of IFF and $(\mathbb{R}^3, l_v)$-lift are identical. Experience with ISF's lets us believe that in practice an ISF has an $(\mathbb{R}^3, l_v)$-lift. Thus, by the ISF conjecture in Chapter 3, we expect that an IFF exists in practice if j is topologically transitive, i.e., that (j, A) has an H-normal form with $H \trianglelefteq SO(2)$. However it is well-known, as pointed out in Section 8.7.2, that $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ exist such that f takes values only in $l_v(SO(3); (0, 0, 1)^t)$ and which have no $(\mathbb{R}^3, l_v)$-lift. In any case, the NFT gives insight into invariant (E, l)-fields which have an (E, l)-lift. In fact if an $f \in \mathcal{C}(\mathbb{T}^d, E)$ has an (E, l)-lift and then, by the NFT, f can only be an invariant (E, l)-field of a spin-orbit system (j, A) if $(j, A) \in \mathcal{CB}_{H(x)}(d, j)$ where

$H(x) := Iso(E,l;x)$. Thus if $H(x)$ is "large enough", e.g., $SO(2) \trianglelefteq H(x)$ then chances are that f is an invariant (E,l)-field of (j,A) and if $H(x)$ is small then chances are that f is not an invariant (E,l)-field of (j,A). We will come back to this point of view in Section 8.4.1 where we will deal with large and small isotropy groups. We will also consider lifts in more detail in the CST in Section 8.7.2.

It is also clear that the NFT deals with arbitrary $SO(3)$-spaces (E,l). Moreover it can also be easily seen that it deals with arbitrary subgroups H of $SO(3)$ since every subgroup H of $SO(3)$ is an isotropy group of some $SO(3)$-space (E,l) [He1].

The following remark compares the NFT for x, x' belonging to the same (E,l)-orbit and at the same time it provides us with the useful formula (8.32).

Remark:

(7) Let (E,l) be an $SO(3)$-space, let $x \in E$ and let $x' \in l(SO(3);x)$, i.e., let $r \in SO(3)$ such that $x' = l(r;x)$. We define $H := Iso(E,l;x)$, $H' := Iso(E,l;x')$. Then, by (2.45),

$$\begin{aligned} H' = Iso(E,l;x') &= Iso(E,l;l(r;x)) = \{r' \in SO(3) : l(r';l(r;x)) = l(r;x)\} \\ &= \{r' \in SO(3) : l(r'r;x) = l(r;x)\} = \{r' \in SO(3) : l(r^t;l(r'r;x)) = x\} \\ &= \{r' \in SO(3) : l(r^t r' r;x) = x\} = \{r r' r^t : r' \in SO(3), l(r';x) = x\} \\ &= r\{r' \in SO(3) : l(r';x) = x\} r^t = r Iso(E,l;x) r^t = r H r^t \ , \end{aligned} \tag{8.32}$$

whence isotropy groups on the same orbit are conjugate.

We now consider the NFT. So let $(j,A) \in \mathcal{SOS}(d,j)$ and $T \in \mathcal{TF}_H(j,A)$ whence, by the NFT, the function $f \in \mathcal{C}(\mathbb{T}^d, E)$ defined by $f := l(\cdot;x) \circ T$ is an invariant (E,l)-field of (j,A).

Because of (8.32) and Remark 3 in Chapter 5 the function $T' \in \mathcal{C}(\mathbb{T}^d, SO(3))$ defined by $T'(z) := T(z) r^t$ belongs to $\mathcal{TF}_{H'}(j,A)$. Thus, by the NFT, the function $f' \in \mathcal{C}(\mathbb{T}^d, E)$, defined by $f'(z) := l(T'(z);x')$ is an invariant (E,l)-field of (j,A). However $f'(z) = l(T'(z);x') = l(T(z)r^t;x') = l(T(z);l(r^t;x')) = l(T(z);x) = f(z)$ whence, as perhaps expected, the function f is independent of the chosen "reference point" x. □

The following remark shows that Theorem 5.4b is a special case of the NFT.

Remark:

(8) We now show that Theorem 5.4b is a special case of the NFT when $(E,l) = (\mathbb{R}^3, l_v)$. We first define, for $\lambda \in [0,\infty)$, $S_\lambda := \lambda(0,0,1)^t$. Let $x = S_\lambda = \lambda(0,0,1)^t$ where $\lambda > 0$. First, if $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ then f in (8.29) is the function $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ defined by $f(z) := l_v(T(z), S_\lambda) = T(z) S_\lambda$. Secondly, by (2.45) and (8.4),

$$\begin{aligned} Iso(\mathbb{R}^3, l_v; S_\lambda) &= \{r \in SO(3) : l_v(r; \lambda(0,0,1)^t) = \lambda(0,0,1)^t\} \\ &= \{r \in SO(3) : r(0,0,1)^t = (0,0,1)^t\} = SO(2) \ , \end{aligned} \tag{8.33}$$

where in the third equality we used the $SO(2)$-Lemma, Theorem 5.4a. We conclude by Remark 2 that in the case, where $(E,l) = (\mathbb{R}^3, l_v)$ and $x = S_\lambda$ with $\lambda > 0$, the NFT reads as follows: If $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ and $(j,A) \in \mathcal{SOS}(d,j)$ then $T \in \mathcal{TF}_{SO(2)}(j,A)$ iff

TS_λ is an invariant polarization field of (j, A). Thus, in the special case $\lambda = 1$, the NFT reads as follows: If $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ and $(j, A) \in \mathcal{SOS}(d, j)$ then $T \in \mathcal{TF}_{SO(2)}(j, A)$ iff the third column of T is an ISF of (j, A). Thus indeed Theorem 5.4b is the NFT in the special case where $(E, l) = (\mathbb{R}^3, l_v)$ and $x = (0, 0, 1)^t$. We also see that $l(T(z); x)$ generalizes the concept of "third column of $T(z)$" from $\mathbb{R}^3, l_v, (0, 0, 1)^t$ to E, l, x. For more details on the isotropy groups of $(\mathbb{R}^3, l_v)$ see Section 8.3.3. □

The following remark shows that Theorem 6.4a is a special cases of the NFT.

Remark:

(9) We now show that Theorem 6.4a is the special case of the NFT for which $(E, l) = (SO(3), l_{SOR})$ and $x = I_{3\times3}$ where the $SO(3)$-space $(SO(3), l_{SOR})$ is defined in terms of the function $l_{SOR} : SO(3) \times SO(3) \to SO(3)$ given by

$$l_{SOR}(r'; r) := r'r \ , \tag{8.34}$$

where $r, r' \in SO(3)$. It is a simple exercise to show that $(SO(3), l_{SOR})$ is an $SO(3)$-space.

To make our point we first note that, if $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$, then f in (8.29) is the function $f \in \mathcal{C}(\mathbb{T}^d, SO(3))$ defined by $f(z) := l_{SOR}(T(z), I_{3\times3}) = T(z)$, i.e., $f = T$. Secondly, by (2.45), (6.4) and (8.34),

$$\begin{aligned} Iso(SO(3), l_{SOR}; I_{3\times3}) &= \{r \in SO(3) : l_{SOR}(r; I_{3\times3}) = I_{3\times3}\} \\ &= \{r \in SO(3) : r = I_{3\times3}\} = G_0 \ . \end{aligned} \tag{8.35}$$

We conclude that in the case where $(E, l) = (SO(3), l_{SOR})$ and $x = I_{3\times3}$ the NFT reads as follows: If $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ and $(j, A) \in \mathcal{SOS}(d, j)$ then $T \in \mathcal{TF}_{G_0}(j, A)$ iff T is an invariant $(SO(3), l_{SOR})$-field of (j, A). On the other hand, by (8.3), the $(SO(3), l_{SOR})$-stationarity equation of (j, A) reads as $T \circ j = AT$ so that indeed Theorem 6.4a is the NFT in the special case where $(E, l) = (SO(3), l_{SOR})$ and $x = I_{3\times3}$.

We also conclude from Theorem 6.4a and Remark 5 in Chapter 6 that every continuous solution of the $(SO(3), l_{SOR})$-stationarity equation of (j, A) is a uniform IFF of (j, A).

For $(E, l) = (SO(3), l_{SOR})$ and arbitrary $x = r_0 \in SO(3)$ it is an easy exercise to show that the NFT reads as follows: If $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ and $(j, A) \in \mathcal{SOS}(d, j)$ then $T \in \mathcal{TF}_{G_0}(j, A)$ iff T satisfies, for all $z \in \mathbb{T}^d$, $T(j(z))r_0 = A(z)T(z)r_0$. Of course since r_0 cancels out, this is equivalent to the case $r_0 = I_{3\times3}$, i.e., equivalent to Theorem 6.4a. □

Remarks 8 and 9 illustrate how, by feeding in appropriate objects, the NFT is capable of covering seemingly disparate aspects of the dynamics. In fact with the ISF and SOR, two kinds of invariance are covered by the NFT by suitable choice of (E, l).

8.2.5 The decomposition method. Invariant sets for (E, l) particle-spin dynamics

Recall from Definition 2.6 that $l(SO(3); x)$ is the (E, l)-orbit of x and that the (E, l)-orbits form a partition E/l of E. Since a particle-spin trajectory satisfies

$$\begin{pmatrix} z(n) \\ x(n) \end{pmatrix} = \begin{pmatrix} j^n(z(0)) \\ l\Big(\Psi[j, A](n; z(0)); x(0) \Big) \end{pmatrix} , \tag{8.36}$$

we see that $(z(n), x(n))$ belongs to $\mathbb{T}^d \times l(SO(3); x(0))$ for all $n \in \mathbb{Z}$. Thus, for every $x \in E$, the set $\mathbb{T}^d \times l(SO(3); x)$ is invariant under the particle-spin motion. This implies, as the following lemma shows, that for the description of the particle-spin motion we can replace $L[E, l, j, A]$ by the $L[l(SO(3); x), l_{dec}[x], j, A]$ where, for arbitrary $x \in E$, the function $l_{dec}[x] : SO(3) \times l(SO(3); x) \to l(SO(3); x)$ is defined as a restriction of the function l, i.e.,

$$l_{dec}[x](r; y) := l(r; y) , \tag{8.37}$$

where $y \in l(SO(3); x), r \in SO(3)$. It is easy to show that $l_{dec}[x]$ is a group action of the group $SO(3)$ on the subset $l(SO(3); x)$ of E. In fact since $l_{dec}[x]$ is a restriction of l, it is even easy to show that $(l(SO(3); x), l_{dec}[x])$ is an $SO(3)$-space (recall Definition 2.6). We now summarize this in a lemma for further reference.

Lemma 8.2 *Let (E, l) be an $SO(3)$-space. For every $x \in E$, $(l(SO(3); x), l_{dec}[x])$ is a transitive $SO(3)$-space. Let $(j, A) \in \mathcal{SOS}(d, j)$ and let $(z(\cdot), x(\cdot))$ be an (E, l)-trajectory of (j, A). Then $(z(\cdot), x(\cdot))$ is also an $\Big(l(SO(3); x(0)), l_{dec}[x(0)] \Big)$-trajectory of (j, A).* □

Each set $\mathbb{T}^d \times l(SO(3); x)$ is invariant under the particle-spin motion of (8.1) so that the "decomposition method" decomposes $\mathbb{T}^d \times E$ into the $\mathbb{T}^d \times l(SO(3); x)$. Thus for particle-spin motion, the $SO(3)$-space (E, l) can be replaced by the $SO(3)$-spaces $(l(SO(3); l_{dec}[x])$ where x ranges over E.

The following remark considers the special case where $(E, l) = (\mathbb{R}^3, l_v)$.

Remark:

(10) In the special case where $(E, l) = (\mathbb{R}^3, l_v)$ the (E, l)-orbits are spheres. In fact it follows from (8.4) and Definition 2.4 that the $(\mathbb{R}^3, l_v)$-orbit of an arbitrary $S \in \mathbb{R}^3$ reads as

$$l_v(SO(3), S) = \{S' \in \mathbb{R}^3 : |S'| = |S|\} = \mathbb{S}^2_{|S|} , \tag{8.38}$$

where $\mathbb{S}^2_\lambda := \{S \in \mathbb{R}^3 : |S| = \lambda\}$ with $\lambda \in [0, \infty)$. Thus the $(\mathbb{R}^3, l_v)$-orbits are the spheres $\mathbb{S}^2_\lambda$ of radius $\lambda \in [0, \infty)$ around $(0, 0, 0)^t$. Moreover $R_v := \{S_\lambda : \lambda \in [0, \infty)\}$ is a representing set of the partition $\mathbb{R}^3/l_v$ of $\mathbb{R}^3$ (recall from Remark 8 that $S_\lambda = \lambda(0, 0, 1)^t$). Note that all values of λ are of physical importance as can be seen for example in Section 8.6 where spin vectors appear as coefficients in the density matrix functions of spin-1/2 and spin-1 particles. □

For the field dynamics the situation is analogous. For the following it is useful to keep in mind from Section 8.2.1 that if $f \in \mathcal{C}(\mathbb{T}^d, E)$ and $z \in \mathbb{T}^d$ then $(z, f(z))$ is mapped into $(j(z), f'(z))$ where the field f evolves into the field f' where $f'(z) := l(A(j^{-1}(z)); f(j^{-1}(z)))$ and that $f' = f$ iff f is invariant.

Let $f \in \mathcal{C}(\mathbb{T}^d, E)$ and let us define for every $x \in E$ the inverse image of $l(SO(3); x)$ under the function f, i.e., the set

$$\Sigma_x[E, l, f] := f^{-1}(l(SO(3); x)) := \{z \in \mathbb{T}^d : f(z) \in l(SO(3); x)\} , \tag{8.39}$$

where the second equality is just the definition of the inverse image. Then the nonempty among the $\Sigma_x[E, l, f]$ form a partition of $\mathbb{T}^d$ which tells us how the values of f are distributed over the various (E, l)-orbits. Note that $\Sigma_x[E, l, f]$ is nonempty iff f takes a value in $l(SO(3); x)$. For each x such that $\Sigma_x[E, l, f]$ is nonempty, we define the function $f_x : \Sigma_x[E, l, f] \to l(SO(3); x)$ by

$$f_x(z) := f(z) . \tag{8.40}$$

Remark:

(11) Let us illustrate (8.39) and (8.40) in the special case, where $(E, l) = (\mathbb{R}^3, l_v)$. Due to (8.40) and for every f in $\mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ and every $S \in \mathbb{R}^3$ the values of f_S lie on the sphere $\mathbb{S}^2_{|S|}$. To give a concrete example we consider the function $f : \mathbb{T}^1 \to \mathbb{R}^3$ defined by $f(z) \equiv f(\phi + \tilde{\mathbb{Z}}) := \cos(\phi)(0, 0, 1)^t$ where $\phi \in z$. It follows from (8.39), for every $S \in \mathbb{R}^3$, that

$$\begin{aligned} \Sigma_S[\mathbb{R}^3, l_v, f] &= \{z \in \mathbb{T}^1 : f(z) \in l_v(SO(3); S)\} = \{z \in \mathbb{T}^1 : |f(z)| = |S|\} \\ &= \{\pi_1(\phi) : \phi \in \mathbb{R}, |f(\pi_1(\phi))| = |S|\} = \{\pi_1(\phi) : \phi \in \mathbb{R}, |\cos(\phi)| = |S|\} , \end{aligned}$$

where in the second equality we used Remark 10. Clearly $\Sigma_S[\mathbb{R}^3, l_v, f]$ is nonempty iff $|S| \leq 1$ whence f takes values in the infinitely many $(\mathbb{R}^3, l_v)$-orbits $\mathbb{S}^2_\lambda$ $(0 \leq \lambda \leq 1)$. Note also, by (2.13), that $f(\pi_d(\phi)) = \cos(\phi)(0, 0, 1)^t$ whence $f \circ \pi_d$ is continuous so that, by the Torus Lemma, Lemma 2.1, f is continuous. □

The following lemma shows us how the time evolution of f changes the $\Sigma_x[E, l, f]$.

Lemma 8.3 *Let (E, l) and $x \in E$ and let $(j, A) \in \mathcal{SOS}(d, j)$. Let us map $f \in \mathcal{C}(\mathbb{T}^d, E)$ under (j, A) into $f' \in \mathcal{C}(\mathbb{T}^d, E)$ which is given by (8.2). Then $\Sigma_x[E, l, f']$ is the image of $\Sigma_x[E, l, f]$ under j, i.e., $\Sigma_x[E, l, f'] = j(\Sigma_x[E, l, f])$ (for the notion of "image", see also Appendix A.1). Moreover if f is an invariant (E, l)-field of (j, A) then $\Sigma_x[E, l, f] = j(\Sigma_x[E, l, f])$.*

Proof of Lemma 8.3: By (8.2) and (8.39)

$$\begin{aligned} \Sigma_x[E, l, f'] &= \{z \in \mathbb{T}^d : f'(z) \in l(SO(3); x)\} \\ &= \{z \in \mathbb{T}^d : l(A(j^{-1}(z)); f(j^{-1}(z))) \in l(SO(3); x)\} \\ &= \{z \in \mathbb{T}^d : f(j^{-1}(z)) \in l(SO(3); x)\} = \{j(z') : z' \in \mathbb{T}^d, f(z') \in l(SO(3); x)\} \end{aligned}$$

$$= j(\{z' \in \mathbb{T}^d : f(z') \in l(SO(3); x)\}) = j(\Sigma_x[E, l, f]) \; , \tag{8.41}$$

where in the third equality we used the fact that l is a group action and where in the fifth and sixth equalities we dealt with images under j.

If f is, in addition, an invariant (E, l)-field of (j, A) then $f' = f$ whence (8.41) implies $\Sigma_x[E, l, f] = j(\Sigma_x[E, l, f])$ which proves the second claim. □

With Lemma 8.3 we see that if f is an invariant (E, l)-field of (j, A) then, at least locally, f_x behaves like an invariant (E, l)-field since, for all $z \in \Sigma_x[E, l, f]$, we have $f_x(j(z)) = l(A(z); f_x(z))$ and, by (8.37),

$$f_x(j(z)) = l_{dec}[x](A(z); f_x(z)) \; . \tag{8.42}$$

With (8.42) it is thus natural to generalize the notion of invariant field to a notion of a "localized" invariant field whose domain is $\Sigma_x[E, l, f]$. This also implies that if $f \in \mathcal{C}(\mathbb{T}^d, E)$ takes only values in one (E, l)-orbit, say $l(SO(3); x)$, then f_x is an invariant $(l(SO(3); x), l_{dec}[x])$-field of (j, A) on $\mathbb{T}^d$ iff f is an invariant (E, l)-field of (j, A). Indeed our focus in this work is on invariant fields which take values in just one orbit. This restriction of ours is for brevity and because (see Lemma 8.4 below) the most important invariant fields have this property.

Thus for field motion, the $SO(3)$-space (E, l) can be replaced by the $SO(3)$-spaces $(l(SO(3); l_{dec}[x])$ where x ranges over E. The benefit of this is, as we will see further below, that the $(l(SO(3); x), l_{dec}[x])$-stationarity equations are easier to handle than the (E, l)-stationarity equations and allow us to use methods which are not available for (E, l) when (E, l) is not transitive (we use these methods in the Decomposition Theorem, Theorem 8.9, of Section 8.3).

We now will show that invariant (E, l)-fields often take values in only one (E, l)-orbit. In fact the following lemma is a straightforward generalization of Theorem 3.2 (see also Remark 13 below).

Lemma 8.4 *Let (E, l) be an $SO(3)$-space and let E be Hausdorff (for the notion of "Hausdorff" see Appendix A.5). Let also $(j, A) \in \mathcal{SOS}(d, j)$ such that j is topologically transitive. Then every invariant (E, l)-field of (j, A) takes values in only one (E, l)-orbit.*

Proof of Lemma 8.4: Let f be an invariant (E, l)-field of (j, A). We pick a $z_0 \in \mathbb{T}^d$ such that the set $B := \{j^n(z_0) : n \in \mathbb{Z}\}$ is dense in $\mathbb{T}^d$. Because the nonempty among the $\Sigma_x[E, l, f]$ form a partition of $\mathbb{T}^d$ we can pick an $x \in E$ such that $z_0 \in \Sigma_x[E, l, f]$ whence, by Lemma 8.3 and since f is invariant, $B \subset \Sigma_x[E, l, f]$. On the other hand, the continuous function $l(\cdot; x) : SO(3) \to E$ has the range $l(SO(3); x)$ whence, since $SO(3)$ is compact, the range $l(SO(3); x)$ of this function is a compact subset of E [Mu]. Because E is Hausdorff the compact subset $l(SO(3); x)$ of E is a closed subset of E [Mu] whence, since f is continuous, it follows from (8.39) that $\Sigma_x[E, l, f]$ is a closed subset of $\mathbb{T}^d$. Because B is a dense subset of $\mathbb{T}^d$, we get $\mathbb{T}^d = \overline{B} \subset \overline{\Sigma_x[E, l, f]}$ so that, since $\Sigma_x[E, l, f]$ is a closed subset of $\mathbb{T}^d$, we conclude that $\mathbb{T}^d \subset \Sigma_x[E, l, f]$ which implies that $\mathbb{T}^d = \Sigma_x[E, l, f]$. Thus, by the definition of $\Sigma_x[E, l, f]$, we conclude that f takes values only in $l(SO(3); x)$. □

The above lemma tells us that invariant fields, which take values in only one orbit, are of major importance and thus in the sequel most of our theorems are stated for that situation.

Note also that the contexts of the NFT and of Lemma 8.4 overlap since the NFT makes statements about invariant fields which take values in only one (E,l)-orbit (see (8.29)).

Remark:

(12) Consider the $SO(3)$-space (E,l) where E is Hausdorff. Clearly one can apply Lemma 8.4 when $j = \mathcal{P}_\omega$ with $(1,\omega)$ nonresonant. Perhaps surprisingly one can even use Lemma 8.4 to get analogous results when $j = \mathcal{P}_\omega$ and $(1,\omega)$ is resonant. In fact in such an approach invariant n-turn fields and tori with dimension smaller than d play a role [He1]. $\square$

Since $\mathbb{R}^3$ is Hausdorff, we can apply Lemma 8.4 to $(E,l) = (\mathbb{R}^3, l_v)$ as the following remarks shows.

Remark:

(13) Let $(j,A) \in \mathcal{SOS}(d,j)$ with j topologically transitive and let f be an invariant polarization field of (j,A), i.e., by Remark 1, an invariant $(\mathbb{R}^3, l_v)$-field of (j,A). Since $\mathbb{R}^3$ is Hausdorff we can apply Lemma 8.4 and conclude that f takes values in only one $(\mathbb{R}^3, l_v)$-orbit, say $\mathbb{S}^2_{|S|}$ whence $|f(z)| = |S|$. Note that we already proved that unknowingly in Theorem 3.2. Of course $\Sigma_S[\mathbb{R}^3, l_v, f] = \mathbb{T}^d$. $\square$

By iterating the above procedure which led to $\Sigma_x[E,l,f]$ and f_x one arrives at the function $F_x : \bigcup_{n\in\mathbb{Z}} (\{n\} \times j^n(\Sigma_x[E,l,f])) \to l(SO(3);x)$ defined by

$$F_x(n,z) := F(n,z) \,, \tag{8.43}$$

where $F : \mathbb{Z} \times \mathbb{T}^d \to E$ is the (E,l)-field trajectory of (j,A) with initial value $F(0,\cdot) = f(\cdot)$. Since F is a field trajectory we find, by (8.12) and (8.43), that

$$F_x(n,z) = l\Big(\Psi[j,A](n; L[j](-n;z)); f_x(L[j](-n;z))\Big) \,. \tag{8.44}$$

Clearly a necessary condition for F_x to be time-independent is that, for all x, $j(\Sigma_x[E,l,f]) = \Sigma_x[E,l,f]$.

8.2.6 The Second ToA Transformation Rule

The Second ToA Transformation Rule tells us how the motions of different $SO(3)$-spaces (E,l) are related. This point of view will render the decomposition method into a useful tool in Sections 8.3-8.5. While in the First ToA Transformation Rule (E,l) and j are held fixed and A is transformed, in the Second ToA Transformation Rule j and A are held fixed and (E,l) is transformed into another $SO(3)$-space.

Let (E_1,l_1) and (E_2,l_2) be $SO(3)$-spaces and suppose there exists a topological $SO(3)$-map β from (E_1,l_1) to (E_2,l_2), i.e., $\beta \in \mathcal{C}(E_1,E_2)$ and $\beta(l_1(r;x)) = l_2(r;\beta(x))$. Let also $(j,A) \in \mathcal{SOS}(d,j)$ be fixed but arbitrary.

Consider the mappings (8.1) for these $SO(3)$-spaces, i.e.,

$$\begin{pmatrix} z \\ x \end{pmatrix} \mapsto \begin{pmatrix} z' \\ x' \end{pmatrix} = \begin{pmatrix} j(z) \\ l_1(A(z);x) \end{pmatrix} , \tag{8.45}$$

$$\begin{pmatrix} \zeta \\ \xi \end{pmatrix} \mapsto \begin{pmatrix} \zeta' \\ \xi' \end{pmatrix} = \begin{pmatrix} j(\zeta) \\ l_2(A(\zeta);\xi) \end{pmatrix} . \tag{8.46}$$

From (8.45) $\beta(x') = \beta(l_1(A(z);x)) = l_2(A(z);\beta(x))$. Thus if ξ in (8.46) is $\beta(x)$ then $\xi' = l_2(A(z);\beta(x)) = \beta(x')$. It follows that if $(z(n),x(n))$ is the solution of the IVP of (8.45) with $(z(0),x(0)) = (z_0,x_0)$ then $(\zeta(n),\xi(n)) = (z(n),\beta(x(n)))$ is the solution of the IVP of (8.46) with $(\zeta(0),\xi(0)) = (z(0),\beta(x(0))) = (z_0,\beta(x_0))$.

If β is not one-one, two IVP's for (8.45) can give rise to the same IVP of (8.46). If β is not onto, then some IVP's of (8.46) are not related to any IVP of (8.45), e.g., pick $\xi(0)$ not in the range of β. The most interesting case is if β is a surjection, then every solution of the IVP of (8.46) is related to a solution of an IVP of (8.45). Thus the (E_1,l_1)-particle-spin motion gives insights into all (E_2,l_2)-particle-spin motions. If β is a homeomorphism then every IVP of (8.46) can be written in terms of an IVP of (8.45) (and vice versa).

Remark:

(14) We will identify and study several important examples of the function β in later sections. We already mention two examples of the function β here. First of all the function β defined by (8.75) and (8.76) maps $\mathbb{S}^2_\lambda$ onto $\mathbb{S}^2_\mu$ and demonstrates how β can provide a way to connect and relate motions of the same spin variable. Moreover the function β defined by (8.111) in Section 8.5.1 maps $\mathbb{S}^2_\lambda$ into $\mathbb{R}^{3\times 3}$ and is a striking example of how β can connect and relate very different spin variables. The Isotropy-Conjugacy Lemma in Section 8.3.1 will show how examples like these are enabled by certain subgroups of $SO(3)$. □

The following remark summarizes the above Second ToA Transformation Rule for particle-spin motion in a concise way:

Remark:

(15) Consider the $SO(3)$-spaces (E_1,l_1) and (E_2,l_2) and let β be a topological $SO(3)$-map from (E_1,l_1) to (E_2,l_2). Moreover let $(j,A) \in \mathcal{SOS}(d,j)$. Using Definition 4.3 it is an easy exercise to show that

$$\mathcal{P}[E_2,l_2,j,A] \circ \beta_{tot} = \beta_{tot} \circ \mathcal{P}[E_1,l_1,j,A] , \tag{8.47}$$

where the function $\beta_{tot} \in \mathcal{C}(\mathbb{T}^d \times E_1, \mathbb{T}^d \times E_2)$ is defined, for $z \in \mathbb{T}^d, x \in E_1$, by $\beta_{tot}(z,x) := (z,\beta(x))$. If β is a homeomorphism then $\beta_{tot} \in \text{Homeo}(\mathbb{T}^d \times E_1, \mathbb{T}^d \times E_2)$ and

$$\beta_{tot}^{-1} \circ \mathcal{P}[E_2,l_2,j,A] = \mathcal{P}[E_1,l_1,j,A] \circ \beta_{tot}^{-1} . \tag{8.48}$$

□

With fields we proceed analogously. Consider the mappings (8.2) for the $SO(3)$-spaces (E_1,l_1) and (E_2,l_2):

$$f \mapsto f' := l_1\Big(A \circ j^{-1}; f \circ j^{-1}\Big) , \tag{8.49}$$

$$g \mapsto g' := l_2\Big(A \circ j^{-1}; g \circ j^{-1}\Big) . \qquad (8.50)$$

From (8.49) $\beta(f'(z)) = \beta(l_1\Big(A(j^{-1}(z)); f(j^{-1}(z))\Big)) = l_2\Big(A(j^{-1}(z)); \beta(f(j(z)))\Big)$. Thus, if

$$g = \beta \circ f \ , \qquad (8.51)$$

then $g' = \beta \circ f'$.

We now formulate this and some consequences as a theorem which will become important for the decomposition method (see Section 8.3).

Theorem 8.5 *Let (E_1, l_1) and (E_2, l_2) be $SO(3)$-spaces and suppose there exists a topological $SO(3)$-map β from (E_1, l_1) to (E_2, l_2). If $f \in \mathcal{C}(\mathbb{T}^d, E_1)$ and $g \in \mathcal{C}(\mathbb{T}^d, E_2)$ then the field mappings (8.49),(8.50) satisfy:*

a) If $g = \beta \circ f$ then $g' = \beta \circ f'$.

b) If $g = \beta \circ f$ and f is an invariant (E_1, l_1)-field of (j, A) then g is an invariant (E_2, l_2)-field of (j, A).

c) If β is a homeomorphism and $g = \beta \circ f$ is an invariant (E_2, l_2)-field of (j, A) then f is an invariant (E_1, l_1)-field of (j, A).

d) If F is the solution of the IVP of (8.49) with $F(0, z) = F_0(z)$ then G given by $G(n, z) = \beta(F(n, z))$ is the solution of the IVP of (8.50) with $G(0, z) = \beta(F(0, z)) = \beta(F_0(z))$. □

If β is not onto then some IVP's of (8.50) have solutions which are not related to any IVP of (8.49). If β is not one-one, then two IVP's of (8.49) can give rise to the same IVP of (8.50).

The following remark summarizes the above Second ToA Transformation Rule for the field motion in a concise way:

Remark:

(16) Let (E_1, l_1) and (E_2, l_2) be $SO(3)$-spaces and let β be a topological $SO(3)$-map from (E_1, l_1) to (E_2, l_2). Let also $(j, A) \in \mathcal{SOS}(d, j)$. Using Definition 4.3 it is a simple exercise to show that

$$\tilde{\mathcal{P}}[E_2, l_2, j, A] \circ \tilde{\beta} = \tilde{\beta} \circ \tilde{\mathcal{P}}[E_1, l_1, j, A] \ , \qquad (8.52)$$

where the function $\tilde{\beta} : \mathcal{C}(\mathbb{T}^d, E_1) \to \mathcal{C}(\mathbb{T}^d, E_2)$ is defined, for $f \in \mathcal{C}(\mathbb{T}^d, E_1)$, by $\tilde{\beta}(f) := \beta \circ f$. If β is a homeomorphism then $\tilde{\beta}$ is a bijection and

$$\tilde{\beta}^{-1} \circ \tilde{\mathcal{P}}[E_2, l_2, j, A] = \tilde{\mathcal{P}}[E_1, l_1, j, A] \circ \tilde{\beta}^{-1} \ . \qquad (8.53)$$

□

Two questions naturally arise: when do topological $SO(3)$-maps exist and what form do they take? This is the subject of Section 8.3.

8.3 The Isotropy-Conjugacy Lemma (ICL) and the Decomposition Theorem (DT)

In this section we address the two questions from the end of Section 8.2.6 for the important case when (E_1, l_1) and (E_2, l_2) both are transitive, i.e., have only one orbit. Most importantly we also address these questions in the case of the decomposition of any given $SO(3)$-spaces (E, l) and (E', l') where $(E_1, l_1) = (l(SO(3); x), l_{dec}[x])$ and $(E_2, l_2) = (l'(SO(3); x'), l'_{dec}[x'])$. All this is achieved by the ICL which we then apply to the field dynamics, leading us to the DT.

8.3.1 The Isotropy-Conjugacy Lemma

The first question from the end of Section 8.2.6 is answered by the following proposition which relates topological $SO(3)$-maps with isotropy groups.

Proposition 8.6 *Let (E_1, l_1) and (E_2, l_2) be transitive $SO(3)$-spaces and let E_1, E_2 be Hausdorff. For arbitrary $x_1 \in E_1$ and $x_2 \in E_2$ the following hold.*

a) A topological $SO(3)$-map from (E_1, l_1) to (E_2, l_2) exists iff $Iso(E_1, l_1; x_1) \trianglelefteq Iso(E_2, l_2; x_2)$.

b) The $SO(3)$-spaces (E_1, l_1) and (E_2, l_2) are isomorphic iff $Iso(E_1, l_1; x_1), Iso(E_2, l_2; x_2)$ are conjugate. □

The reader finds the proof of this proposition at the end of this section. In fact this proposition is a simple corollary of the ICL.

In our applications we start with $SO(3)$-spaces (E, l) and (E', l') which are not transitive (for example, $(\mathbb{R}^3, l_v)$) and we work with the decompositions $(E_1, l_1) = (l(SO(3); x), l_{dec}[x])$ and $(E_2, l_2) = (l'(SO(3); x'), l'_{dec}[x'])$. To formulate the ICL we make the following definition:

Definition 8.7 *Let (E, l) and (E', l') be $SO(3)$-spaces and let $x \in E$ and $x' \in E'$. We denote by $B(E, l, E', l'; x, x')$ the set of all topological $SO(3)$-maps from $(l(SO(3); x), l_{dec}[x])$ to $(l'(SO(3), x'), l'_{dec}[x'])$. In the case where $(E', l') = (E, l)$ we abbreviate $B(E, l, E', l'; x, x')$ by $B(E, l; x, x')$.*

If H and H' are subsets of $SO(3)$ then we define

$$N(H, H') := \{r \in SO(3) : rHr^t \subset H'\} . \tag{8.54}$$

Note that, by Definition 5.2, $N(H, H')$ is nonempty iff $H \trianglelefteq H'$.

If $Iso(E, l; x) \trianglelefteq Iso(E', l'; x')$, i.e., if $N(Iso(E, l; x), Iso(E', l'; x'))$ is nonempty then we can pick $r_0 \in N(Iso(E, l; x), Iso(E', l'; x'))$ and so we get $r_0 Iso(E, l; x) r_0^t \subset Iso(E', l'; x')$. Then we define the function $\hat{\beta}[E, l, E', l'; x, x', r_0] : l(SO(3); x) \to l'(SO(3); x')$ by

$$\hat{\beta}[E, l, E', l'; x, x', r_0](l(r_1; x)) := l'(r_1 r_0^t; x') . \tag{8.55}$$

That $\hat{\beta}[E, l, E', l'; x, x', r_0]$ is a function, i.e., is single-valued, is shown below. Clearly $\hat{\beta}[E, l, E', l'; x, x', r_0]$ is a surjection. In the case where $(E', l') = (E, l)$ we abbreviate $\hat{\beta}[E, l, E', l'; x, x', r_0]$ by $\hat{\beta}[E, l; x, x', r_0]$. The ICL will show us that all elements of $B(E, l, E', l'; x, x')$ are of the form $\hat{\beta}[E, l, E', l'; x, x', r_0]$ if E and E' are Hausdorff. □

To show that $\hat{\beta}[E,l,E',l';x,x',r_0]$ is a function, i.e., is single-valued, let $r_0 \in N(Iso(E,l;x),Iso(E',l';x'))$. If $l(r_1;x) = l(r_2;x)$ then $h := r_1^t r_2 \in Iso(E,l;x)$ whence we get, by (8.55),

$$\hat{\beta}[E,l,E',l';x,x',r_0](l(r_2;x)) = l'(r_2 r_0^t;x') = l'(r_1 h r_0^t;x') = l'(r_1 r_0^t r_0 h r_0^t;x') \\ = l'(r_1 r_0^t; l'(r_0 h r_0^t;x')) = l'(r_1 r_0^t;x') = \hat{\beta}[E,l,E',l';x,x',r_0](l(r_1;x)) \; ,$$

where in the fifth equality we used that $r_0 Iso(E,l;x) r_0^t \subset Iso(E',l';x')$. Thus indeed $\hat{\beta}[E,l,E',l';x,x',r_0]$ is a function.

If H and H' are nonempty subsets of $SO(3)$ then it is an easy exercise to show that

$$N(H,H') = \bigcap_{h\in H} \bigcup_{h'\in H'} N(\{h\},\{h'\}) = \bigcap_{h\in H} N(\{h\},H') \; . \tag{8.56}$$

The sets $N(H,H')$ are well-known and will become convenient below.

Lemma 8.8 *(ICL)*
Let (E,l) and (E',l') be $SO(3)$-spaces and let E,E' be Hausdorff. Let also $x \in E$ and $x' \in E'$. Then the following hold.

a) $B(E,l,E',l';x,x')$ is nonempty iff $Iso(E,l;x) \trianglelefteq Iso(E',l';x')$. Moreover

$$B(E,l,E',l';x,x') = \left\{ \hat{\beta}[E,l,E',l';x,x',r_0] : r_0 \in N\Big(Iso(E,l;x), Iso(E',l';x') \Big) \right\} . \tag{8.57}$$

b) Let $Iso(E,l;x) \trianglelefteq Iso(E',l';x')$ and pick a $r_0 \in N(Iso(E,l;x),Iso(E',l';x'))$. Let also $y \in l(SO(3);x), y' \in l'(SO(3),x')$, i.e., $r_1,r_2 \in SO(3)$ exist such that $y = l(r_1;x)$ and $y' = l'(r_2;x')$. Then $(r_2 r_0 r_1^t) \in N(Iso(E,l;y),Iso(E',l';y'))$ and $Iso(E,l;y) \trianglelefteq Iso(E',l';y')$ as well as $\hat{\beta}[E,l,E',l';y,y',r_2 r_0 r_1^t] = \hat{\beta}[E,l,E',l';x,x',r_0]$. Moreover one can choose y,y' such that $Iso(E,l;y) \subset Iso(E',l';y')$.
Remark: Since $B(E,l,E',l';x,x') = B(E,l,E',l';y,y')$, the choice of y,y' such that $Iso(E,l;y) \subset Iso(E',l';y')$ can be helpful for the computation of $B(E,l,E',l';x,x')$ since in that case $I_{3\times3} \in N(Iso(E,l;y),Iso(E',l';y'))$ and since $\hat{\beta}[E,l,E',l';y,y',I_{3\times3}]$ is easy to handle. In fact in all our applications we make use of this choice of y,y'.

c) $Iso(E,l;x), Iso(E',l';x')$ are conjugate iff the $SO(3)$-spaces $(l'(SO(3),x'), l'_{dec}[x'])$, $(l(SO(3);x), l_{dec}[x])$ are isomorphic. Also, for every $r_0 \in SO(3)$ such that $r_0 Iso(E,l;x) r_0^t = Iso(E',l';x')$, $\hat{\beta}[E,l,E',l';x,x',r_0]$ is an isomorphism from $(l(SO(3);x), l_{dec}[x])$ to $(l'(SO(3),x'), l'_{dec}[x'])$. Moreover if $Iso(E,l;x), Iso(E',l';x')$ are conjugate then one can choose $y \in l(SO(3);x), y' \in l'(SO(3),x')$ such that $Iso(E,l;y) = Iso(E',l';y')$.
Remark: In all our applications we make use of this choice of y,y'.

Proof of Lemma 8.8: See Appendix B.1. The Hausdorff property of E,E' is needed for proving (8.57) and, as in the proof of Lemma 8.4, the compactness of $SO(3)$ is used as well.
□

The following remark mentions some interesting facts which are not addressed by Lemma 8.8 (in order to keep its proof short) and which will be confirmed by our examples.

Remark:

(17) Let (E, l) be an $SO(3)$-space, let E be Hausdorff and $x \in E$. We first mention the trivial fact that $Iso(E, l; x)$ is closed (this follows from the fact that the singleton $\{x\}$ is a closed subset of the Hausdorff space E and that $Iso(E, l; x)$ is the inverse image of $\{x\}$ under the continuous function $l(\cdot; x)$). Let also (E', l') be an $SO(3)$-space, let E' be Hausdorff and $x' \in E'$. Since $Iso(E, l; x), Iso(E', l'; x')$ are closed and $SO(3)$ is compact it follows [Ka, 1.71,1.72] that either all elements of $B(E, l, E', l'; x, x')$ are isomorphisms or none of them (by Lemma 8.8c the former case occurs iff $Iso(E, l; x), Iso(E', l'; x')$ are conjugate and then $B(E, l, E', l'; x, x')$ is the set of isomorphisms from $(l(SO(3); x), l_{dec}[x])$ to $(l'(SO(3), x'), l'_{dec}[x']))$. Moreover since $Iso(E, l; x)$ is closed and $SO(3)$ is compact it follows [Ka, 1.70] that

$$N(Iso(E, l; x), Iso(E, l; x)) = \{r_0 \in SO(3) : r_0 Iso(E, l; x) r_0^t = Iso(E, l; x)\} ,$$

which implies that $N(Iso(E, l; x), Iso(E, l; x))$ is a subgroup of $SO(3)$ and that $Iso(E, l; x), Iso(E', l'; x')$ are conjugate iff $Iso(E, l; x) \trianglelefteq Iso(E', l'; x')$ and $Iso(E', l'; x') \trianglelefteq Iso(E, l; x)$. The latter fact implies that if $Iso(E, l; x), Iso(E', l'; x')$ are not conjugate then either $B(E, l, E', l'; x, x')$ or $B(E', l', E, l; x', x)$ is empty (or both). □

Proof of Proposition 8.6: The claims follow by setting $(E, l) = (E_1, l_1)$ and $(E', l') = (E_2, l_2)$ in Lemma 8.8a and 8.8c and using Definition 8.7. □

8.3.2 The Decomposition Theorem

In this section we first state the DT which is the main corollary to Lemma 8.8. Then we show how Lemma 8.8 and the DT turn the decomposition method into a useful instrument of classifying field motions.

Theorem 8.9 *(DT)*
Let (E, l) and (E', l') be a $SO(3)$-spaces where the topological spaces E, E' are Hausdorff and let $x, x' \in E$. Moreover let $(j, A) \in \mathcal{SOS}(d, j)$. Then the following hold.

a) Let $Iso(E, l; x) \trianglelefteq Iso(E', l'; x')$ and pick $r_0 \in N(Iso(E, l; x), Iso(E', l'; x'))$. Let $f \in \mathcal{C}(\mathbb{T}^d, E)$ take values only in the (E, l)-orbit $l(SO(3); x)$ of x and let $f' \in \mathcal{C}(\mathbb{T}^d, E)$ be defined by $f' := \tilde{\mathcal{P}}[E, l, j, A](f)$. Let the functions $g, g' \in \mathcal{C}(\mathbb{T}^d, E')$ be defined by $g(z) := \hat{\beta}[E, l, E', l'; x, x', r_0](f(z))$ and $g' := \tilde{\mathcal{P}}[E', l', j, A](g)$. Then $g'(z) = \hat{\beta}[E, l, E', l'; x, x', r_0](f'(z))$. Remark: f' takes values only in $l(SO(3); x)$. Also, g, g' take values only in $l'(SO(3); x')$. Moreover if $f = f'$ then $g = g'$, i.e., if f is an invariant (E, l)-field of (j, A) then g is an invariant (E', l')-field of (j, A).

b) Let $Iso(E', l'; x'), Iso(E, l; x)$ be conjugate, i.e., $r_0 \in SO(3)$ exists such that $r_0 Iso(E, l; x) r_0^t = Iso(E', l'; x')$. Let $f \in \mathcal{C}(\mathbb{T}^d, E)$ be a function which takes values only in the

(E,l)*-orbit of* x. *Let the function* $g \in \mathcal{C}(\mathbb{T}^d, E')$ *be defined by* $g(z) := \hat{\beta}[E,l,E',l';x,x',r_0](f(z))$. *Then* f *is an invariant* (E,l)*-field of* (j,A) *iff* g *is an invariant* (E',l')*-field of* (j,A).
Remark: Thus the invariant (E',l')*-fields which take values only in* $l'(SO(3);x')$ *are redundant since they can be referred to invariant* (E,l)*-fields.*

Proof of Theorem 8.9: See Appendix B.2. □

Of course since Theorem 8.9 deals with functions which take values in only one orbit, it is naturally applied in the situation when j is topologically transitive. While the claims of Theorem 8.9 are focused on fields, it is easy to see how the corresponding statements for the particle-spin trajectories would look like.

If f in Theorem 8.9a is an invariant field then, as the theorem tells us, this is a sufficient condition for g to be invariant, too. However this is not a necessary condition as we will see by the example of the 2-snake model in Section 8.5.2. In fact Theorem 8.9a will play an active role for the 2-snake model for which we will apply it in the situation of an $f \in \mathcal{C}(\mathbb{T}^1, \mathbb{R}^3)$ which is not an ISF (in fact the 2-snake model does not have an ISF).

Remark:

(18) Clearly the central task when applying the DT to (E,l) and (E',l') is to determine for every $x \in E, x' \in E'$ whether $Iso(E,l;x) \trianglelefteq Iso(E',l';x')$, i.e., whether $N\Big(Iso(E,l;x), Iso(E',l';x')\Big)$ is nonempty. If $Iso(E,l;x), Iso(E',l';x')$ are conjugate, $l'(SO(3);x')$-valued (E',l')-fields are redundant whence, in this situation, the elements of $B(E,l,E',l';x,x')$ are of no great importance in the present work (note also that by Remark 17 that in this case all elements of $B(E,l,E',l';x,x')$ are isomorphisms). Moreover, by Remark 7, isotropy groups on the same orbit are conjugate whence the above strategy amounts to restricting ourselves to those $x \in E$ which belong a representing set of the partition E/l of E and to those $x' \in E'$ which belong a representing set of the partition E'/l' of E'. We will apply this strategy in the following to several important choices of $(E,l),(E',l')$. □

Our first application of the DT and of the strategy of Remark 18 is the case of spin-orbit resonance, i.e., when $(E,l) = (SO(3), l_{SOR})$ and where (E',l') is kept arbitrary, i.e., (E',l') is an $SO(3)$-space and E' is Hausdorff. This case will show that, on spin-orbit resonance, invariant (E',l')-fields always exist. Note that $SO(3)$ is Hausdorff whence indeed we can apply the DT. It is clear, by (8.34), that the $SO(3)$-space $(SO(3), l_{SOR})$ is transitive whence we only have one orbit and so we choose, as in Remark 9, $x = I_{3\times3}$ and recall from (8.35) that $Iso(SO(3), l_{SOR}; I_{3\times3}) = G_0$. Thus $Iso(SO(3), l_{SOR}; I_{3\times3}) = G_0 \subset Iso(E',l';x')$ whence $Iso(SO(3), l_{SOR}; I_{3\times3}) \trianglelefteq Iso(E',l';x')$. To compute $B(SO(3), l_{SOR}, E', l'; I_{3\times3}, x')$ we first note, by (8.54), that

$$N(Iso(SO(3), l_{SOR}, I_{3\times3}), Iso(E',l';x')) = N(G_0, Iso(E',l';x')) = SO(3) \; , \tag{8.58}$$

whence, by Lemma 8.8a,

$$B(SO(3), l_{SOR}, E', l'; I_{3\times3}, x') = \{\hat{\beta}[SO(3), l_{SOR}, E', l'; I_{3\times3}, x', r_0] : r_0 \in SO(3)\} \; . \tag{8.59}$$

If $r_0, r \in SO(3)$ then, by (8.34),(8.55),

$$\hat\beta[SO(3), l_{SOR}, E', l'; I_{3\times 3}, x', r_0](r) = \hat\beta[SO(3), l_{SOR}, E', l'; I_{3\times 3}, x', r_0](l_{SOR}(r; I_{3\times 3})) \\ = l'(rr_0^t; x') \; . \tag{8.60}$$

Thus if $r_0, r_1 \in SO(3)$ then $\hat\beta[SO(3), l_{SOR}, E', l'; I_{3\times 3}, x', r_0] = \hat\beta[SO(3), l_{SOR}, E', l'; I_{3\times 3}, x', r_1]$ iff, for all $r \in SO(3)$, $l'(rr_0^t; x') = l'(rr_1^t; x')$, i.e., iff $r_0 r_1^t \in Iso(E', l'; x')$. Thus $\hat\beta[SO(3), l_{SOR}, E', l'; I_{3\times 3}, x', r_0] = \hat\beta[SO(3), l_{SOR}, E', l'; I_{3\times 3}, x', r_1]$ iff $r_1^t \in r_0^t Iso(E', l'; x')$. In other words, $B(SO(3), l_{SOR}, E', l'; I_{3\times 3}, x')$ has as many elements as there are left cosets $rIso(E', l'; x')$. To apply the DT let $(j, A) \in \mathcal{SOS}(d, j)$, let $f \in \mathcal{C}(\mathbb{T}^d, SO(3))$ and let $g \in \mathcal{C}(\mathbb{T}^d, E')$ be defined by $g(z) := \hat\beta[SO(3), l_{SOR}, E', l'; I_{3\times 3}, x', r_0](f(z))$ for fixed but arbitrary $r_0 \in SO(3)$. Since $(SO(3), l_{SOR})$ is transitive, all values of f are in the $(SO(3), l_{SOR})$-orbit of $I_{3\times 3}$. Note also that g takes values only in the (E', l')-orbit of x'. Let f be an invariant $(SO(3), l_{SOR})$-field of (j, A) (note, by Remark 9, such an f is a uniform IFF of (j, A), i.e., it exists iff (j, A) is on spin-orbit resonance). It follows from Theorem 8.9a that g is an invariant (E', l')-field of (j, A). Thus on spin-orbit resonance invariant (E', l')-fields always exist if E' is Hausdorff. In the subcase $(E', l') = (\mathbb{R}^3, l_v)$ this result is not surprising because if we pick $x' = (0, 0, 1)^t$ then, by the IFF Theorem, the third column of the uniform IFF f is an ISF! We finally look if we can apply Theorem 8.9b as well. In fact, for every $r \in SO(3)$, $rI_{3\times 3}r^t = I_{3\times 3}$ whence G_0 is conjugate to $Iso(E', l'; x')$ only in the exceptional case when $Iso(E', l'; x') = G_0$ (e.g., if $(E', l') = (SO(3), l_{SOR})$).

8.3.3 Applying the Decomposition Theorem in the case $(E, l) = (E', l') = (\mathbb{R}^3, l_v)$

In this section we apply the DT to the case $(E, l) = (E', l') = (\mathbb{R}^3, l_v)$. This case serves to illustrate the basic technique because it is more involved than our first example above. However, unlike the above example, it does not add to our basic knowledge about spin motion. Thus the reader who is interested in the more important applications of the DT in Sections 8.4-8.5 may go straight to those sections.

We will see that the representing set $R_v = \{S_\lambda : \lambda \in [0, \infty)\} = \{\lambda(0, 0, 1)^t : \lambda \in [0, \infty)\}$ of the partition $\mathbb{R}^3/l_v$ of $\mathbb{R}^3$ is very convenient (see also Remark 10). Choosing R_v we must determine for given $\lambda, \mu \in [0, \infty)$ whether $Iso(\mathbb{R}^3, l_v; S_\lambda) \trianglelefteq Iso(\mathbb{R}^3, l_v; S_\mu)$ or whether $Iso(\mathbb{R}^3, l_v; S_\lambda), Iso(\mathbb{R}^3, l_v; S_\mu)$ are even conjugate.

To compute the isotropy groups we use (2.45) and (8.4) to get, for $\lambda \in [0, \infty)$,

$$Iso(\mathbb{R}^3, l_v; S_\lambda) = \{r \in SO(3) : l_v(r; S_\lambda) = S_\lambda\} \\ = \{r \in SO(3) : \lambda r(0, 0, 1)^t = \lambda(0, 0, 1)^t\} = \begin{cases} SO(2) & \text{if } \lambda > 0 \\ SO(3) & \text{if } \lambda = 0 \; , \end{cases} \tag{8.61}$$

where again we used the $SO(2)$-Lemma, Theorem 5.4a. Since the group $SO(2)$ is convenient to deal with, we see by (8.61) that it was prescient to have chosen the S_λ to be the elements of R_v. With (8.61) we are led to consider the following four separate cases: $\lambda > 0, \mu > 0$; $\lambda = 0, \mu = 0$; $\lambda = 0, \mu > 0$; $\lambda > 0, \mu = 0$.

We first consider the case when $\lambda, \mu > 0$ and we will find that $B(\mathbb{R}^3, l_v; S_\lambda, S_\mu)$ has only two elements. Thus, by (8.61), $Iso(\mathbb{R}^3, l_v; S_\lambda) = SO(2) = Iso(\mathbb{R}^3, l_v; S_\mu)$ whence

$I_{3\times 3} Iso(\mathbb{R}^3, l_v; S_\lambda) I^t_{3\times 3} = Iso(\mathbb{R}^3, l_v; S_\mu)$ so that
$Iso(\mathbb{R}^3, l_v; S_\lambda)$ and $Iso(\mathbb{R}^3, l_v; S_\mu)$ are conjugate. To compute $B(\mathbb{R}^3, l_v; S_\lambda, S_\mu)$ we first note, by (8.61), that

$$N(Iso[\mathbb{R}^3, l_v; S_\lambda], Iso[\mathbb{R}^3, l_v; S_\mu]) = N(SO(2), SO(2)) \ . \tag{8.62}$$

To compute $N(SO(2), SO(2))$ let $r_0 \in N(SO(2), SO(2))$ so that, by (8.54), $r_0 SO(2) r_0^t \subset SO(2)$ whence, by (5.5), for every $\nu \in \mathbb{R}$ there exists an $\nu' \in \mathbb{R}$ such that $r_0 \exp(2\pi\nu\mathcal{J}) r_0^t = \exp(2\pi\nu'\mathcal{J})$. Thus

$$\exp(2\pi\nu\mathcal{J}) r_0^t (0,0,1)^t = r_0^t (0,0,1)^t \ , \tag{8.63}$$

and it is clear that if $r_0 \in N(SO(2), SO(2))$ then (8.63) holds for every $\nu \in \mathbb{R}$. This implies that if $r_0 \in N(SO(2), SO(2))$ then (8.63) holds for $\nu = 1/2$, i.e.,

$$\mathrm{diag}(-1,-1,1) r_0^t (0,0,1)^t = r_0^t (0,0,1)^t \ , \tag{8.64}$$

where, for $y_1, y_2, y_3 \in \mathbb{R}$, we use the abbreviation
$\mathrm{diag}(y_1, y_2, y_3) := \begin{pmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & y_3 \end{pmatrix}$. It follows from (8.64) that $r_0^t(0,0,1)^t$ is a normalized eigenvector of $\mathrm{diag}(-1,-1,1)$ with eigenvalue 1 whence $r_0^t(0,0,1)^t = \pm(0,0,1)^t$. If $r_0^t(0,0,1)^t = (0,0,1)^t$ then, by the $SO(2)$-Lemma, $r_0 \in SO(2)$ and if $r_0^t(0,0,1)^t = -(0,0,1)^t$ then, by the $SO(2)$-Lemma, $r_0 \in \mathcal{K}_1 SO(2)$ where

$$\mathcal{K}_0 := I_{3\times 3} \ , \ \ \mathcal{K}_1 := \mathrm{diag}(1,-1,-1) \ , \ \ \mathcal{K}_2 := \mathrm{diag}(-1,-1,1) \ , \ \ \mathcal{K}_3 := \mathrm{diag}(-1,1,-1) \ , \tag{8.65}$$

and where $\mathcal{K}_0, \mathcal{K}_2, \mathcal{K}_3$ will come into play later. Thus we have shown that
$N(SO(2), SO(2)) \subset (SO(2) \bowtie \mathbb{Z}_2)$ where

$$SO(2) \bowtie \mathbb{Z}_2 := \{rr' : r \in \mathbb{Z}_2, r' \in SO(2)\} \ , \quad \mathbb{Z}_2 := \{\mathcal{K}_0, \mathcal{K}_1\} \ . \tag{8.66}$$

It is a simple exercise to show that $SO(2) \bowtie \mathbb{Z}_2$ is a subgroup of $SO(3)$, the so-called "knit product" (or "Zappa-Szep product") of the subgroups $SO(2)$ and $\mathbb{Z}_2$ of $SO(3)$. It is also an easy exercise to show that $N(SO(2), SO(2)) \supset (SO(2) \bowtie \mathbb{Z}_2)$ whence, by (8.62),

$$N(Iso[\mathbb{R}^3, l_v; S_\lambda], Iso[\mathbb{R}^3, l_v; S_\mu]) = N(SO(2), SO(2)) = SO(2) \bowtie \mathbb{Z}_2 \ . \tag{8.67}$$

Thus $N(Iso[\mathbb{R}^3, l_v; S_\lambda], Iso[\mathbb{R}^3, l_v; S_\mu])$ is a group. Since $Iso(\mathbb{R}^3, l_v; S_\lambda)$ and $Iso(\mathbb{R}^3, l_v; S_\mu)$ are conjugate, this group property is no surprise due to Remark 17. It follows from (8.66), (8.67) and Lemma 8.8a that

$$\begin{aligned} B(\mathbb{R}^3, l_v; S_\lambda, S_\mu) &= B(\mathbb{R}^3, l_v, \mathbb{R}^3, l_v; S_\lambda, S_\mu) = \{\hat\beta[\mathbb{R}^3, l_v; S_\lambda, S_\mu, r_0] : r_0 \in (SO(2) \bowtie \mathbb{Z}_2)\} \\ &= \{\hat\beta[\mathbb{R}^3, l_v; S_\lambda, S_\mu, r_0] : r_0 \in SO(2)\} \cup \{\hat\beta[\mathbb{R}^3, l_v; S_\lambda, S_\mu, r_0] : r_0 \in \mathcal{K}_1 SO(2)\} \ . \end{aligned} \tag{8.68}$$

We now show that both sets on the rhs of (8.68) are singletons. If $r_0 \in SO(2)$ and $r \in SO(3)$ then, by (8.55),

$$\hat\beta[\mathbb{R}^3, l_v; S_\lambda, S_\mu, r_0](l_v(r; S_\lambda)) = l_v(r r_0^t; S_\mu) = l_v(r; l_v(r_0^t; S_\mu))$$

$$= l_v(r; S_\mu) = \hat{\beta}[\mathbb{R}^3, l_v; S_\lambda, S_\mu, I_{3\times 3}](l_v(r; S_\lambda)) \ , \tag{8.69}$$

i.e.,

$$\{\hat{\beta}[\mathbb{R}^3, l_v; S_\lambda, S_\mu, r_0] : r_0 \in SO(2)\} = \{\hat{\beta}[\mathbb{R}^3, l_v; S_\lambda, S_\mu, I_{3\times 3}]\} \ , \tag{8.70}$$

where in the third equality of (8.69) we used the relation $SO(2) = Iso(\mathbb{R}^3, l_v; S_\mu)$ from (8.61). In analogy to (8.69), if $r_0 \in \mathcal{K}_1 SO(2)$, i.e., $r_0 = \mathcal{K}_1 r_1$ with $r_1 \in SO(2)$ and if $r \in SO(3)$ then, by (8.55) and (8.65),

$$\begin{aligned}
\hat{\beta}[\mathbb{R}^3, l_v; S_\lambda, S_\mu, r_0](l_v(r; S_\lambda)) &= l_v(rr_0^t; S_\mu) = l_v(rr_1^t \mathcal{K}_1; S_\mu) \\
&= l_v(rr_1^t; l_v(\mathcal{K}_1; S_\mu)) = -l_v(rr_1^t; S_\mu) = -l_v(r; l_v(r_1^t; S_\mu)) = -l_v(r; S_\mu) \\
&= l_v(r; l_v(\mathcal{K}_1; S_\mu)) = l_v(r\mathcal{K}_1; S_\mu) = \hat{\beta}[\mathbb{R}^3, l_v; S_\lambda, S_\mu, \mathcal{K}_1](l_v(r; S_\lambda)) \ ,
\end{aligned} \tag{8.71}$$

i.e.,

$$\{\hat{\beta}[\mathbb{R}^3, l_v; S_\lambda, S_\mu, r_0] : r_0 \in \mathcal{K}_1 SO(2)\} = \{\hat{\beta}[\mathbb{R}^3, l_v; S_\lambda, S_\mu, \mathcal{K}_1]\} \ . \tag{8.72}$$

In the sixth equality of (8.71) we again used the relation $SO(2) = Iso(\mathbb{R}^3, l_v; S_\mu)$ from (8.61). We conclude from (8.68),(8.70) and (8.72) that $B(\mathbb{R}^3, l_v; S_\lambda, S_\mu)$ has only two elements:

$$B(\mathbb{R}^3, l_v; S_\lambda, S_\mu) = \{\hat{\beta}[\mathbb{R}^3, l_v; S_\lambda, S_\mu, \mathcal{K}_0], \hat{\beta}[\mathbb{R}^3, l_v; S_\lambda, S_\mu, \mathcal{K}_1]\} \ . \tag{8.73}$$

We now take a closer look at these two elements and we first note that if $r_0 \in (SO(2) \bowtie \mathbb{Z}_2)$ then, by (8.4) and (8.55),

$$\hat{\beta}[\mathbb{R}^3, l_v; S_\lambda, S_\mu, r_0](rS_\lambda) = rr_0^t S_\mu \ . \tag{8.74}$$

It follows from (8.74) and Remark 10 that, for $r \in SO(3)$, $\hat{\beta}[\mathbb{R}^3, l_v; S_\lambda, S_\mu, I_{3\times 3}](\lambda r(0,0,1)^t) = \mu r(0,0,1)^t$, whence, for every S in the domain $\mathbb{S}^2_\lambda$ of $\hat{\beta}[\mathbb{R}^3, l_v; S_\lambda, S_\mu, I_{3\times 3}]$,

$$\hat{\beta}[\mathbb{R}^3, l_v; S_\lambda, S_\mu, I_{3\times 3}](S) = \frac{\mu}{\lambda} S \ . \tag{8.75}$$

It also follows from (8.74) and Remark 10 that, for $r \in SO(3)$, $\hat{\beta}[\mathbb{R}^3, l_v; S_\lambda, S_\mu, \mathcal{K}_1](\lambda r(0,0,1)^t) = \mu r \mathcal{K}_1 (0,0,1)^t = -\mu r(0,0,1)^t$ whence, for every S in the domain $\mathbb{S}^2_\lambda$ of $\hat{\beta}[\mathbb{R}^3, l_v; S_\lambda, S_\mu, \mathcal{K}_1]$,

$$\hat{\beta}[\mathbb{R}^3, l_v; S_\lambda, S_\mu, \mathcal{K}_1](S) = -\frac{\mu}{\lambda} S \ . \tag{8.76}$$

With (8.73), (8.75) and (8.76) it is a simple exercise to show that both elements of $B(\mathbb{R}^3, l_v; S_\lambda, S_\mu)$ are not only topological $SO(3)$-maps but also isomorphisms (and, since $Iso(\mathbb{R}^3, l_v; S_\lambda)$ and $Iso(\mathbb{R}^3, l_v; S_\mu)$ are conjugate, this is predicted by Remark 17). To apply the DT let $(j, A) \in \mathcal{SOS}(d, j)$, let $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^d)$ take values only in the $(\mathbb{R}^3, l_v)$-orbit $\mathbb{S}^2_\lambda$ of S_λ and let $g_0, g_1 \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^d)$ be defined by $g_0(z) := \hat{\beta}[\mathbb{R}^3, l_v; S_\lambda, S_\mu, \mathcal{K}_0](f(z))$ and $g_1(z) := \hat{\beta}[\mathbb{R}^3, l_v; S_\lambda, S_\mu, \mathcal{K}_1](f(z))$. Then g_0 and g_1 take values only in the $(\mathbb{R}^3, l_v)$-orbit $\mathbb{S}^2_\mu$ of S_μ. Moreover, by Theorem 8.9b, f

is an invariant polarization field of (j, A) iff g_0 is an invariant polarization field of (j, A) and f is an invariant polarization field of (j, A) iff g_1 is an invariant polarization field of (j, A). This completes our treatment of the first case.

We now consider the case when $\lambda = 0, \mu = 0$. Then, by (8.61), $Iso(\mathbb{R}^3, l_v; S_\lambda) = SO(3) = Iso(\mathbb{R}^3, l_v; S_\mu)$ whence $I_{3\times3} Iso(\mathbb{R}^3, l_v; S_\lambda) I^t_{3\times3} = Iso(\mathbb{R}^3, l_v; S_\mu)$ so that $Iso(\mathbb{R}^3, l_v; S_\lambda)$ and $Iso(\mathbb{R}^3, l_v; S_\mu)$ are conjugate. Since the $(\mathbb{R}^3, l_v)$-orbit $\mathbb{S}^2_0$ of $S_0 = (0, 0, 0)^t$ only contains S_0, the only element of $B(\mathbb{R}^3, l_v; S_\lambda, S_\mu)$ is the constant $(0, 0, 0)^t$-valued function. Since $Iso(\mathbb{R}^3, l_v; S_\lambda)$ and $Iso(\mathbb{R}^3, l_v; S_\mu)$ are conjugate it is no surprise that the only element of $B(\mathbb{R}^3, l_v; S_\lambda, S_\mu)$ is an isomorphism (see Remark 17).

We now consider the case when $\lambda = 0, \mu > 0$. Thus, by (8.61), $Iso(\mathbb{R}^3, l_v; S_\lambda) = SO(3), Iso(\mathbb{R}^3, l_v; S_\mu) = SO(2)$. Since, for every $r_0 \in SO(3)$, one has $r_0 SO(3) r_0^t = SO(3)$ we conclude from (8.54) that $N(SO(3), SO(2))$ is empty so that $N(Iso(\mathbb{R}^3, l_v; S_\lambda), Iso(\mathbb{R}^3, l_v; S_\mu))$ is empty which implies, by Lemma 8.8a, that $B(\mathbb{R}^3, l_v; S_\lambda, S_\mu)$ is empty. Note also that the only subgroup of $SO(3)$ which is conjugate to $SO(3)$ is $SO(3)$. Thus $SO(2)$ and $SO(3)$ are not conjugate so that the emptiness of $N(SO(3), SO(2))$ is predicted by Remark 17.

We finally consider the case when $\lambda > 0, \mu = 0$. Then, by (8.61), $Iso(\mathbb{R}^3, l_v; S_\lambda) = SO(2), Iso(\mathbb{R}^3, l_v; S_\mu) = SO(3)$ whence, by (8.54), $N(Iso(\mathbb{R}^3, l_v; S_\lambda), Iso(\mathbb{R}^3, l_v; S_\mu)) = SO(3)$. Recalling from above that $SO(2)$ and $SO(3)$ are not conjugate we observe that $Iso(\mathbb{R}^3, l_v; S_\lambda)$ and $Iso(\mathbb{R}^3, l_v; S_\mu)$ are not conjugate. As in the second case, the only element of $B(\mathbb{R}^3, l_v; S_\lambda, S_\mu)$ is the constant $(0, 0, 0)^t$-valued function. Since $Iso(\mathbb{R}^3, l_v; S_\lambda) \trianglelefteq Iso(\mathbb{R}^3, l_v; S_\mu)$ and since $Iso(\mathbb{R}^3, l_v; S_\lambda)$ and $Iso(\mathbb{R}^3, l_v; S_\mu)$ are not conjugate it is no surprise that the only element of $B(\mathbb{R}^3, l_v; S_\lambda, S_\mu)$ is a topological $SO(3)$-map which is not an isomorphism (see Remark 17).

We finally mention some further features of the example $(E, l) = (E', l') = (\mathbb{R}^3, l_v)$:

Remark:

(19) It follows from (8.61) that for given $\lambda, \mu \in [0, \infty)$ either $Iso(\mathbb{R}^3, l_v; S_\lambda) \trianglelefteq Iso(\mathbb{R}^3, l_v; S_\mu)$ or $Iso(\mathbb{R}^3, l_v; S_\mu) \trianglelefteq Iso(\mathbb{R}^3, l_v; S_\lambda)$. This is quite remarkable since in general two subgroups of $SO(3)$ are not related by $\trianglelefteq$. Thus more generally, by Lemma 8.8b, for arbitrary $S, S' \in \mathbb{R}^3$, either $Iso(\mathbb{R}^3, l_v; S) \trianglelefteq Iso(\mathbb{R}^3, l_v; S')$ or $Iso(\mathbb{R}^3, l_v; S') \trianglelefteq Iso(\mathbb{R}^3, l_v; S)$. We also see by (8.61) that if $Iso(\mathbb{R}^3, l_v; S_\lambda) \trianglelefteq Iso(\mathbb{R}^3, l_v; S_\mu)$ then $Iso(\mathbb{R}^3, l_v; S_\lambda) \subset Iso(\mathbb{R}^3, l_v; S_\mu)$. The latter inclusion is another reason why we have chosen the S_λ to be the elements of R_v (note also that this inclusion is predicted by Lemma 8.8b). Since the S_λ are the elements of a representing set of the partition $\mathbb{R}^3/l_v$ of $\mathbb{R}^3$ it also follows from (8.32) and (8.61) that every isotropy group of $(\mathbb{R}^3, l_v)$ is either conjugate to $SO(2)$ or to $SO(3)$. □

We have thus shown in the simple example of the this section how the DT classifies invariant fields in terms of the isotropy groups of the $SO(3)$-spaces (E, l) and (E', l') at hand and how Definition 8.7 and the ICL play a key role. So, for example, for the case $\lambda > 0, \mu > 0$, we decomposed (E, l_v) into the two (E, l_v) orbits $l_v(SO(3); S_\lambda)$ and $l_v(SO(3); S_\mu)$ (both of them spheres of nonzero radius). Then we showed how invariant fields with values confined to these spheres are related and thus classified via the functions $\hat{\beta}[\mathbb{R}^3, l_v; S_\lambda, S_\mu, I_{3\times3}]$ and $\hat{\beta}[\mathbb{R}^3, l_v; S_\lambda, S_\mu, \mathcal{K}_1]$ which in fact are the only ones there are. A key role was played by

$N(Iso[\mathbb{R}^3, l_v; S_\lambda], Iso[\mathbb{R}^3, l_v; S_\mu])$ which turned out to be the subgroup $SO(2) \rtimes \mathbb{Z}_2$ of $SO(3)$. This will also play a major role in Section 8.4 (but for a different reason). We also saw that the invariant fields with values confined to the sphere S_μ are redundant since both betas are isomorphisms (and this implied that the subset $N(Iso[\mathbb{R}^3, l_v; S_\lambda], Iso[\mathbb{R}^3, l_v; S_\mu])$ of $SO(3)$ is a group).

8.4 Applying the ToA to (E_t, l_t)

The $SO(3)$-space $(E, l) = (\mathbb{R}^3, l_v)$ is needed for describing polarized beams of arbitrary nonzero spin. However, it does not always suffice. In particular, for spin-1 particles like deuterons [BV2] we need a framework for handling the spin tensor variable M which is a real, symmetric and traceless 3×3 matrix. See [BV2] for the dynamics of M under the influence of the T-BMT equation. This inspires (8.77) below which leads to the correct 1-turn map in (8.78). See [BV2] and Section 8.6.2 for the way in which M appears in the spin-1 density matrix function. So in this section we introduce the $SO(3)$-space (E_t, l_t) to encompass the spin tensor and allow us to use the ToA for spin-1 particles [BV2]. As in Section 8.3 the focus is on the field motion.

We will proceed as follows. In Section 8.4.1, after defining (E_t, l_t), we will obtain the representing set R_t of the partition E_t/l_t of E_t and compute the isotropy groups of (E_t, l_t) allowing us to apply the DT in the case $(E, l) = (E', l') = (E_t, l_t)$. Then in Section 8.4.2 we apply the NFT to (E_t, l_t).

8.4.1 Basic properties of (E_t, l_t)

We define $E_t := \{M \in \mathbb{R}^{3\times3} : M^t = M, Tr[M] = 0\}$ and equip E_t with the subspace topology from $\mathbb{R}^{3\times3}$. Thus, and since $\mathbb{R}^{3\times3}$ with its natural topology is a Hausdorff space, E_t is a Hausdorff space, too. We also define the function $l_t : SO(3) \times E_t \to E_t$ by

$$l_t(r; M) := rMr^t \, , \tag{8.77}$$

with $r \in SO(3), M \in E_t$. It is an easy exercise to show that (E_t, l_t) is an $SO(3)$-space. Note that matrices, which belong to the same (E_t, l_t)-orbit, are similar, in particular they have the same number of distinct eigenvalues. If $M \in E_t$ then we denote by $\#(M)$ the number of its distinct eigenvalues.

The 1-turn map (8.1) in the present case is $\mathcal{P}[E_t, l_t, j, A]$, given by

$$\mathcal{P}[E_t, l_t, j, A](z, M) = \begin{pmatrix} j(z) \\ A(z)MA(z) \end{pmatrix} , \tag{8.78}$$

and the 1-turn field map (8.2) by

$$\tilde{\mathcal{P}}[E_t, l_t, j, A](f) = (AfA^t) \circ j^{-1} \, . \tag{8.79}$$

In the following remark we compute the (E_t, l_t)-orbits, giving us the partition E_t/l_t of E_t.

Remark:

(20) It follows from (8.77), Definition 2.4 and some simple Linear Algebra [He1] that the (E_t, l_t)-orbit of an arbitrary $M \in E_t$ reads as

$$l_t(SO(3); M) = \{M' \in E_t : (\det(M'), Tr[M'^2]) = (\det(M), Tr[M^2])\} . \tag{8.80}$$

Note that (8.80) follows easily from the fact that the characteristic polynomial of M is the function $\det(M - xI_{3\times 3}) = -x^3 + \frac{1}{2}Tr[M^2]x + \det(M)$ and that this polynomial is the same for all elements of the (E_t, l_t)-orbit of M. We now define

$$R_t := \Big\{ \mathrm{diag}(y_1, y_2, -y_1 - y_2) : (y_1, y_2) \in (\Lambda_1 \cup \Lambda_2 \cup \Lambda_3) \Big\} \subset E_t , \tag{8.81}$$

where $\Lambda_1 := \{(0,0)\}, \Lambda_2 := \{(y,y) : 0 \neq y \in \mathbb{R}\}, \Lambda_3 := \{(y,y') : y \in (0,\infty), y' \in (-y/2, y)\}$. The matrices $\mathrm{diag}(y_1, y_2, -y_1 - y_2)$ with $(y_1, y_2) \in \Lambda_j$ have a simple interpretation: they are those matrices M in R_t which have $\#(M) = j$. This implies, since matrices of the same (E_t, l_t)-orbit are similar, that an arbitrary matrix $M \in E_t$ has $\#(M) = j$ if its (E_t, l_t)-orbit contains a matrix $M' = \mathrm{diag}(y_1, y_2, -y_1 - y_2)$ with $(y_1, y_2) \in \Lambda_j$ [He1] (of course M' is unique). By (8.80), each element of R_t belongs to a different (E_t, l_t)-orbit. Moreover, by some simple Linear Algebra, one can show [He1] that every element M of E_t belongs to the (E_t, l_t)-orbit of some $M' \in R_t$. In other words, R_t is a representing set of the partition E_t/l_t of E_t (recall from Remark 10 that R_v is a representing set of the partition $\mathbb{R}^3/l_v$ of $\mathbb{R}^3$). As with R_v, the choice R_t is very convenient as will become clear below. □

Note that the above technique of using $\det(M), Tr[M^2]$ as "invariants" of M is also used sometimes for the emittance matrix in four-dimensional linear beam optics.

Remark 20 allows us, in the following remark, to parametrize the elements of E_t in terms of normalized vectors.

Remark:

(21) By the above, the set of those M in E_t for which $\#(M) = j$ is given by

$$\{r\mathrm{diag}(y_1, y_2, -y_1 - y_2)r^t : r \in SO(3), (y_1, y_2) \in \Lambda_j\} , \tag{8.82}$$

and, for $(y_1, y_2) \in \mathbb{R}^2$,

$$\begin{aligned} \mathrm{diag}(y_1, y_2, -y_1 - y_2) &= y_1 I_{3\times 3} + \mathrm{diag}(0, y_2 - y_1, -2y_1 - y_2) \\ &= y_1 I_{3\times 3} + (y_2 - y_1)(0,1,0)(0,1,0)^t - (2y_1 + y_2)(0,0,1)(0,0,1)^t . \end{aligned} \tag{8.83}$$

□

The following remark shows the impact of Lemma 8.4 on invariant (E_t, l_t)-fields.

Remark:

(22) Let $(j, A) \in \mathcal{SOS}(d, j)$ where j is topologically transitive. Let f be an invariant (E_t, l_t)-field of (j, A). Then, by Lemma 8.4, f takes values in only one (E_t, l_t)-orbit, say $l_t(SO(3); M)$. By Remark 20 we can choose M to belong to R_t, i.e.,

$M = \mathrm{diag}(y_1, y_2, -y_1 - y_2)$ where $(y_1, y_2) \in \Lambda_i$ with $\#(M) = i$. Thus, by Remark 21, a function $T : \mathbb{T}^d \to SO(3)$ exists such that $f(z) = l_t(T(z); M)$ and

$$f(z) = y_1 I_{3\times 3} + (y_2 - y_1)\tilde{k}(z)\tilde{k}^t(z) - (2y_1 + y_2)k(z)k^t(z) , \tag{8.84}$$

where the functions $k, \tilde{k} : \mathbb{T}^d \to \mathbb{R}^3$ are defined by

$$k(z) := T(z)(0, 0, 1)^t , \quad \tilde{k}(z) := T(z)(0, 1, 0)^t . \tag{8.85}$$

Of course $\#(f(z)) = i$ for all $z \in \mathbb{T}^d$. In the special case where $i = 2$, i.e., $y_1 = y_2 =: y$ with $0 \neq y \in \mathbb{R}$, (8.84) reads as

$$f(z) = yI_{3\times 3} - 3yk(z)k^t(z) , \tag{8.86}$$

and in the special case where $i = 1$, i.e., $M = \mathrm{diag}(0, 0, 0)$, (8.84) reads as $f(z) = \mathrm{diag}(0, 0, 0)$. □

Using (2.45),(5.5),(8.66), (8.77) and (8.81) it is a simple exercise [He1] to show that for $M \in R_t$

$$Iso(E_t, l_t; M) = \begin{cases} SO(3) & \text{if } \#(M) = 1 \\ SO(2) \bowtie \mathbb{Z}_2 & \text{if } \#(M) = 2 \\ SO_{diag}(3) & \text{if } \#(M) = 3 , \end{cases} \tag{8.87}$$

where

$$SO_{diag}(3) := SO(3) \cap \{\mathrm{diag}(y_1, y_2, y_3) : y_1, y_2, y_3 \in \mathbb{R}\} = \{\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3\} . \tag{8.88}$$

Clearly $SO_{diag}(3)$ is a subgroup of $SO(3)$ and it is the set of diagonal matrices in $SO(3)$. Note that, by (8.66) and (8.88),

$$SO_{diag}(3) \subset (SO(2) \bowtie \mathbb{Z}_2) \subset SO(3) . \tag{8.89}$$

Since the groups $SO(2) \bowtie \mathbb{Z}_2$ and $SO_{diag}(3)$ are conveniently handled, we see by (8.87) that it was prescient to have chosen R_t as in (8.81).

The following remark mentions some implications of (8.87).

Remark:

(23) We conclude from (8.87) and (8.89) that, if $M, M' \in R_t$, then $Iso(E_t, l_t; M) \trianglelefteq Iso(E_t, l_t; M')$ iff $\#(M) \geq \#(M')$ (and, by Lemma 8.8b, this holds for arbitrary $M, M' \in E_t$ since R_t is a representing set of E_t/l_t). Thus, quite remarkably we see that for all $M, M' \in E_t$ either $Iso(E_t, l_t; M) \trianglelefteq Iso(E_t, l_t; M')$ or $Iso(E_t, l_t; M') \trianglelefteq Iso(E_t, l_t; M)$.

We also see, by (8.87) and (8.89), that if $M, M' \in R_t$ and $Iso(E_t, l_t; M) \trianglelefteq Iso(E_t, l_t; M')$ then $Iso(E_t, l_t; M) \subset Iso(E_t, l_t; M')$. The latter inclusion is another reason why we have chosen R_t as in (8.81) (note also that this inclusion is predicted by Lemma 8.8b). We recall from Section 8.3.3 that $SO(3)$ is only conjugate to itself. Also, by Definition 2.2, the finite group $SO_{diag}(3)$ is not conjugate to infinite groups and the infinite group

$SO(2) \bowtie \mathbb{Z}_2$ is not conjugate to finite groups. Thus it follows from (8.87) that, if $M, M' \in R_t$, then $Iso(E_t, l_t; M)$ and $Iso(E_t, l_t; M')$ are conjugate iff $\#(M) = \#(M')$ (and this holds for arbitrary $M, M' \in E_t$ since R_t is a representing set of E_t/l_t). In fact for arbitrary $M \in E_t$ we have $\#(M) = 1$ iff $Iso(E_t, l_t; M)$ is conjugate to $SO(3)$, we have $\#(M) = 2$ iff $Iso(E_t, l_t; M)$ is conjugate to $SO(2) \bowtie \mathbb{Z}_2$, and we have $\#(M) = 3$ iff $Iso(E_t, l_t; M)$ is conjugate to $SO_{diag}(3)$. □

Recalling the remarks after the NFT, the isotropy groups in (8.87) will give us insight into the possibility of invariant (E_t, l_t)-fields. In fact if (j, A) has an ISF then, as we will see in Section 8.5.1, (j, A) has an invariant (E_t, l_t)-field whose values are matrices M with $\#(M) = 2$. This follows from the fact that $SO(2) \subset (SO(2) \bowtie \mathbb{Z}_2)$. Thus, by the remarks after the NFT, we believe that in practice invariant (E_t, l_t)-fields exist whose values are matrices M with $\#(M) = 2$. On the other hand, by (8.87) and the remarks after the NFT, an invariant (E_t, l_t)-field which has an (E_t, l_t)-lift and whose values are matrices M with $\#(M) = 3$ can only exist if (j, A) has an $SO_{diag}(3)$-normal form (for the notion of "lift", see the remarks after the NFT). However $SO_{diag}(3)$ is "small" since it contains only four elements thus we expect that (j, A) has an $SO_{diag}(3)$-normal form only if it has a G_0-normal form, i.e., if (j, A) is on spin-orbit resonance. Since in practice we expect that the assumption of an (E_t, l_t)-lift is not strong, we expect that in practice invariant (E_t, l_t)-fields whose values are matrices M with $\#(M) = 3$ only exist on spin-orbit resonance. On spin-orbit resonance those invariant (E_t, l_t)-fields exist, as is explained after Remark 18. Note also that the $\text{diag}(0, 0, 0)$-valued function is always an invariant (E_t, l_t)-field whence invariant (E_t, l_t)-fields always exist whose values are matrices M with $\#(M) = 1$. Note that $\#(M) = 1$ is of physical importance as can be seen for example in Section 8.6 where spin tensors M appear as coefficients in the density matrix functions of spin-1 particles.

We now apply the DT to the case $(E, l) = (E', l') = (E_t, l_t)$. This is interesting since it deals with invariant (E_t, l_t)-fields and since the latter have been much less studied than invariant polarization fields. We follow the strategy of Remark 18 and use again the convenient representing set R_t of the partition E_t/l_t. Thus we have to determine for given $M, M' \in R_t$ whether $Iso(E_t, l_t; M) \trianglelefteq Iso(E_t, l_t; M')$ or whether $Iso(E_t, l_t; M)$ and $Iso(E_t, l_t; M')$ are even conjugate. In fact by Remark 23 we know that $Iso(E_t, l_t; M) \trianglelefteq Iso(E_t, l_t; M')$ iff $\#(M) \geq \#(M')$ and that $Iso(E_t, l_t; M)$ and $Iso(E_t, l_t; M')$ are conjugate iff $\#(M) = \#(M')$. Recalling Remark 18, the latter case compares invariant (E_t, l_t)-fields which have identical behavior. To keep the discussion short we confine ourselves to the case where $i = 3, k = 2$ and which is interesting since it involves the two most important isotropy groups of (E_t, l_t): $SO_{diag}(3)$ and $SO(2) \bowtie \mathbb{Z}_2$.

The computations are analogous to those in Section 8.3.3. So we focus here on the results and leave the somewhat lengthy Linear Algebra [He1] to the reader. In fact, by Definition 8.7, one gets

$$\begin{aligned} N([Iso(E_t, l_t; M), Iso(E_t, l_t; M')) &= N(SO_{diag}(3), SO(2) \bowtie \mathbb{Z}_2) \\ &= (SO(2) \bowtie \mathbb{Z}_2) \cup (SO(2) \bowtie \mathbb{Z}_2)\mathcal{K}_4 \cup (SO(2) \bowtie \mathbb{Z}_2)\mathcal{K}_5 \,, \end{aligned} \tag{8.90}$$

where

$$\mathcal{K}_4 := \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \quad \mathcal{K}_5 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} . \tag{8.91}$$

Thus, by Lemma 8.8a, and some further Linear Algebra one gets

$$B(E_t, l_t, M, M') = B(E_t, l_t, E_t, l_t, M, M') = \{\hat{\beta}[E_t, l_t; M, M', \mathcal{K}_0], \hat{\beta}[E_t, l_t; M, M', \mathcal{K}_4], \\ \hat{\beta}[E_t, l_t; M, M', \mathcal{K}_5]\} . \tag{8.92}$$

Since $SO(2) \subset (SO(2) \bowtie \mathbb{Z}_2)$ it is no surprise by Lemma 8.8a that $B(E_t, l_t, M, M')$ is nonempty. We now can apply the DT to the current case. So let $f \in \mathcal{C}(\mathbb{T}^d, E_t)$ take values only in the (E_t, l_t)-orbit of M, i.e., let a function $T : \mathbb{T}^d \to SO(3)$ exist such that

$$f(z) := l_t(T(z), M) . \tag{8.93}$$

Then f satisfies (8.84), i.e., $f(z) = y_1 I_{3\times3} + (y_2 - y_1)\tilde{k}(z)\tilde{k}^t(z) - (2y_1 + y_2)k(z)k^t(z)$ where the functions $k, \tilde{k} : \mathbb{T}^d \to \mathbb{R}^3$ are defined by (8.85). Also let $g_n \in \mathcal{C}(\mathbb{T}^d, E_t)$ be defined by $g_n(z) := \hat{\beta}[E_t, l_t; M, M', \mathcal{K}_n](f(z))$ where $n = 0, 4, 5$. Then

$$g_0(z) = yI_{3\times3} - 3yk(z)k^t(z) , \tag{8.94}$$
$$g_4(z) = -2yI_{3\times3} + 3y\tilde{k}(z)\tilde{k}^t(z) + 3yk(z)k^t(z) , \tag{8.95}$$
$$g_5(z) = yI_{3\times3} - 3y\tilde{k}(z)\tilde{k}^t(z) . \tag{8.96}$$

Moreover if $(j, A) \in \mathcal{SOS}(d, j)$ and f is an invariant (E_t, l_t)-field of (j, A) then g_0, g_4, g_5 are invariant (E_t, l_t)-fields of (j, A). Note that g_0, g_4, g_5 take values only in $l_t(SO(3); M')$. Thus each value of the functions g_0, g_4, g_5 is a matrix with two distinct eigenvalues. It is thus easy to show, by part b) of the DT, that g_0, g_4, g_5 are equivalent, i.e., are related by isomorphisms of $SO(3)$-spaces.

8.4.2 Applying the Normal Form Theorem to (E_t, l_t)

We have already applied the NFT to the cases of $(\mathbb{R}^3, l_v)$ and $(SO(3), l_{SOR})$ in Remarks 8 and 9. Now with Section 8.4.1 we are equipped to apply the Normal Form Theorem to (E_t, l_t).

Theorem 8.10 *Let $M \in E_t$ have $\#(M) = i$, i.e., let $r \in SO(3)$ exist such that $M = l_t(r; \mathrm{diag}(y_1, y_2, -y_1 - y_2))$ where $(y_1, y_2) \in \Lambda_i$. Moreover let $(j, A) \in \mathcal{SOS}(d, j)$ and $T \in \mathcal{TF}_H(j, A)$ where we define $H := Iso(E_t, l_t; M)$. Then $f \in \mathcal{C}(\mathbb{T}^d, E_t)$, defined by (8.93), i.e., $f(z) := l_t(T(z), M)$, is an invariant (E_t, l_t)-field of (j, A). Moreover $\#(f(z)) = i$ and*

$$f(z) = y_1 I_{3\times3} + (y_2 - y_1)\tilde{l}(z)\tilde{l}^t(z) - (2y_1 + y_2)l(z)l^t(z) , \tag{8.97}$$

where the functions $l, \tilde{l} \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ are defined by

$$l(z) := T(z)r(0, 0, 1)^t , \quad \tilde{l}(z) := T(z)r(0, 1, 0)^t . \tag{8.98}$$

Remark 1: Note that $|l(z)| = |\tilde{l}(z)| = 1$.
Remark 2: In the special case where $i = 2$, *i.e., where* $y_1 = y_2 =: y$ *with* $0 \neq y \in \mathbb{R}$, *(8.97) reads as*

$$f(z) = yI_{3\times3} - 3yl(z)l^t(z) \ , \tag{8.99}$$

and in the special case where $i = 1$, *i.e.,* $M = \mathrm{diag}(0,0,0)$, *(8.97) reads as* $f(z) = \mathrm{diag}(0,0,0)$.

Proof of Theorem 8.10: That f in (8.93) is an invariant (E_t, l_t)-field of (j, A) follows from the NFT, Theorem 8.1. Moreover, by (8.77) and (8.93)

$$f(z) = l_t(T(z), M) = l_t(T(z), l_t(r; \mathrm{diag}(y_1, y_2, -y_1 - y_2)))$$
$$= l_t(T(z)r; \mathrm{diag}(y_1, y_2, -y_1 - y_2)) = T(z)\, r\, \mathrm{diag}(y_1, y_2, -y_1 - y_2)\, r^t\, T(z) \ ,$$

whence (8.97) follows from (8.83) and (8.98). The remaining claim follows from Remark 21.
□

Since $SO(2) \subset (SO(2) \bowtie \mathbb{Z}_2)$, the case $i = 2$ in Theorem 8.10 shows the impact of IFF's on invariant (E_t, l_t)-fields as the following remark demonstrates.

Remark:

(24) Let $M \in E_t$ have $\#(M) = 2$, i.e., let $r \in SO(3)$ exist such that $M = l_t(r; \mathrm{diag}(y, y, -2y))$ where $0 \neq y \in \mathbb{R}$. Also let $(j, A) \in \mathcal{SOS}(d, j)$ and T be an IFF of (j, A), i.e., by Section 5.2, $T \in \mathcal{TF}_{SO(2)}(j, A)$. We define $H := Iso(E_t, l_t; \mathrm{diag}(y, y, -2y)) = SO(2) \bowtie \mathbb{Z}_2$ and $H' := Iso(E_t, l_t; M)$ where we also used (8.87). It follows from (8.32) that $H' = rHr^t$. On the other hand, by (8.66), $SO(2) \subset (SO(2) \bowtie \mathbb{Z}_2)$ whence, by Remark 2 in Chapter 5, $\mathcal{TF}_{SO(2)}(j, A) \subset \mathcal{TF}_H(j, A)$ so that $T \in \mathcal{TF}_H(j, A)$. Now define the function $T' \in \mathcal{C}(\mathbb{T}^d, SO(3))$ by $T'(z) := T(z)r^t$. Thus, by Remark 7, $T' \in \mathcal{TF}_{H'}(j, A)$ whence T' is a generalized IFF of (j, A) and, by Theorem 8.10, the function $f \in \mathcal{C}(\mathbb{T}^d, E)$, defined by $f(z) := l_t(T'(z), M)$ is an invariant (E_t, l_t)-field of (j, A). Note also that $f(z) = l_t(T(z), \mathrm{diag}(y, y, -2y))$. This demonstrates how IFF's lead to $(SO(2) \bowtie \mathbb{Z}_2)$-normal forms and invariant (E_t, l_t)-fields. □

8.5 Applying the Decomposition Theorem in the case $(E, l) = (\mathbb{R}^3, l_v)$ and $(E', l') = (E_t, l_t)$

In this section we apply the DT to the case $(E, l) = (\mathbb{R}^3, l_v)$ and $(E', l') = (E_t, l_t)$ and thereby illustrate the connection between invariant (E_t, l_t)-fields and invariant polarization fields. Moreover, we give insights into a model with two Siberian snakes (the "2-snake model").

We proceed as follows. In Section 8.5.1 we apply the DT to arrive at Theorem 8.11. Then in Section 8.5.2 we consider the 2-snake model which has normalized, piecewise continuous solutions of the $(\mathbb{R}^3, l_v)$-stationarity equation but none of them continuous whence the 2-snake model has no ISF. However we will show that it has a nonzero invariant (E_t, l_t)-field whose values are matrices M with $\#(M) = 2$ (and the latter will be derived from Theorem 8.11).

8.5.1 A corollary to the DT

To apply the DT in the case where $(E, l) = (\mathbb{R}^3, l_v)$ and $(E', l') = (E_t, l_t)$ we follow the strategy of Remark 18 and again use the convenient representing sets R_v and R_t of $\mathbb{R}^3/l_v$ and E_t/l_t respectively. Thus we have to determine for given $S_\lambda \in R_v, M \in R_t$ whether $Iso(\mathbb{R}^3, l_v; S_\lambda) \trianglelefteq Iso(E_t, l_t; M)$ or whether $Iso(\mathbb{R}^3, l_v; S_\lambda), Iso(E_t, l_t; M)$ are conjugate. In fact, by (8.61) and (8.87) and Remarks 19 and 23, we only have to consider three cases defined as follows. In the first case $\lambda = 0$ and $\#(M) = 1$, in the second case $\lambda > 0$ and $\#(M) = 1$, and in the third case $\lambda > 0$ and $\#(M) = 2$. In the remaining case $N(Iso(\mathbb{R}^3, l_v; S_\lambda), Iso(E_t, l_t; M)) = \emptyset$ whence $B(\mathbb{R}^3, l_v, E_t, l_t; S_0, M) = \emptyset$. The following remark considers the first two cases.

Remark:

(25) In the first case, where $S_0 = (0, 0, 0)^t, M = \mathrm{diag}(0, 0, 0)$, we have, by (8.61) and (8.87), $Iso(\mathbb{R}^3, l_v; S_0) = SO(3) = Iso(E_t, l_t; M)$ whence $Iso(\mathbb{R}^3, l_v; S_0)$ and $Iso(E_t, l_t; M)$ are conjugate and $B(\mathbb{R}^3, l_v, E_t, l_t; S_0, M)$ is a singleton containing the constant, $\mathrm{diag}(0, 0, 0)$-valued function which, by the DT, results in the constant $\mathrm{diag}(0, 0, 0)$-valued invariant (E_t, l_t)-field. Since $Iso(E_t, l_t; M)$ and $Iso(E_t, l_t; M')$ are conjugate it is no surprise that the only element of $B(\mathbb{R}^3, l_v, E_t, l_t; S_0, M)$ is an isomorphism (see Remark 17).

In the second case where $\lambda > 0, M = \mathrm{diag}(0, 0, 0)$ we have, by (8.61) and (8.87), $Iso(\mathbb{R}^3, l_v; S_\lambda) = SO(2) \subset SO(3) = Iso(E_t, l_t; M)$. Since $SO(3)$ is only conjugate to itself, $SO(2), SO(3)$ are not conjugate whence, according to our strategy, we will compute the elements of $B(\mathbb{R}^3, l_v, E_t, l_t; S_\lambda, M)$. By (8.54) we have $N(SO(2), SO(3)) = SO(3)$. If $r_0 \in SO(3)$ then, by (8.4), (8.55) and (8.77), $\hat{\beta}[\mathbb{R}^3, l_v, E_t, l_t; S_\lambda, M, r_0](rS_\lambda) = rr_0^t M r_0 r^t = \mathrm{diag}(0, 0, 0)$ whence, for every $S \in \mathbb{S}^2_\lambda$,

$$\hat{\beta}[\mathbb{R}^3, l_v, E_t, l_t; S_\lambda, M, r_0)(S) = \mathrm{diag}(0, 0, 0) \; . \tag{8.100}$$

Thus $\hat{\beta}[\mathbb{R}^3, l_v, E_t, l_t; S_\lambda, M, r_0)$ is independent of the choice of r_0 and so $B(\mathbb{R}^3, l_v, E_t, l_t; S_\lambda, M)$ contains only one element:

$$B(\mathbb{R}^3, l_v, E_t, l_t; S_\lambda, M) = \{\hat{\beta}[\mathbb{R}^3, l_v, E_t, l_t; S_\lambda, M, I_{3\times 3}]\} \; . \tag{8.101}$$

Thus the only element of $B(\mathbb{R}^3, l_v, E_t, l_t; S_\lambda, M)$ is the constant $\mathrm{diag}(0, 0, 0)$-valued function which, by the DT, results in the constant $\mathrm{diag}(0, 0, 0)$-valued invariant (E_t, l_t)-field. Since $Iso(\mathbb{R}^3, l_v; S_\lambda) \trianglelefteq Iso(E_t, l_t; M)$ and since $Iso(\mathbb{R}^3, l_v; S_\lambda)$ and $Iso(E_t, l_t; M)$ are not conjugate it is no surprise that the only element of $B(\mathbb{R}^3, l_v, E_t, l_t; S_\lambda, M)$ is a topological $SO(3)$-map which is not an isomorphism (recall Remark 17). $\square$

We finally consider the third case where $\lambda > 0, \#(M) = 2$ and with this case we can fulfill the above mentioned aims in the situation where $(E, l) = (\mathbb{R}^3, l_v)$ and $(E', l') = (E_t, l_t)$. In the present case where $\lambda > 0, M = \mathrm{diag}(y, y, -2y)$ with $0 \neq y \in \mathbb{R}$ we have, by (8.61), (8.87), $Iso(\mathbb{R}^3, l_v; S_\lambda) = SO(2) \subset (SO(2) \bowtie \mathbb{Z}_2) = Iso(E_t, l_t; M)$ whence

$$N(Iso(\mathbb{R}^3, l_v; S_\lambda), Iso(E_t, l_t; M)) = N(SO(2), SO(2) \bowtie \mathbb{Z}_2) \; . \tag{8.102}$$

Since $SO(2)$ is Abelian and $SO(2) \bowtie \mathbb{Z}_2$ is not Abelian, $SO(2), SO(2) \bowtie \mathbb{Z}_2$ are not conjugate whence, according to our strategy, we will compute the elements of $B(\mathbb{R}^3, l_v, E_t, l_t; S_\lambda, M)$, i.e., by Lemma 8.8a and (8.102) we have to compute $N(SO(2), SO(2) \bowtie \mathbb{Z}_2)$. It is a simple exercise to show, by (5.5),(8.56) and (8.66), that

$$N(SO(2), SO(2) \bowtie \mathbb{Z}_2) \supset (SO(2) \bowtie \mathbb{Z}_2) , \tag{8.103}$$

and we now show that $N(SO(2), SO(2) \bowtie \mathbb{Z}_2) = SO(2) \bowtie \mathbb{Z}_2$, i.e., we will show that the inclusion, which is the converse to (8.103), holds too. For this we use (8.56) and (8.66) to obtain

$$\begin{aligned} N(SO(2), SO(2) \bowtie \mathbb{Z}_2) &= \bigcap_{h \in SO(2)} \ \bigcup_{h' \in (SO(2) \bowtie \mathbb{Z}_2)} N(\{h\}, \{h'\}) \\ &= \bigcap_{h \in SO(2)} \ \bigcup_{h' \in (SO(2)} \Big(N(\{h\}, \{h'\}) \cup N(\{h\}, \{\mathcal{K}_1 h'\}) \Big) \\ &= \bigcap_{h \in SO(2)} \Big(N(\{h\}, SO(2)) \cup N(\{h\}, \mathcal{K}_1 SO(2)) \Big) \\ &\subset \bigcap_{h \in (SO(2) \setminus \{I_{3\times 3}, \exp(\pi \mathcal{J})\})} \Big(N(\{h\}, SO(2)) \cup N(\{h\}, \mathcal{K}_1 SO(2)) \Big) . \end{aligned} \tag{8.104}$$

It is an easy exercise to show, by (5.5),(8.56) and (8.65) and the $SO(2)$-Lemma, that if $h \in (SO(2) \setminus \{I_{3\times 3}, \exp(\pi \mathcal{J})\})$ then $N(\{h\}, \mathcal{K}_1 SO(2)) = \emptyset$ whence, by (8.104),

$$N(SO(2), SO(2) \bowtie \mathbb{Z}_2) \subset \bigcap_{h \in (SO(2) \setminus \{I_{3\times 3}, \exp(\pi \mathcal{J})\})} N(\{h\}, SO(2)) . \tag{8.105}$$

It is also a simple exercise to show, by (5.5),(8.56) and the $SO(2)$-Lemma, that if $h \in (SO(2) \setminus \{I_{3\times 3}, \exp(\pi \mathcal{J})\})$ then $N(\{h\}, SO(2)) \subset (SO(2) \bowtie \mathbb{Z}_2)$ whence, by (8.105),

$$N(SO(2), SO(2) \bowtie \mathbb{Z}_2) \subset (SO(2) \bowtie \mathbb{Z}_2) . \tag{8.106}$$

We conclude from (8.103) and (8.106) that $N(SO(2), SO(2) \bowtie \mathbb{Z}_2) = SO(2) \bowtie \mathbb{Z}_2$ whence, by (8.102),

$$N(Iso(\mathbb{R}^3, l_v; S_\lambda), Iso(E_t, l_t; M)) = SO(2) \bowtie \mathbb{Z}_2 , \tag{8.107}$$

so that, by Lemma 8.8a,

$$B(\mathbb{R}^3, l_v, E_t, l_t; S_\lambda, M) = \{\hat{\beta}[\mathbb{R}^3, l_v, E_t, l_t; S_\lambda, M, r_0] : r_0 \in SO(2) \bowtie \mathbb{Z}_2)\} . \tag{8.108}$$

We will now see that $B(\mathbb{R}^3, l_v, E_t, l_t; S_\lambda, M)$ is a singleton. In fact if $r_0 \in (SO(2) \bowtie \mathbb{Z}_2)$ and $r \in SO(3)$ then, by (8.55),

$$\begin{aligned} \hat{\beta}[\mathbb{R}^3, l_v, E_t, l_t; S_\lambda, M, r_0](l_v(r; S_\lambda)) &= l_t(r r_0^t; M) = l_t(r; l_t(r_0^t; M)) \\ &= l_t(r; M) = \hat{\beta}[\mathbb{R}^3, l_v, E_t, l_t; S_\lambda, M, I_{3\times 3}](l_v(r; S_\lambda)) , \end{aligned} \tag{8.109}$$

whence, by (8.108),

$$B(\mathbb{R}^3, l_v, E_t, l_t; S_\lambda, M) = \{\hat{\beta}[\mathbb{R}^3, l_v, E_t, l_t; S_\lambda, M, I_{3\times3}]\} \ , \tag{8.110}$$

where in the third equality of (8.109) we used the second case from (8.87) where $SO(2) \bowtie \mathbb{Z}_2 = Iso(E_t, l_t; M)$.

We now apply the DT to the current case.

Theorem 8.11 *Let $(j, A) \in \mathcal{SOS}(d, j)$. Let $\lambda \in (0, \infty)$ and let $M \in R_t$ have $\#(M) = 2$, i.e., $M = \mathrm{diag}(y, y, -2y)$ where $0 \neq y \in \mathbb{R}$. Then $B(\mathbb{R}^3, l_v, E_t, l_t; S_\lambda, M)$ is given by (8.110) and for every S in its domain its only element, $\hat{\beta}[\mathbb{R}^3, l_v, E_t, l_t; S_\lambda, M, I_{3\times3}]$, satisfies*

$$\hat{\beta}[\mathbb{R}^3, l_v, E_t, l_t; S_\lambda, M, I_{3\times3}](S) = yI_{3\times3} - \frac{3y}{\lambda^2} SS^t \ . \tag{8.111}$$

Let $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ take values only in the $(\mathbb{R}^3, l_v)$-orbit of S_λ. Let the function $g \in \mathcal{C}(\mathbb{T}^d, E_t)$ be defined by $g(z) := \hat{\beta}[\mathbb{R}^3, l_v, E_t, l_t; S_\lambda, M, I_{3\times3}](f(z))$. Then

$$g(z) = yI_{3\times3} - \frac{3y}{\lambda^2} f(z)f^t(z) \ . \tag{8.112}$$

Let us apply the 1-turn field map, i.e., let the function $f' \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ be defined by $f' := \tilde{\mathcal{P}}[j, A](f)$ and the function $g' \in \mathcal{C}(\mathbb{T}^d, E_t)$ be defined by $g' := \tilde{\mathcal{P}}[E_t, l_t, j, A](g)$. Then

$$g'(z) = yI_{3\times3} - \frac{3y}{\lambda^2} f'(z)f'^t(z) \ , \tag{8.113}$$

and $g'(z) = \hat{\beta}[\mathbb{R}^3, l_v, E_t, l_t; S_\lambda, M, I_{3\times3}](f'(z))$.
Remark: If $f = f'$ then $g = g'$. In other words if f is an invariant polarization field of (j, A) then g is an invariant (E_t, l_t)-field of (j, A). In particular, if $\lambda = 1$ and f is an ISF of (j, A) then g is an invariant (E_t, l_t)-field of (j, A).

Proof of Theorem 8.11: Using (8.55) and (8.110) and Remark 21 we get (8.111). Moreover (8.112) follows from (8.111). The remaining claims follow from Theorem 8.9a. □

In the special case $\lambda = 1, y = 1/\sqrt{6}$, (8.112) is the expression for the invariant tensor field in [BV2]. So we have independently reconstructed the invariant tensor field of [BV2] by using the DT!

8.5.2 The 2-snake model

In this section we consider a model describing the spin-orbit system of a flat storage ring which has two thin-lens Siberian Snakes with mutually perpendicular axes of spin rotation placed at $\theta = 0$ and $\theta = \pi$. With this layout, the spin tune, ν_0, on the design orbit, of the ring is 1/2. Here we are interested in the situation where, in the absence of snakes, the spin motion is dominated by the effect of a single harmonic in the Fourier expansion of the radial component of the $\mathbf{\Omega}(\theta, J, \phi(\theta))$, mentioned in the Introduction, and due to vertical betatron motion. This case is often called the "single resonance model". The combination of the single resonance model and two snakes considered in this section has been studied extensively. See for example [BV1, Vo] and the references therein. The interest in this model stems from the effect on the polarization of the so-called "snake resonances". These occur at vertical betatron tunes of $1/2, 1/6, 5/6, 1/10, 3/10\ldots$ Note that the term snake resonance is a misnomer since it does not refer to the proper definition of spin-orbit resonance given in (6.17). Our main interest here is in the fact that at snake resonance, there is no ISF of the kind that we define in this paper. We have already mentioned this situation in Section 6.2. For further background material see [BV1].

Here we focus on the simplest case, namely that with vertical betatron tune, $\omega = 1/2$, and we denote the resulting spin-orbit system by $(\mathcal{P}_{1/2}, A_{2S})$. Of course a real bunch is not stable at $\omega = 1/2$ but this does not play a role in the present section. We prove two claims. We first show that $(\mathcal{P}_{1/2}, A_{2S})$ has a 2-turn ISF, defined below, and a normalized, piecewise continuous solution of the $(\mathbb{R}^3, l_v)$-stationarity equation but no ISF. Secondly we apply the DT via Theorem 8.11 to construct, out of the two 2-turn ISF's, an invariant nonzero (E_t, l_t)-field of $(\mathcal{P}_{1/2}, A_{2S})$.

We first define $(\mathcal{P}_{1/2}, A_{2S})$. For this we define the function $A_{2S} \in \mathcal{C}(\mathbb{T}^1, SO(3))$, for $\epsilon \in (\mathbb{R} \setminus \mathbb{Z})$ with [BV1, Vo] by

$$A_{2S}(\phi + \tilde{\mathbb{Z}}) := \begin{pmatrix} 1 - 2c^2(\phi) & 2b(\phi)c(\phi) & 2a(\phi)c(\phi) \\ 2b(\phi)c(\phi) & 1 - 2b^2(\phi) & -2a(\phi)b(\phi) \\ -2a(\phi)c(\phi) & 2a(\phi)b(\phi) & 2a^2(\phi) - 1 \end{pmatrix} , \tag{8.114}$$

where the functions $a, b, c \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ are defined by

$$\begin{aligned} &a(\phi) := -2\sin^2(\pi\epsilon/2)\sin(\phi)\cos(\phi) , \quad b(\phi) := -2\sin(\pi\epsilon/2)\cos(\pi\epsilon/2)\cos(\phi) , \\ &c(\phi) := 2\sin^2(\pi\epsilon/2)\cos^2(\phi) - 1 . \end{aligned} \tag{8.115}$$

Note that

$$a^2 + b^2 + c^2 = 1 , \tag{8.116}$$

and that we exclude ϵ from being an integer because in that case $(\mathcal{P}_{1/2}, A_{2S})$ would have an ISF [He1]. Note also that, by the Torus Lemma, Lemma 2.1, A_{2S} is continuous since the continuous functions a, b, c are 2π-periodic.

Since $\mathcal{P}_{1/2}^2 = id_{\mathbb{T}^d}$ we will prove both of our claims by computing the so-called 2-turn ISF's of $(\mathcal{P}_{1/2}, A_{2S})$. We call an invariant 2-turn $(\mathbb{R}^3, l_v)$-field of $(\mathcal{P}_{1/2}, A_{2S})$ a "2-turn ISF of $(\mathcal{P}_{1/2}, A_{2S})$" if it is normalized. Thus, noting that with $\omega = 1/2$, a particle returns to the same z over two turns, an $h \in \mathcal{C}(\mathbb{T}^1, \mathbb{R}^3)$ is a 2-turn ISF of $(\mathcal{P}_{1/2}, A_{2S})$ iff

$$\tilde{\mathcal{P}}[\mathcal{P}_{1/2}, A_{2S}]^2(h) = h , \tag{8.117}$$

$$|h| = 1 . \tag{8.118}$$

In fact we will see that $(\mathcal{P}_{1/2}, A_{2S})$ has just two 2-turn ISF's namely $h = k$ and $h = -k$ where k will be defined below. It is clear that every ISF is a 2-turn ISF and we will show that in fact neither k nor $-k$ is an ISF of $(\mathcal{P}_{1/2}, A_{2S})$ which implies that no ISF exists. We will then apply Theorem 8.11 to $h = k$ and will thereby obtain an invariant (E_t, l_t)-field of $(\mathcal{P}_{1/2}, A_{2S})$. The case $h = -k$ will result in the same invariant (E_t, l_t)-field.

To begin our computations we first rewrite (8.117), by using (3.7) and (3.8), into

$$h(z) = \Psi[\mathcal{P}_{1/2}, A_{2S}](2; z)h(z) \ . \tag{8.119}$$

Thus a $h \in \mathcal{C}(\mathbb{T}^1, \mathbb{R}^3)$ is a 2-turn ISF of $(\mathcal{P}_{1/2}, A_{2S})$ iff (8.118),(8.119) are fullfilled. Note that, very conveniently, (8.119) is an eigenvalue problem for $h(z)$ and it is here that we used that $\omega = 1/2$. To obtain the 2–turn spin transfer matrix function in (8.119) we first conclude from (8.114) and (8.115) that

$$A_{2S}((\phi+\pi)+\tilde{\mathbb{Z}}) = \begin{pmatrix} 1-2c^2(\phi) & -2b(\phi)c(\phi) & 2a(\phi)c(\phi) \\ -2b(\phi)c(\phi) & 1-2b^2(\phi) & 2a(\phi)b(\phi) \\ -2a(\phi)c(\phi) & -2a(\phi)b(\phi) & 2a^2(\phi)-1 \end{pmatrix} . \tag{8.120}$$

We also conclude from (2.36), (8.114) and (8.120) that the 2–turn spin transfer matrix function reads as

$$\begin{aligned} &\Psi[\mathcal{P}_{1/2}, A_{2S}](2; \phi+\tilde{\mathbb{Z}}) = A_{2S}((\phi+\pi)+\tilde{\mathbb{Z}})A_{2S}(\phi+\tilde{\mathbb{Z}}) \\ &= \begin{pmatrix} 1-8c^2(\phi)+8c^4(\phi) & 4b(\phi)c(\phi)(1-2c^2(\phi)) & 4a(\phi)c(\phi)(1-2c^2(\phi)) \\ -4b(\phi)c(\phi)(1-2c^2(\phi)) & 1-8b^2(\phi)c^2(\phi) & -8a(\phi)b(\phi)c^2(\phi) \\ -4a(\phi)c(\phi)(1-2c^2(\phi)) & -8a(\phi)b(\phi)c^2(\phi) & 1-8a^2(\phi)c^2(\phi) \end{pmatrix} . \end{aligned} \tag{8.121}$$

Since ϵ is not an integer, $|\sin(\pi\epsilon/2)|$ equals neither 0 or 1, and so we define the 2π-periodic function $K \in \mathcal{C}(\mathbb{R}, \mathbb{R}^3)$ by

$$K(\phi) := \frac{\cos(\pi\epsilon/2)}{|\cos(\pi\epsilon/2)|\sqrt{1-\sin^2(\pi\epsilon/2)\cos^2(\phi)}} \Big(0, \sin(\pi\epsilon/2)\sin(\phi), -\cos(\pi\epsilon/2)\Big) . \tag{8.122}$$

By the Torus Lemma, Lemma 2.1, a unique function $k \in \mathcal{C}(\mathbb{T}^1, \mathbb{R}^3)$ exists such that

$$k = K \circ \pi_1 \ . \tag{8.123}$$

It is easy to show that (8.118) and (8.119) are fullfilled for $h = k$, i.e.,

$$k(z) = \Psi[\mathcal{P}_{1/2}, A_{2S}](2; z)k(z) \ , \tag{8.124}$$

$$|k(z)| = 1 \ . \tag{8.125}$$

Thus indeed k and $-k$ are 2-turn ISF's of $(\mathcal{P}_{1/2}, A_{2S})$. Let $h \in \mathcal{C}(\mathbb{T}^1, \mathbb{R}^3)$ be an arbitrary 2-turn ISF of $(\mathcal{P}_{1/2}, A_{2S})$, i.e., let h satisfy (8.118) and (8.119).

To show that either $h = k$ or $h = -k$ let $R \neq I_{3\times3}$ be a matrix in $SO(3)$. Then R has a real eigenvector $v \in \mathbb{R}^3$ with eigenvalue 1 and such that $|v| = 1$ whence $r \in SO(3)$ exists such that $v = r(0,0,1)^t$. Thus $r^tRr(0,0,1)^t = (0,0,1)^t$ whence, by the $SO(2)$-Lemma, $r^tRr \in SO(2)$ so that a $\nu \in [0,1)$ exists such that $R = r\exp(2\pi\nu\mathcal{J})r^t$. This implies, since $R \neq I_{3\times3}$, that $\nu \neq 0$. Thus if $w, w' \in \mathbb{R}^3$ are real eigenvectors of r^tRr with the eigenvalue 1 and $|w| = |w'| = 1$ then $|w \cdot w'| = 1$ whence if $v, v' \in \mathbb{R}^3$ are real eigenvectors of R with the eigenvalue 1 and $|v| = |v'| = 1$ then $|v \cdot v'| = 1$.

Defining the set

$$M := \{z \in \mathbb{T}^1 : \Psi[\mathcal{P}_{1/2}, A_{2S}](2; z) = I_{3\times3}\} \ , \tag{8.126}$$

we observe that, if $z \in (\mathbb{T}^1 \setminus M)$, then $\Psi[\mathcal{P}_{1/2}, A_{2S}](2; z) \neq I_{3\times3}$. Thus, and since by (8.118), (8.119), (8.124) and (8.125), $h(z), k(z)$ are real eigenvectors of $\Psi[\mathcal{P}_{1/2}, A_{2S}](2; z)$ with eigenvalue 1 and $|h(z)| = |k(z)| = 1$ we conclude that, if $z \in (\mathbb{T}^1 \setminus M)$, then $\lambda(z) = 1$ where the function $\lambda : \mathbb{T}^1 \to \mathbb{R}$ is defined by $\lambda(z) := |h(z) \cdot k(z)|$. To show that $\lambda(z) = 1$ for all $z \in \mathbb{T}^1$ we only have to show that λ is a constant function. We thus compute, by (8.115) and (8.121),

$$M = \{\phi + \tilde{\mathbb{Z}} : \phi \in \mathbb{R}, c(\phi)(c^2(\phi) - 1) = 0\} = \{\phi + \tilde{\mathbb{Z}} : \phi \in \mathbb{R}, \cos^2(\phi) = \frac{1}{2\sin^2(\pi\epsilon/2)}\} , \tag{8.127}$$

whence M consists of only finitely many points. Since $\lambda(z) = 1$ on $\mathbb{T}^1 \setminus M$ and since M has only finitely many points we conclude that λ is a continuous function with only finitely many values. Since $\mathbb{T}^1$ is path-connected and λ is continuous we use the same argument as in the proof of Theorem 7.1b and conclude that the range of λ is an interval whence λ is constant so that $\lambda(z) = |h(z) \cdot k(z)| = 1$ holds for *every* $z \in \mathbb{T}^1$. Thus, and since $|h(z)| = |k(z)| = 1$, either $h = k$ or $h = -k$. So we have shown that the only 2-turn ISF's are $h = k$ and $h = -k$.

To show that neither k nor $-k$ is an ISF we compute, by (8.114) and (8.122),

$$A_{2S}(\phi + \tilde{\mathbb{Z}})K(\phi) = -K(\phi + \pi) , \tag{8.128}$$

whence, by (2.17) and (8.123), $A_{2S}(z)k(z) = -k(\mathcal{P}_{1/2}(z))$ so that, by (3.3),

$$\tilde{\mathcal{P}}[\mathcal{P}_{1/2}, A_{2S}](k) = -k , \tag{8.129}$$

which implies, by Definition 3.1, that k is not an ISF of $(\mathcal{P}_{1/2}, A_{2S})$. Thus $-k$ is not an ISF of $(\mathcal{P}_{1/2}, A_{2S})$ either which completes the proof that the two only 2-turn ISF's of $(\mathcal{P}_{1/2}, A_{2S})$ are not ISF's of $(\mathcal{P}_{1/2}, A_{2S})$. We conclude, by our earlier remarks, that $(\mathcal{P}_{1/2}, A_{2S})$ has no ISF. This proves the first claim.

Remark:

(26) While $(\mathcal{P}_{1/2}, A_{2S})$ has no ISF, it is easy to construct a normalized, piecewise continuous solution of the $(\mathbb{R}^3, l_v)$-stationarity equation (see also [BV2]). In fact defining $\tilde{K} : \mathbb{R} \to \mathbb{R}^3$ by

$$\tilde{K}(\phi) := \begin{cases} K(\phi) & \text{if } \phi \in \bigcup_{n\in\mathbb{Z}}[2\pi n, 2\pi n + \pi) \\ -K(\phi) & \text{if } \phi \in \bigcup_{n\in\mathbb{Z}}[2\pi n + \pi, 2\pi n + 2\pi) , \end{cases} \tag{8.130}$$

we observe, by the Torus Lemma in Section 2.2, that a unique function $\tilde{k} : \mathbb{T}^1 \to \mathbb{R}^3$ exists such that $\tilde{k} = \tilde{K} \circ \pi_1$. It is a simple exercise to show that $\tilde{k}$ is a normalized piecewise continuous solution of the $(\mathbb{R}^3, l_v)$-stationarity equation of $(\mathcal{P}_{1/2}, A_{2S})$. Of course, $\tilde{k}$ is not an ISF of $(\mathcal{P}_{1/2}, A_{2S})$ since $(\mathcal{P}_{1/2}, A_{2S})$ has no ISF. In fact it is an easy exercise to show, by (8.122) and (8.130), that $\tilde{k}$ is discontinuous at $z = \pi_1(0)$ and $z = \pi_1(\pi)$. This is an example of a consequence of a lack of topological transitivity of j mentioned just after Theorem 3.2.

As mentioned at the end of Section 8.2.2 since A, j, l are continuous we require that invariant fields be continuous. However this requirement is a matter of choice. In fact if one would impose the weaker condition of Borel measurability then $\tilde{k}$ would be an ISF. In fact, as mentioned in Section 6.2, the requirement of continuity was relaxed in [BV1]. □

We now apply Theorem 8.11 in the case $\lambda = 1$ for the function $f := k$. In fact, in the notation of Theorem 8.11, it follows from (8.129) that

$$f' = \tilde{\mathcal{P}}[\mathcal{P}_{1/2}, A_{2S}](f) = \tilde{\mathcal{P}}[\mathcal{P}_{1/2}, A_{2S}](k) = -k \; , \tag{8.131}$$

Using the notation of Theorem 8.11, the functions $g, g' \in \mathcal{C}(\mathbb{T}^d, E_t)$ are given by

$$\begin{aligned} g(z) &= \hat{\beta}[\mathbb{R}^3, l_v, E_t, l_t; S_1, M, I_{3\times 3}](f(z)) \\ &= \hat{\beta}[\mathbb{R}^3, l_v, E_t, l_t; S_1, M, I_{3\times 3}](k(z)) = yI_{3\times 3} - 3yk(z)k^t(z) \; , \end{aligned} \tag{8.132}$$

$$\begin{aligned} g'(z) &= \hat{\beta}[\mathbb{R}^3, l_v, E_t, l_t; S_1, M, I_{3\times 3}](f'(z)) \\ &= \hat{\beta}[\mathbb{R}^3, l_v, E_t, l_t; S_1, M, I_{3\times 3}](-k(z)) = yI_{3\times 3} - 3yk(z)k^t(z) \; , \end{aligned} \tag{8.133}$$

whence $g = g'$ so that g is an invariant (E_t, l_t)-field of $(\mathcal{P}_{1/2}, A_{2S})$. The same holds for $f = -k$ since, by repeating the above construction of g and replacing $f = k$ by $f = -k$ we get the same g. This completes the proof of the second claim.

So although $(\mathcal{P}_{1/2}, A_{2S})$ has no ISF, there is a nontrivial invariant tensor field. This is also expected from [BV1] by noting that the discontinuities at $z = \pi_1(0)$ and $z = \pi_1(\pi)$ involve a simple change of sign. Then since the invariant tensor field is quadratic in $\tilde{k}$, those discontinuities do not cause discontinuities in the invariant tensor field.

8.6 Applying the ToA to density matrix functions

8.6.1 Spin-1/2 particles. Applying the ToA to $(E^{1/2}_{dens}, l^{1/2}_{dens})$

In this section we introduce the $SO(3)$-space $(E^{1/2}_{dens}, l^{1/2}_{dens})$ to enable the use of the ToA for the study of the spin-1/2 density matrix function employed for describing polarized beams of spin-1/2 particles [BV2]. As in Sections 8.3-8.5 the focus is on the field motion.

We define

$$E^{1/2}_{dens} := \{R \in \mathbb{C}^{2\times 2} : R^\dagger = R, Tr[R] = 1\} \; , \tag{8.134}$$

where $R^\dagger$ denotes the hermitian conjugate of the matrix R and we equip $E^{1/2}_{dens}$ with the subspace topology from $\mathbb{C}^{2\times 2}$. Thus, and since $\mathbb{C}^{2\times 2}$ with its natural topology is a Hausdorff space, $E^{1/2}_{dens}$ is a Hausdorff space, too. Following a standard parametrization we define the function $\beta^{1/2}_{dens} : \mathbb{R}^3 \to E^{1/2}_{dens}$ for $S \in \mathbb{R}^3$ by

$$\beta^{1/2}_{dens}(S) := \frac{1}{2}\Big(I_{2\times 2} + \sum_{i=1}^{3} S_i \sigma_i\Big) \; , \tag{8.135}$$

where the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ are defined by

$$\sigma_1 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_2 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \sigma_3 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \tag{8.136}$$

and where S_i denotes the i-th component of S. Since every σ_k is hermitian, i.e., $\sigma_k^\dagger = \sigma_k$, by (8.135), $\beta_{dens}^{1/2}(S)$ is hermitian too. Moreover since, by (8.136) and for $i, k = 1, 2, 3$,

$$Tr[\sigma_i] = 0 \ , \quad Tr[\sigma_i \sigma_k] = 2\delta_{ik} \ , \tag{8.137}$$

we find from (8.135) that $Tr[\beta_{dens}^{1/2}(S)] = 1$. So $\beta_{dens}^{1/2}(S)$ is a hermitian matrix of trace 1, i.e., indeed $\beta_{dens}^{1/2}$ is a function into $E_{dens}^{1/2}$. To show that $\beta_{dens}^{1/2}$ is a homeomorphism we first note, by (8.136), that if $R \in \mathbb{C}^{2\times 2}$ is hermitian then real numbers S'_0, S'_1, S'_2, S'_3 exist such that $R = S'_0 I_{2\times 2} + \sum_{i=1}^3 S'_i \sigma_i$ whence, by (8.134) and (8.137) and if $R \in E_{dens}^{1/2}$, we get $S'_0 = 1/2$ so that $R = \beta_{dens}^{1/2}(S)$ for some $S \in \mathbb{R}^3$. Thus the function $\beta_{dens}^{1/2}$ is onto $E_{dens}^{1/2}$, i.e.,

$$E_{dens}^{1/2} = \{\beta_{dens}^{1/2}(S) : S \in \mathbb{R}^3\} \ . \tag{8.138}$$

It also follows from (8.135) and (8.137) that, for $S \in \mathbb{R}^3$ and $i = 1, 2, 3$,

$$S_i = Tr[\sigma_i \beta_{dens}^{1/2}(S)] \ . \tag{8.139}$$

Thus S is uniquely determined by $\beta_{dens}^{1/2}(S)$ whence $\beta_{dens}^{1/2}$ is one-one so that we conclude that $\beta_{dens}^{1/2}$ is a bijection. Since $\beta_{dens}^{1/2}$ is a bijection it follows from (8.135) that its inverse, $(\beta_{dens}^{1/2})^{-1}$, is defined for $R \in E_{dens}^{1/2}$ by

$$(\beta_{dens}^{1/2})^{-1}(R) := S \ , \quad S_i := Tr[\sigma_i R] \ , \tag{8.140}$$

where S_i denotes the i-th component of S. Moreover, by (8.135) and (8.140), both $\beta_{dens}^{1/2}$ and $(\beta_{dens}^{1/2})^{-1}$ are continuous functions whence $\beta_{dens}^{1/2} \in \mathrm{Homeo}(\mathbb{R}^3, E_{dens}^{1/2})$, a fact which plays a key role in this section.

We now define the function $l_{dens}^{1/2} : SO(3) \times E_{dens}^{1/2} \to E_{dens}^{1/2}$ by

$$l_{dens}^{1/2}(r; R) := \beta_{dens}^{1/2}\Big(l_v(r; (\beta_{dens}^{1/2})^{-1}(R))\Big) \ , \tag{8.141}$$

i.e.,

$$l_{dens}^{1/2}(r; \beta_{dens}^{1/2}(S)) := \beta_{dens}^{1/2}(l_v(r; S)) = \beta_{dens}^{1/2}(rS) \ , \tag{8.142}$$

with $r \in SO(3), R \in E_{dens}^{1/2}$ and $S \in \mathbb{R}^3$. Since $(\mathbb{R}^3, l_v)$ is an $SO(3)$-space and $\beta_{dens}^{1/2} \in \mathrm{Homeo}(\mathbb{R}^3, E_{dens}^{1/2})$ it follows from (8.141) that $(E_{dens}^{1/2}, l_{dens}^{1/2})$ is an $SO(3)$-space and that $\beta_{dens}^{1/2}$ is an isomorphism from the $SO(3)$-space $(\mathbb{R}^3, l_v)$ to the $SO(3)$-space $(E_{dens}^{1/2}, l_{dens}^{1/2})$.

Due to (8.1), the 1-turn particle-spin map $\mathcal{P}[E_{dens}^{1/2}, l_{dens}^{1/2}, j, A]$ is given by

$$\mathcal{P}[E_{dens}^{1/2}, l_{dens}^{1/2}, j, A](z, R) = \begin{pmatrix} j(z) \\ l_{dens}^{1/2}(A(z); R) \end{pmatrix} \ , \tag{8.143}$$

where $z \in \mathbb{T}^d, R \in E^{1/2}_{dens}$. Because the $SO(3)$-spaces $(\mathbb{R}^3, l_v)$ and $(E^{1/2}_{dens}, l^{1/2}_{dens})$ are isomorphic, we get easy insight into $\mathcal{P}[E^{1/2}_{dens}, l^{1/2}_{dens}, j, A]$ by using the Second ToA Transformation Rule. In fact recalling Section 8.2.6 we find via (8.47) that $\mathcal{P}[E^{1/2}_{dens}, l^{1/2}_{dens}, j, A] \circ \beta^{1/2}_{dens,tot} = \beta^{1/2}_{dens,tot} \circ \mathcal{P}[\mathbb{R}^3, l_v, j, A]$ whence and since, by Remark 1, $\mathcal{P}[\mathbb{R}^3, l_v, j, A] = \mathcal{P}[j, A]$ we get

$$\mathcal{P}[E^{1/2}_{dens}, l^{1/2}_{dens}, j, A] \circ \beta^{1/2}_{dens,tot} = \beta^{1/2}_{dens,tot} \circ \mathcal{P}[j, A] \ , \tag{8.144}$$

i.e.,

$$\begin{aligned} \mathcal{P}[E^{1/2}_{dens}, l^{1/2}_{dens}, j, A](z, \beta^{1/2}_{dens}(S)) &= \mathcal{P}[E^{1/2}_{dens}, l^{1/2}_{dens}, j, A](\beta^{1/2}_{dens,tot}(z, S)) \\ &= \begin{pmatrix} j(z) \\ \beta^{1/2}_{dens}(A(z)S) \end{pmatrix} , \end{aligned} \tag{8.145}$$

where $z \in \mathbb{T}^d, S \in \mathbb{R}^3$ and where the function $\beta^{1/2}_{dens,tot} \in \text{Homeo}(\mathbb{T}^d \times \mathbb{R}^3, \mathbb{T}^d \times E^{1/2}_{dens})$ is defined by $\beta^{1/2}_{dens,tot}(z, S) := (z, \beta^{1/2}_{dens}(S))$.

We now come to our main focus, the fields, which in the case $(E, l) = (E^{1/2}_{dens}, l^{1/2}_{dens})$ are also called spin-1/2 density matrix functions and which are functions $\rho : \mathbb{T}^d \to E^{1/2}_{dens}$ whence, by (8.138), $\rho = \beta^{1/2}_{dens} \circ f$ where the function $f : \mathbb{T}^d \to \mathbb{R}^3$ is defined by $f(z) := (\beta^{1/2}_{dens})^{-1}(\rho(z))$. Thus, using (8.135) and (8.140), we get

$$f_i(z) = Tr[\rho(z)\sigma_i] \ , \tag{8.146}$$

$$\rho(z) = \beta^{1/2}_{dens}(f(z)) = \frac{1}{2}\Big(I_{2\times 2} + \sum_{i=1}^{3} f_i(z)\sigma_i\Big) \ , \tag{8.147}$$

where $f_i(z)$ denotes the i-th component of $f(z)$. Of course since $\beta^{1/2}_{dens} \in \text{Homeo}(\mathbb{R}^3, E^{1/2}_{dens})$, ρ is continuous iff f is continuous. We call an invariant $(E^{1/2}_{dens}, l^{1/2}_{dens})$-field an "equilibrium spin-1/2 density matrix function". Due to (8.2), the 1-turn field map $\tilde{\mathcal{P}}[E^{1/2}_{dens}, l^{1/2}_{dens}, j, A]$ is given by

$$\tilde{\mathcal{P}}[E^{1/2}_{dens}, l^{1/2}_{dens}, j, A](\rho) = l^{1/2}_{dens}\Big(A \circ j^{-1}; \rho \circ j^{-1}\Big) \ . \tag{8.148}$$

Because the $SO(3)$-spaces $(\mathbb{R}^3, l_v)$ and $(E^{1/2}_{dens}, l^{1/2}_{dens})$ are isomorphic, we get easy insight into $\tilde{\mathcal{P}}[E^{1/2}_{dens}, l^{1/2}_{dens}, j, A]$ by using once again the Second ToA Transformation Rule. In fact recalling Section 8.2.6 we find via (8.52) that $\tilde{\mathcal{P}}[E^{1/2}_{dens}, l^{1/2}_{dens}, j, A] \circ \tilde{\beta}^{1/2}_{dens} = \tilde{\beta}^{1/2}_{dens} \circ \tilde{\mathcal{P}}[\mathbb{R}^3, l_v, j, A]$ whence and since, by Remark 1, $\tilde{\mathcal{P}}[\mathbb{R}^3, l_v, j, A] = \tilde{\mathcal{P}}[j, A]$ we get

$$\tilde{\mathcal{P}}[E^{1/2}_{dens}, l^{1/2}_{dens}, j, A] \circ \tilde{\beta}^{1/2}_{dens} = \tilde{\beta}^{1/2}_{dens,tot} \circ \tilde{\mathcal{P}}[j, A] \ , \tag{8.149}$$

where the function $\tilde{\beta}^{1/2}_{dens} : \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3) \to \mathcal{C}(\mathbb{T}^d, E^{1/2}_{dens})$ is defined, for $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$, by $\tilde{\beta}^{1/2}_{dens}(f) := \beta^{1/2}_{dens} \circ f$. It thus follows by Remark 16 that an $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ is an invariant polarization field of (j, A) iff $\beta^{1/2}_{dens} \circ f$ is an invariant $(E^{1/2}_{dens}, l^{1/2}_{dens})$-field of (j, A).

We thus have proved:

Theorem 8.12 *The function $\beta^{1/2}_{dens}$ belongs to* Homeo$(\mathbb{R}^3, E^{1/2}_{dens})$. *Let $\rho : \mathbb{T}^d \to E^{1/2}_{dens}$. Then a unique function $f : \mathbb{T}^d \to \mathbb{R}^3$ exists such that $\rho = \beta^{1/2}_{dens} \circ f$, i.e., (8.147) holds where $f_i(z)$ denotes the i-th component of $f(z)$. Moreover ρ is continuous iff f is continuous. Moreover, let $(j, A) \in \mathcal{SOS}(d, j)$. Then ρ is an equilibrium spin-1/2 density matrix function of (j, A), i.e., is an invariant $(E^{1/2}_{dens}, l^{1/2}_{dens})$-field of (j, A) iff f is an invariant polarization field of (j, A).* □

Because of Theorem 8.12 the study of equilibrium spin-1/2 density matrix functions effectively amounts to the study of invariant polarization fields. See [BV2] too. Since invariant polarization fields are studied in other parts of this work the remainder of this section can be brief and so we leave the application of the NFT, DT etc. as an exercise to the reader and conclude this section with Remarks 27 and 28.

Recalling that $E^{1/2}_{dens}$ is Hausdorff we can address topological transitivity by applying Lemma 8.4 to $(E, l) = (E^{1/2}_{dens}, l^{1/2}_{dens})$ in the following remark:

Remark:

(27) Recalling that $\beta^{1/2}_{dens}$ is an isomorphism from $(\mathbb{R}^3, l_v)$ to $(E^{1/2}_{dens}, l^{1/2}_{dens})$ we note, by Definition 4.3, that $\beta^{1/2}_{dens}$ maps each $(\mathbb{R}^3, l_v)$-orbit, i.e., each sphere $\mathbb{S}^2_\lambda$ onto an $(E^{1/2}_{dens}, l^{1/2}_{dens})$-orbit and that $(\beta^{1/2}_{dens})^{-1}$ maps each $(E^{1/2}_{dens}, l^{1/2}_{dens})$-orbit onto an $(\mathbb{R}^3, l_v)$-orbit. Let $(j, A) \in \mathcal{SOS}(d, j)$ with j topologically transitive and let ρ be an equilibrium density matrix function of (j, A). Since $E^{1/2}_{dens}$ is Hausdorff we can apply Lemma 8.4 and we conclude that ρ takes values in only one $(E^{1/2}_{dens}, l^{1/2}_{dens})$-orbit whence by the above $f := (\beta^{1/2}_{dens})^{-1} \circ \rho$ takes values in only one $(\mathbb{R}^3, l_v)$-orbit. However this is no surprise since from Theorem 8.12 we know that f is an invariant polarization field of (j, A) whence, by applying Lemma 8.4 to $(E, l) = (\mathbb{R}^3, l_v)$, we see once again that f takes values in only one $(\mathbb{R}^3, l_v)$-orbit. □

The following remark sketches how one uses spin-1/2 density matrices for the statistical description of a bunch of spin-1/2 particles:

Remark:

(28) One can describe a bunch of spin-1/2 particles statistically by a function $\rho_{tot} : \mathbb{Z} \times \mathbb{T}^d \times \Lambda \to \mathbb{C}^{2\times 2}$ of the form $\rho_{tot}(n, z, J) := (1/2\pi)^d \rho_{eq}(J) \rho_{spin}(n, z, J)$ where $(1/2\pi)^d \rho_{eq}$ describes the equilibrium particle distribution in the bunch and where $\rho_{spin} : \mathbb{Z} \times \mathbb{T}^d \times \Lambda \to E^{1/2}_{dens}$ has the property that each of the functions $\rho_{spin}(n, \cdot, J)$ moves as a spin-1/2 density matrix function, i.e., moves into $\tilde{\mathcal{P}}[E^{1/2}_{dens}, l^{1/2}_{dens}, j, A](\rho_{spin}(n, \cdot, J))$ after one turn. Clearly we deal here with the Schroedinger picture. Of course $\rho_{spin} = \beta^{1/2}_{dens} \circ f$, i.e., $\rho_{spin}(n, z, J) = \frac{1}{2}(I_{2\times 2} + \sum_{i=1}^3 f_i(n, z, J)\sigma_i)$ where f_i is the i-th component of f. Note that the domain, Λ, of the action variable J was introduced in Section 7.1.

Let $\mathcal{O} : \mathbb{T}^d \times \Lambda \to \mathbb{C}^{2\times 2}$ be a "physical observable", i.e., let every value $\mathcal{O}(z, J)$ be a hermitian 2×2-matrix whence $\mathcal{O}(z, J) = g_0(z, J) + \sum_{i=1}^3 m_i(z, J)\sigma_i$ where $m_0, m_1, m_2, m_3 : \mathbb{T}^d \times \Lambda \to \mathbb{R}$. Then the "expectation value" $< \mathcal{O} > (n)$ of $\mathcal{O}$ at time

n is defined by

$$< \mathcal{O} > (n) := (1/2\pi)^d \int_{[0,2\pi]^d\times\Lambda} Tr\Big[\rho_{tot}(n,\pi_d(\phi),J)\mathcal{O}(\pi_d(\phi),J)\Big] d\phi dJ$$
$$= \int_{[0,2\pi]^d\times\Lambda} \rho_{eq}(J)\Big(m_0(\pi_d(\phi),J) + \sum_{i=1}^{3} m_i(\pi_d(\phi),J) f_i(n,\pi_d(\phi),J)\Big) d\phi dJ \; , \tag{8.150}$$

where in the second equality we used (8.137). For example, in case of the spin observable, i.e., $\mathcal{O}_i(z,J) := \sigma_i$, we get from (8.150)

$$< \mathcal{O}_i > (n) = (1/2\pi)^d \int_{[0,2\pi]^d\times\Lambda} \rho_{eq}(J) f_i(n,\pi_d(\phi),J) d\phi dJ \; , \tag{8.151}$$

which is the i-th component of the polarization vector of the bunch, i.e., the bunch polarization is $P(n) = \sqrt{\sum_{i=1}^3 (< \mathcal{O}_i > (n))^2}$ which we used in Section 7.1 for the definition of $P(n)$ in eq. (7.8). At equilibrium, $\rho_{spin}(n,\cdot,J) = \rho_{spin}(0,\cdot,J)$ and $\rho_{spin}(0,\cdot,J)$ is an equilibrium spin-1/2 density matrix function of (j,A).

The choice $(E^{1/2}_{dens}, l^{1/2}_{dens})$ and the above theory of ρ_{tot} follows from the semiclassical treatment of Dirac's equation in terms of Wigner functions where the particle-variables z and J are purely classical (see [BHHV] and the references therein). □

8.6.2 Spin-1 particles. Applying the ToA to (E^1_{dens}, l^1_{dens})

In this section we introduce the $SO(3)$-space (E^1_{dens}, l^1_{dens}) to enable the use of the ToA for the study of the density matrix function to be employed for polarized beams of spin-1 particles [BV2]. As in Sections 8.3-8.5 the focus is on the field motion.

To accomplish this we first introduce the particle-spin motion and field motion of spin-1 particles which can be described by the ToA in terms of the $SO(3)$-space $(E_{v\times t}, l_{v\times t})$ where

$$E_{v\times t} := \mathbb{R}^3 \times E_t \; , \tag{8.152}$$

and where the function $l_{v\times t} : SO(3)\times E_{v\times t} \to E_{v\times t}$ is defined by

$$l_{v\times t}(r;S,M) := (l_v(r;S), l_t(r;M)) = (rS, rMr^t) \; , \tag{8.153}$$

with $r \in SO(3), S \in \mathbb{R}^3, M \in E_t$. We equip $E_{v\times t}$ with the subspace topology from $\mathbb{R}^3\times\mathbb{R}^{3\times 3}$. Thus, and since $\mathbb{R}^3\times\mathbb{R}^{3\times 3}$ with its natural topology is a Hausdorff space, $E_{v\times t}$ is a Hausdorff space, too. Since $(\mathbb{R}^3, l_v)$ and (E_t, l_t) are $SO(3)$-spaces it follows from (8.152) and (8.153) that $(E_{v\times t}, l_{v\times t})$ is an $SO(3)$-space. Using (8.1) and (8.153) it is a simple exercise to show that the 1-turn particle-spin map satisfies

$$\mathcal{P}[E_{v\times t}, l_{v\times t}, j, A](r;z,S,M) = \Big(j(z), l_v(A(z);S), l_t(A(z);M)\Big) \; . \tag{8.154}$$

Clearly if $F : \mathbb{T}^d \to E_{v\times t}$ is a function, then unique functions $f_F : \mathbb{T}^d \to \mathbb{R}^3$ and $T_F : \mathbb{T}^d \to E_t$ exist such that

$$F(z) = (f_F(z), T_F(z)) \;, \tag{8.155}$$

and it is an easy exercise to show that F is continuous iff f_F and T_F are continuous. Using (8.2), (8.153) and (8.155) it is simple to show that the 1-turn field map satisfies

$$\begin{aligned}\tilde{\mathcal{P}}[E_{v\times t}, l_{v\times t}, j, A](F) &= l_{v\times t}\Big(A\circ j^{-1}; F\circ j^{-1}\Big)\\ &= \Big(l_v(A\circ j^{-1}; f_F\circ j^{-1}), l_t(A\circ j^{-1}; T_F\circ j^{-1})\Big) = \Big(\tilde{\mathcal{P}}[j,A](f_F), \tilde{\mathcal{P}}[E_t, l_t, j, A](t_F)\Big) \;,\end{aligned} \tag{8.156}$$

where $F \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3 \times E_t)$. If f_F is an invariant polarization field of (j, A) and T_F is an invariant (E_t, l_t)-field of (j, A) then, by (8.156), $\tilde{\mathcal{P}}[E_{v\times t}, l_{v\times t}, j, A](F) =$ $(\tilde{\mathcal{P}}[j,A](f_F), \tilde{\mathcal{P}}[E_t, l_t, j, A](t_F)) = (f_F, t_F) = F$ whence F is an invariant $(E_{v\times t}, l_{v\times t})$-field of (j, A). Conversely if F is an invariant $(E_{v\times t}, l_{v\times t})$-field of (j, A) then, by (8.156), $(f_F, t_F) = F = \tilde{\mathcal{P}}[E_{v\times t}, l_{v\times t}, j, A](F) = (\tilde{\mathcal{P}}[j,A](f_F), \tilde{\mathcal{P}}[E_t, l_t, j, A](t_F))$ whence f_F is an invariant polarization field of (j, A) and T_F is an invariant (E_t, l_t)-field of (j, A). We thus have proven that F is an invariant $(E_{v\times t}, l_{v\times t})$-field of (j, A) iff f_F is an invariant polarization field of (j, A) and T_F is an invariant (E_t, l_t)-field of (j, A). This completes our outline of the particle-spin and field motion of spin-1 particles and we can now study spin-1 density matrices.

We define

$$E^1_{dens} := \{R \in \mathbb{C}^{3\times 3} : R^\dagger = R, Tr[R] = 1\} \;, \tag{8.157}$$

where $R^\dagger$ denotes the hermitian conjugate of the matrix R and we equip E^1_{dens} with the subspace topology from $\mathbb{C}^{3\times 3}$. Thus, and since $\mathbb{C}^{3\times 3}$ with its natural topology is a Hausdorff space, E^1_{dens} is a Hausdorff space, too. Following a standard parametrization [BV2] we define the function $\beta^1_{dens} : E_{v\times t} \to E^1_{dens}$ for $S \in \mathbb{R}^3, M \in E_t$ by

$$\beta^1_{dens}(S, M) := \frac{1}{3}\Big(I_{3\times 3} + \sum_{i=1}^3 S_i\mathfrak{J}_i + \sqrt{\frac{3}{2}}\sum_{i,k=1}^3 M_{ik}(\mathfrak{J}_i\mathfrak{J}_k + \mathfrak{J}_k\mathfrak{J}_i)\Big) \;, \tag{8.158}$$

where the matrices $\mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3$ are defined by

$$\mathfrak{J}_1 := \sqrt{\frac{1}{2}}\begin{pmatrix} 0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix} \;,\quad \mathfrak{J}_2 := \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix} \;,\quad \mathfrak{J}_3 := \sqrt{\frac{1}{2}}\begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} . \tag{8.159}$$

Clearly every $\mathfrak{J}_k$ is hermitian, i.e., $\mathfrak{J}_k^\dagger = \mathfrak{J}_k$ whence, by (8.158), $\beta^1_{dens}(S, M)$ is hermitian too. For $i, k = 1, 2, 3$,

$$Tr[\mathfrak{J}_i] = 0 \;,\quad Tr[\mathfrak{J}_i\mathfrak{J}_k] = 2\delta_{ik} \;, \tag{8.160}$$

whence, by (8.158) and since $Tr[M] = 0$, we have $Tr[\beta^1_{dens}(S, M)] = 1$. So $\beta^1_{dens}(S, M)$ is a hermitian matrix of trace 1, i.e., it is indeed a function into E^1_{dens}. To show that β^1_{dens}

is a homeomorphism we first note [BV2] that by (8.159), and if $R \in \mathbb{C}^{3\times 3}$ is hermitian, then real numbers S_0', S_1', S_2', S_3' and an $M' \in E_t$ exist such that $R = S_0' I_{3\times 3} + \sum_{i=1}^3 S_i'\mathfrak{J}_i + \sum_{i,k=1}^3 M_{ik}'(\mathfrak{J}_i\mathfrak{J}_k + \mathfrak{J}_k\mathfrak{J}_i)$ whence, by (8.157) and (8.160) and if $R \in E^1_{dens}$, we get $S_0' = 1/3$ so that $R = \beta^1_{dens}(S, M)$ for some $S \in \mathbb{R}^3$ and $M \in E_t$. Thus the function β^1_{dens} is onto E^1_{dens}, i.e.,

$$E^1_{dens} = \{\beta^1_{dens}(S, M) : S \in \mathbb{R}^3, M \in E_t\} \,. \tag{8.161}$$

It also follows [BV2] from (8.158) and (8.159) that, for $S \in \mathbb{R}^3, M \in E_t$ and $i, k = 1, 2, 3$,

$$S_i = Tr[\mathfrak{J}_i\beta^1_{dens}(S, M)] \;, \quad M_{ik} = -\sqrt{\frac{2}{3}}\delta_{ik} + \sqrt{\frac{3}{8}}Tr[(\mathfrak{J}_i\mathfrak{J}_k + \mathfrak{J}_k\mathfrak{J}_i)\beta^1_{dens}(S, M)] \;, \tag{8.162}$$

where S_i denotes the i-th component of S and where M_{ik} denotes the (ik)-th matrix element of M. Thus S and M are uniquely determined by $\beta^1_{dens}(S, M)$ whence β^1_{dens} is one-one so that we conclude that β^1_{dens} is a bijection. Since β^1_{dens} is a bijection it follows from (8.158) and (8.162) that its inverse, $(\beta^1_{dens})^{-1}$, is defined for $R \in E^1_{dens}$ by

$$(\beta^1_{dens})^{-1}(R) := (S, M) \;, \quad S_i := Tr[\mathfrak{J}_i R] \;, \quad M_{ik} = -\sqrt{\frac{2}{3}}\delta_{ik} + \sqrt{\frac{3}{8}}Tr[(\mathfrak{J}_i\mathfrak{J}_k + \mathfrak{J}_k\mathfrak{J}_i)R] \;, \tag{8.163}$$

where S_i denotes the i-th component of S and where M_{ik} denotes the (ik)-th matrix element of M. Moreover, by (8.158) and (8.163), both β^1_{dens} and $(\beta^1_{dens})^{-1}$ are continuous functions whence $\beta^1_{dens} \in \mathrm{Homeo}(E_{v\times t}, E^1_{dens})$, a fact which plays a key role in this section.

We now define the function $l^1_{dens} : SO(3) \times E^1_{dens} \to E^1_{dens}$ by

$$l^1_{dens}(r; R) := \beta^1_{dens}\Big(l_{v\times t}(r; (\beta^1_{dens})^{-1}(R))\Big) \,, \tag{8.164}$$

i.e., by (8.153),

$$l^1_{dens}(r; \beta^1_{dens}(S, M)) = \beta^1_{dens}(l_{v\times t}(r; S, M)) = \beta^1_{dens}(rS, rMr^t) \,, \tag{8.165}$$

with $r \in SO(3), S \in \mathbb{R}^3, M \in E_t$. Recalling that $(E_{v\times t}, l_{v\times t})$ is an $SO(3)$-space and that $\beta^1_{dens} \in \mathrm{Homeo}(E_{v\times t}, E^1_{dens})$ it follows from (8.164) that (E^1_{dens}, l^1_{dens}) is an $SO(3)$-space and that β^1_{dens} is an isomorphism from the $SO(3)$-space $(E_{v\times t}, l_{v\times t})$ to the $SO(3)$-space (E^1_{dens}, l^1_{dens}).

Due to (8.1), the 1-turn particle-spin map $\mathcal{P}[E^1_{dens}, l^1_{dens}, j, A]$ is given by

$$\mathcal{P}[E^1_{dens}, l^1_{dens}, j, A](z, R) = \begin{pmatrix} j(z) \\ l^1_{dens}(A(z); R) \end{pmatrix} \,, \tag{8.166}$$

where $z \in \mathbb{T}^d, R \in E^1_{dens}$. Because the $SO(3)$-spaces $(E_{v\times t}, l_{v\times t})$ and (E^1_{dens}, l^1_{dens}) are isomorphic, we get easy insight into $\mathcal{P}[E^1_{dens}, l^1_{dens}, j, A]$ by using the Second ToA Transformation Rule. In fact recalling Section 8.2.6 we obtain via (8.47) that

$$\mathcal{P}[E^1_{dens}, l^1_{dens}, j, A] \circ \beta^1_{dens,tot} = \beta^1_{dens,tot} \circ \mathcal{P}[E_{v\times t}, l_{v\times t}, j, A] \,, \tag{8.167}$$

i.e.,

$$\begin{aligned}\mathcal{P}[E^1_{dens}, l^1_{dens}, j, A](z, \beta^1_{dens}(S, M)) &= \mathcal{P}[E^1_{dens}, l^1_{dens}, j, A](\beta^1_{dens,tot}(z, S, M)) \\ &= \begin{pmatrix} j(z) \\ \beta^1_{dens}(A(z)S, A(z)MA^t(z)) \end{pmatrix} , \end{aligned} \tag{8.168}$$

where $z \in \mathbb{T}^d, S \in \mathbb{R}^3, M \in E_t$ and where the function $\beta^1_{dens,tot} \in \text{Homeo}(\mathbb{T}^d \times E_{v\times t}, \mathbb{T}^d \times E^1_{dens})$ is defined, for $z \in \mathbb{T}^d, S \in \mathbb{R}^3, M \in E_t$, by $\beta^1_{dens,tot}(z, S, M) := (z, \beta^1_{dens}(S, M))$.

We now come to our main focus, the fields, which in the case $(E, l) = (E^1_{dens}, l^1_{dens})$ are also called spin-1 density matrix functions and which are functions $\rho : \mathbb{T}^d \to E^1_{dens}$ so that $\rho = \beta^1_{dens} \circ F$ where the function $F : \mathbb{T}^d \to E_{v\times t}$ is defined by $F(z) := (\beta^1_{dens})^{-1}(\rho(z))$. Clearly $F = (f_F, T_F)$ where $f_F : \mathbb{T}^d \to \mathbb{R}^3$ and $T_F : \mathbb{T}^d \to E_t$ are uniquely determined by (8.155). Thus, by (8.158), (8.161) and (8.163), we get

$$(f_F)_i(z) = Tr[\rho(z)\mathfrak{J}_i] , \quad (T_F)_{ik}(z) = -\sqrt{\frac{2}{3}}\delta_{ik} + \sqrt{\frac{3}{8}}Tr[(\mathfrak{J}_i\mathfrak{J}_k + \mathfrak{J}_k\mathfrak{J}_i)\rho(z)] , \tag{8.169}$$

$$\begin{aligned}\rho(z) &= \beta^1_{dens}(f_F(z), T_F(z)) \\ &= \frac{1}{3}\Big(I_{3\times3} + \sum_{i=1}^{3} (f_F)_i(z)\mathfrak{J}_i + \sqrt{\frac{3}{2}} \sum_{i,k=1}^{3} (T_F)_{ik}(z)(\mathfrak{J}_i\mathfrak{J}_k + \mathfrak{J}_k\mathfrak{J}_i)\Big) ,\end{aligned} \tag{8.170}$$

where $(f_F)_i$ denotes the i-th component of f_F and where $(T_F)_{ik}$ denotes the (ik)-th matrix element of T_F. Of course, since $\beta^1_{dens} \in \text{Homeo}(E_{v\times t}, E^1_{dens})$, ρ is continuous iff F is continuous, i.e., iff f_F and T_F are continuous. We call an invariant (E^1_{dens}, l^1_{dens})-field an "equilibrium spin-1 density matrix function". Due to (8.2), the 1-turn field map $\tilde{\mathcal{P}}[E^1_{dens}, l^1_{dens}, j, A]$ is given by

$$\tilde{\mathcal{P}}[E^1_{dens}, l^1_{dens}, j, A](\rho) = l^1_{dens}\Big(A \circ j^{-1}; \rho \circ j^{-1}\Big) . \tag{8.171}$$

Because the $SO(3)$-spaces $(E_{v\times t}, l_{v\times t})$ and (E^1_{dens}, l^1_{dens}) are isomorphic, we get easy insight into $\tilde{\mathcal{P}}[E^1_{dens}, l^1_{dens}, j, A]$ by using once again the Second ToA Transformation Rule. In fact recalling Section 8.2.6 we obtain via (8.52) that

$$\tilde{\mathcal{P}}[E^1_{dens}, l^1_{dens}, j, A] \circ \tilde{\beta}^1_{dens} = \tilde{\beta}^1_{dens} \circ \tilde{\mathcal{P}}[E_{v\times t}, l_{v\times t}] , \tag{8.172}$$

where the function $\tilde{\beta}^1_{dens} : \mathcal{C}(\mathbb{T}^d, E_{v\times t}) \to \mathcal{C}(\mathbb{T}^d, E^1_{dens})$ is defined, for $F \in \mathcal{C}(\mathbb{T}^d, E_{v\times t})$, by $\tilde{\beta}^1_{dens}(F) := \beta^1_{dens} \circ F$. It thus follows by Remark 16 that an $F \in \mathcal{C}(\mathbb{T}^d, E_{v\times t})$ is an invariant $(E_{v\times t}, l_{v\times t})$-field of (j, A) iff $\beta^1_{dens} \circ F$ is an invariant (E^1_{dens}, l^1_{dens})-field of (j, A), that is, an equilibrium spin-1 density matrix function of (j, A). Thus, by the remarks after (8.156), a $\rho \in \mathcal{C}(\mathbb{T}^d, E^1_{dens})$ is an equilibrium spin-1 density matrix function of (j, A) iff f_F is an invariant polarization field of (j, A) and T_F is an invariant (E_t, l_t)-field of (j, A) where $F \in \mathcal{C}(\mathbb{T}^d, E_{v\times t})$ is defined by $F := (\beta^1_{dens})^{-1} \circ \rho$.

We thus have proven:

Theorem 8.13 *The function* β^1_{dens} *belongs to* $\text{Homeo}(E_{v\times t}, E^1_{dens})$. *Let* $\rho : \mathbb{T}^d \to E^1_{dens}$. *Then a unique function* $F : \mathbb{T}^d \to E_{v\times t}$ *exists such that* $\rho = \beta^1_{dens} \circ F$. *Moreover* $F = (f_F, T_F)$ *where* $f_F : \mathbb{T}^d \to \mathbb{R}^3$ *and* $T_F : \mathbb{T}^d \to E_t$ *are uniquely determined by* F *via (8.155). Also* F *is continuous iff* f_F *and* T_F *are continuous. Furthermore* $\rho = \beta^1_{dens} \circ (f_F, T_F)$, *i.e., (8.170) holds where* $(f_F)_i$ *denotes the* i*-th component of* f_F *and* $(T_F)_{ik}$ *denotes the* (ik)*-th matrix element of* T_F. *Moreover* ρ *is continuous iff* f_F *and* T_F *are continuous. In addition let* $(j, A) \in \mathcal{SOS}(d, j)$. *Then* ρ *is an equilibrium spin-*1 *density matrix function of* (j, A), *i.e., is an invariant* (E^1_{dens}, l^1_{dens})*-field of* (j, A) *iff* f_F *is an invariant polarization field of* (j, A) *and* T_F *is an invariant* (E_t, l_t)*-field of* (j, A). □

Because of Theorem 8.13 the study of equilibrium spin-1 density matrix functions effectively amounts to to the study of invariant polarization fields and invariant (E_t, l_t)-fields. Since those invariant fields have been studied in other parts of this work this section has been rather brief and we leave the application of the NFT, DT etc. as an exercise for the reader.

Moreover we leave a remark on how to address topological transitivity by applying Lemma 8.4 to $(E, l) = (E^1_{dens}, l^1_{dens})$ to the reader since it would be analogous to Remark 27. In particular in the case of topological transitivity the values of the invariant (E_t, l_t)-field T_F are matrices with the same number of distinct eigenvalues and the case of three eigenvalues is of interest only on spin-orbit resonance.

Furthermore we leave a remark on how to use spin-1 density matrices for the statistical description of a bunch of spin-1 particles to the reader since it would be analogous to Remark 28.

8.7 The topological spaces $\hat{\mathcal{E}}_d[E, l, x, f]$

We now come to an intriguing feature of the ToA, namely the topological spaces $\hat{\mathcal{E}}_d[E, l, x, f]$ to be defined below. They allow us to study invariant (E, l)-fields f, and in particular each ISF, in terms of $\hat{\mathcal{E}}_d[E, l, x, f]$. This leads to a significant avenue for studying the question of the existence of f and for studying IFF's and the generalized IFF's.

8.7.1 Encapsulating invariant (E, l)-fields in the topological spaces $\hat{\mathcal{E}}_d[E, l, x, f]$. The Invariant Reduction Theorem (IRT)

We first need a definition:

Definition 8.14 *Let* (E, l) *be an* $SO(3)$*-space and* $x, y \in E$ *and let* $f \in \mathcal{C}(\mathbb{T}^d, E)$ *take values only in* $l(SO(3); x)$. *We define* $\mathcal{R}[E, l, x, y] := \{r \in SO(3) : l(r; x) = y\}$. *We also define* $\mathcal{E}_d := \mathbb{T}^d \times SO(3)$ *and* $p_d \in \mathcal{C}(\mathcal{E}_d, \mathbb{T}^d)$ *by* $p_d(z, r) := z$ *as well as*

$$\hat{\mathcal{E}}_d[E, l, x, f] \equiv \hat{\mathcal{E}}_d[f] := \{(z, r) \in \mathcal{E}_d : l(r; x) = f(z)\} = \{(z, r) \in \mathcal{E}_d : r \in \mathcal{R}[E, l, x, f(z)]\}$$
$$= \bigcup_{z\in\mathbb{T}^d} \Big(\{z\} \times \mathcal{R}[E, l, x, f(z)]\Big) \subset \mathcal{E}_d \;, \tag{8.173}$$

and equip $\hat{\mathcal{E}}_d[f]$ *with the subspace topology from* $\mathcal{E}_d$. *Note that* $\mathcal{E}_d$ *is the compact topological space equipped with the product topology from* $\mathbb{T}^d$ *and* $SO(3)$. *We will use the abbreviation* $\hat{\mathcal{E}}_d[f]$ *when* E, l *and* x *are clear from the context.*

We also define the function $\hat{\mathcal{P}}[j,A]:\mathcal{E}_d\to\mathcal{E}_d$ *by*

$$\hat{\mathcal{P}}[j,A](z,r):=\begin{pmatrix} j(z)\\ A(z)r\end{pmatrix}\,, \tag{8.174}$$

where $z\in\mathbb{T}^d$ *and* $r\in SO(3)$*. Note that, by (8.174) and Remark 9,*
$\hat{\mathcal{P}}[j,A]=\mathcal{P}[SO(3),l_{SOR};j,A]$ *whence, recalling Section 8.2.1,* $\hat{\mathcal{P}}[j,A]\in\mathrm{Homeo}(\mathcal{E}_d)$*.* □

Note that if $r\in\mathcal{R}[E,l,x,y]$ then $\mathcal{R}[E,l,x,y]=rIso(E,l,x)$, i.e., every $\mathcal{R}[E,l,x,y]$ is a "copy" of $Iso(E,l,x)$. More precisely, $\mathcal{R}[E,l,x,y]$ is a so-called left coset of the subgroup $Iso(E,l,x)$ of $SO(3)$ with respect to r. It is also a simple exercise to show, by (8.173), that if $\hat{\mathcal{E}}_d[f]=\hat{\mathcal{E}}_d[g]$ then $f=g$.

In the following remark we derive another important property of $\hat{\mathcal{E}}_d[f]$.

Remark:

(29) Let (E,l) be an $SO(3)$-space where E is Hausdorff and let $\hat{\mathcal{E}}_d[f]$ be given as in Definition 8.14. Then the topological space $\hat{\mathcal{E}}_d[f]$ is compact as follows. In fact we note, by (8.173), that

$$\begin{aligned}\hat{\mathcal{E}}_d[f]&=\{(z,r)\in\mathcal{E}_d: l(r;x)=f(z)\}=\{(z,r)\in\mathcal{E}_d: x=l(r^t;f(z))\}\\ &=\{(z,r)\in\mathcal{E}_d: l(r^t;f(z))\in\{x\}\}\subset\mathcal{E}_d\,,\end{aligned} \tag{8.175}$$

whence $\hat{\mathcal{E}}_d[f]$ is the inverse image of $\{x\}$ under a continuous function. Since E is Hausdorff, $\{x\}$ is a closed subset of E whence we conclude that $\hat{\mathcal{E}}_d[f]$ is a closed subset of the compact topological space $\mathcal{E}_d$. This implies that $\hat{\mathcal{E}}_d[f]$ is compact [Mu]. □

With Definition 8.14 we arrive at:

Theorem 8.15 *(IRT) Let* (E,l) *be an* $SO(3)$*-space and* $x\in E$ *and let* $f\in\mathcal{C}(\mathbb{T}^d,E)$ *take values only in* $l(SO(3);x)$*. Then* f *is an invariant* (E,l)*-field of* (j,A) *iff*
$\hat{\mathcal{P}}[j,A](\hat{\mathcal{E}}_d[f])=\hat{\mathcal{E}}_d[f]$*.*

Proof of Theorem 8.15: We conclude from (8.173),(8.174) that

$$\begin{aligned}\hat{\mathcal{P}}[j,A](\hat{\mathcal{E}}_d[f])&=\hat{\mathcal{P}}[j,A](\{(z,r)\in\mathcal{E}_d: l(r;x)=f(z)\})\\ &=\Big\{(j(z),A(z)r):(z,r)\in\mathcal{E}_d, l(r;x)=f(z)\Big\}\\ &=\Big\{(z',r')\in\mathcal{E}_d: l\Big(A^t(j^{-1}(z'))r';x\Big)=f(j^{-1}(z'))\Big\}\\ &=\Big\{(z',r')\in\mathcal{E}_d: l(r';x)=l\Big(A(j^{-1}(z'));f(j^{-1}(z'))\Big)\Big\}\\ &=\{(z',r')\in\mathcal{E}_d: l(r';x)=f'(z')\}=\hat{\mathcal{E}}_d[f']\,,\end{aligned} \tag{8.176}$$

where $f'\in\mathcal{C}(\mathbb{T}^d,E)$ is defined by $f':=\tilde{\mathcal{P}}[E,l,j,A](f)$, i.e. (recall (8.2)),
$f'(z)=l(A(j^{-1}(z));f(j^{-1}(z)))$.

If f is an invariant (E,l)-field of (j,A) then, by Section 8.2.1, $f = f'$ whence, by (8.176), $\hat{\mathcal{P}}[j,A](\hat{\mathcal{E}}_d[f]) = \hat{\mathcal{E}}_d[f]$.

Conversely, let $\hat{\mathcal{P}}[j,A](\hat{\mathcal{E}}_d[f]) = \hat{\mathcal{E}}_d[f]$. Then, by (8.176), $\hat{\mathcal{E}}_d[f'] = \hat{\mathcal{E}}_d[f]$ where $f' := \tilde{\mathcal{P}}[E,l,j,A](f)$. It follows from the remark after Definition 8.14 that $f = f'$ whence f is an invariant (E,l)-field of (j,A). □

We will see below how the topological spaces $\hat{\mathcal{E}}_d[f]$ in Theorem 8.15 can be used for the question of the existence of invariant (E,l)-fields in particular ISF's (recall that an ISF takes values only in $l_v(SO(3);(0,0,1)^t)$).

The terminology "reduction" refers to $\mathcal{E}_d$ being "reduced" to the subspace $\hat{\mathcal{E}}_d[f]$ and the terminology "invariant" refers to the invariance condition: $\hat{\mathcal{P}}[j,A](\hat{\mathcal{E}}_d[f]) = \hat{\mathcal{E}}_d[f]$, i.e., $\hat{\mathcal{E}}_d[f]$ is an invariant subset of $\mathcal{E}_d$ under the function $\hat{\mathcal{P}}[j,A]$. For more details and the definition of invariant reductions, see [Fe, Zi2] and Chapter 9 in [HK1] as well as our comments on bundle theory in Section 8.8. By the bundle aspect of the IRT the concept of ISF is rather deep.

With Definition 8.14 we have encapsulated f into the topological space $\hat{\mathcal{E}}_d[E,l,x,f] \equiv \hat{\mathcal{E}}_d[f]$. Below we will see how one gets insight into $\hat{\mathcal{E}}_d[f]$ in terms of the f-independent topological spaces $\check{\mathcal{E}}_d[z,y]$ which we now define:

Definition 8.16 *Let (E,l) be an $SO(3)$-space and $x,y \in E$. Let also $(j,A) \in \mathcal{SOS}(d,j)$ and $z \in \mathbb{T}^d$. Then we define*

$$\begin{aligned}\check{\mathcal{E}}_d[E,l,j,A,z,x,y] \equiv \check{\mathcal{E}}_d[z,y] &:= \bigcup_{n\in\mathbb{Z}} \hat{\mathcal{P}}[j,A]^n(\{z\}\times\{r\in SO(3): l(r;x)=y\}) \\ &= \bigcup_{n\in\mathbb{Z}}\Big(\{j^n(z)\}\times\{r\in SO(3): l(r;x) = l(\Psi[j,A](n;z);y)\}\Big) \\ &= \bigcup_{n\in\mathbb{Z}}\Big(\{j^n(z)\}\times\mathcal{R}[E,l,x,l(\Psi[j,A](n;z);y)]\Big) \subset \mathcal{E}_d\ ,\end{aligned} \tag{8.177}$$

where in the third equality we used (8.9). Clearly $\check{\mathcal{E}}_d[z,y]$ is nonempty iff x,y belong to the same (E,l)-orbit. We equip $\check{\mathcal{E}}_d[z,y]$ and $\overline{\check{\mathcal{E}}_d[z,y]}$ with the subspace topology from $\mathcal{E}_d$. We will use the abbreviation $\check{\mathcal{E}}_d[z,y]$ when E,l,j,A and x are clear from the context. □

The following corollary to the IRT gives us a first glimpse into how one gets insight into $\hat{\mathcal{E}}_d[f]$ by the topological spaces $\check{\mathcal{E}}_d[z,y]$:

Theorem 8.17 *Let (E,l) be an $SO(3)$-space and $(j,A) \in \mathcal{SOS}(d,j)$. Also let*

$$\hat{\mathcal{E}}_d[f] = \overline{\check{\mathcal{E}}_d[z,y]}\ . \tag{8.178}$$

Then f is an invariant (E,l)-field of (j,A) and j is topologically transitive. Moreover if $(z',r') \in \overline{\check{\mathcal{E}}_d[z,y]}$ then $f(z') = l(r';x)$.

Proof of Theorem 8.17: Since the image of a union is a union of images we conclude from (8.177) that

$$\begin{aligned}\hat{\mathcal{P}}[j,A](\check{\mathcal{E}}_d[z,y]) &= \bigcup_{n\in\mathbb{Z}} \hat{\mathcal{P}}[j,A]^{n+1}(\{z\}\times\{r\in SO(3): l(r;x)=y\}) \\ &= \bigcup_{n\in\mathbb{Z}} \hat{\mathcal{P}}[j,A]^{n}(\{z\}\times\{r\in SO(3): l(r;x)=y\}) = \check{\mathcal{E}}_d[z,y]\ , \end{aligned} \tag{8.179}$$

i.e., $\check{\mathcal{E}}_d[z,y]$ is invariant under $\hat{\mathcal{P}}[j,A]$. Since $\hat{\mathcal{P}}[j,A]$ is a homeomorphism we get from (8.179)

$$\overline{\check{\mathcal{E}}_d[z,y]} = \overline{\hat{\mathcal{P}}[j,A](\check{\mathcal{E}}_d[z,y])} = \hat{\mathcal{P}}[j,A](\overline{\check{\mathcal{E}}_d[z,y]})\ , \tag{8.180}$$

where in the second equality we used [Du, Section III.12]. By (8.180), $\overline{\check{\mathcal{E}}_d[z,y]}$ is invariant under $\hat{\mathcal{P}}[j,A]$ whence, by our assumption (8.178), $\hat{\mathcal{E}}_d[f]$ is invariant under $\hat{\mathcal{P}}[j,A]$ and this implies, by the IRT, that f is an invariant (E,l)-field of (j,A). Using (8.178) and Definition 8.14 we get

$$\mathbb{T}^d = p_d(\hat{\mathcal{E}}_d[f]) = p_d(\overline{\check{\mathcal{E}}_d[z,y]}) \subset \overline{p_d(\check{\mathcal{E}}_d[z,y])} = \overline{\bigcup_{n\in\mathbb{Z}}\{j^n(z)\}} = \overline{\{j^n(z): n\in\mathbb{Z}\}} \subset \mathbb{T}^d\ ,$$

where in the inclusion we used [Du, Section III.8]. Thus $\mathbb{T}^d = \overline{\{j^n(z): n\in\mathbb{Z}\}}$ whence j is topologically transitive (recall Section 2.3). Furthermore if $(z',r')\in\overline{\check{\mathcal{E}}_d[z,y]}$ then, by (8.178), $(z',r')\in\hat{\mathcal{E}}_d[f]$ whence, by Definition 8.14, $f(z')=l(r';x)$. □

Since $\overline{\check{\mathcal{E}}_d[z,y]}$ is a closed subset of the compact topological space $\mathcal{E}_d$ and since it is equipped with the subspace topology from $\mathcal{E}_d$, it is a compact topological space [Mu]. A key motivation of using $\check{\mathcal{E}}_d[z,y]$ for the existence problem of invariant (E,l)-fields is our belief that, unless (8.178) holds, the topological spaces $\overline{\check{\mathcal{E}}_d[z,y]}$ are in general different from the $\hat{\mathcal{E}}_d[f]$ and this may display itself in different homology and homotopy groups and other features of these topological spaces. Thus in this approach to the existence problem one "scans" through all $\overline{\check{\mathcal{E}}_d[z,y]}$ by varying y over $l(SO(3);x)$. However this issue is beyond the scope of this work.

In the following section we will get more insight into the relation between $\hat{\mathcal{E}}_d[f]$ and $\check{\mathcal{E}}_d[z,y]$.

8.7.2 Further properties of the topological spaces $\hat{\mathcal{E}}_d[E,l,x,f]$. The Cross Section Theorem (CST)

In the situation of the NFT one has an $SO(3)$-space (E,l) and an $x\in E$ as well as functions $T\in\mathcal{C}(\mathbb{T}^d,SO(3))$ and $f\in\mathcal{C}(\mathbb{T}^d,E)$ which are related by $f(z)=l(T(z);x)$, i.e., T is an (E,l)-lift of f. Note that a necessary condition to satisfy this relation is that f takes values only in $l(SO(3);x)$. Perhaps surprisingly, it is not a sufficient condition, i.e., there are situations where $f\in\mathcal{C}(\mathbb{T}^d,E)$ takes only values in $l(SO(3);x)$ and where nevertheless no $T\in\mathcal{C}(\mathbb{T}^d,SO(3))$ exists such that $f(z)=l(T(z);x)$. This is quite remarkable since, if f takes values only in $l(SO(3);x)$, then there exists for each z a $T(z)\in SO(3)$ such that $f(z)=l(T(z);x)$, i.e., there always exists a function $T:\mathbb{T}^d\to SO(3)$ such that $f(\cdot)=l(T(\cdot);x)$. Interesting examples arise in the case of Chapters 2-7 where $(E,l)=(\mathbb{R}^3,l_v)$. In fact one can show by using simple arguments from Homotopy Theory [He2] that, if $d \geq 2$, then $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ exist such that f takes values only in $l_v(SO(3); (0,0,1)^t)$, i.e., such that $|f(z)| = 1$ and no continuous $T : \mathbb{T}^d \to SO(3)$ exists such that $f(z) = l_v(T(z); (0,0,1)^t)$ (of course T exists if we relax the continuity condition on T). One can also show by using simple arguments from Homotopy Theory [He2] that for every $f \in \mathcal{C}(\mathbb{T}^1, \mathbb{R}^3)$ such that $|f(z)| = 1$ an $T \in \mathcal{C}(\mathbb{T}^1, SO(3))$ exists such that $f(z) = l_v(T(z); (0,0,1)^t)$.

The central theme of this section is to show how the set $\hat{\mathcal{E}}_d[f]$ gives a simple sufficient and necessary condition on f to be of the form $f(z) = l(T(z); x)$ with $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$. In fact this condition, stated in Theorem 8.19 below, is the existence of a "cross section" of the function $p_d[E, l, x, f]$ (to be defined below). Thus in the situation of the NFT a cross section exists.

We first define $p_d[f]$ and the function $\Upsilon_d[T]$ and its restriction.

Definition 8.18 *Let (E, l) be an $SO(3)$-space and $x \in E$ and let $f \in \mathcal{C}(\mathbb{T}^d, E)$ take values only in $l(SO(3); x)$. Thus $\hat{\mathcal{E}}_d[f]$ is well-defined and we define the function $p_d[E, l, x, f] \in \mathcal{C}(\hat{\mathcal{E}}_d[f], \mathbb{T}^d)$ by*

$$p_d[E, l, x, f](z, r) \equiv p_d[f](z, r) := z \, , \tag{8.181}$$

where $z \in \mathbb{T}^d, r \in SO(3)$. We will use the abbreviation $p_d[f]$ when E, l and x are clear from the context. Clearly $p_d[f]$ is surjective since, by Definition 8.14 and for every $z \in \mathbb{T}^d$, there is an $r \in SO(3)$ such that $(z, r) \in \mathcal{E}_d[f]$.

Given $T : \mathbb{T}^d \to SO(3)$, we define the function $\Upsilon_d[T] : \mathcal{E}_d \to \mathcal{E}_d$ by

$$\Upsilon_d[T](z, r) := (z, T(z)r) \, . \tag{8.182}$$

We also define the surjection $\hat{\Upsilon}_d[E, l, x, T] : \mathbb{T}^d \times Iso(E, l; x) \to \hat{\Upsilon}_d[E, l, x, T](\mathbb{T}^d \times Iso(E, l; x))$ as the restriction of $\Upsilon_d[T]$, i.e.,

$$\hat{\Upsilon}_d[E, l, x, T](z, r) \equiv \hat{\Upsilon}_d[T](z, r) := \Upsilon_d[T](z, r) = (z, T(z)r) \, . \tag{8.183}$$

We will use the abbreviation $\hat{\Upsilon}_d[T]$ when E, l and x are clear from the context. □

It is an easy exercise to show that $\Upsilon_d[T]$ is a bijection and that $\Upsilon_d[T^t]$ is its inverse. Moreover if T is continuous then, by (8.174) and (8.182), $\Upsilon_d[T] = \hat{\mathcal{P}}[id_{\mathbb{T}^d}, T]$ whence in this case, by Definition 8.14, $\Upsilon_d[T] \in \text{Homeo}(\mathcal{E}_d)$. Note also, by (8.183) and Definition 8.14, that the range of $\hat{\Upsilon}_d[T]$ reads as

$$\begin{aligned} \hat{\Upsilon}_d[E, l, x, T](\mathbb{T}^d \times Iso(E, l; x)) &= \bigcup_{z \in \mathbb{T}^d} \Big(\{z\} \times T(z) Iso(E, l; x) \Big) \\ &= \bigcup_{z \in \mathbb{T}^d} \Big(\{z\} \times \mathcal{R}[E, l, x, l(T(z); x)] \Big) \, . \end{aligned} \tag{8.184}$$

Moreover since $\Upsilon_d[T]$ is a bijection, so is $\hat{\Upsilon}_d[T]$ and the latter's inverse $\hat{\Upsilon}_d[T]^{-1}$ is defined by $\hat{\Upsilon}_d[T]^{-1}(z, r) := \Upsilon_d[T^t](z, r)$.

Since $p_d[f]$ is a surjection it has a right-inverse, i.e., a function $\sigma : \mathbb{T}^d \to \hat{\mathcal{E}}_d[f]$ such that $p_d[f] \circ \sigma = id_{\mathbb{T}^d}$. Note that, by Definition 8.14, $p_d[f]$ has more than one right-inverse except for the case when $Iso(E,l;x) = G_0$. Furthermore even though $p_d[f]$ is continuous, it does not always have a continuous right-inverse. A cross section of $p_d[f]$ is, by definition, a continuous right-inverse (see also Appendix A.4) and the following theorem sheds light at the cross sections of $p_d[f]$.

Theorem 8.19 *(CST)*
Let (E,l) be an $SO(3)$-space and let $x \in E$. Abbreviating $H := Iso(E,l;x)$ the following hold.

a) Let $f \in \mathcal{C}(\mathbb{T}^d, E)$ take values only in $l(SO(3);x)$. Then $p_d[f]$ has a cross section iff a $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ exists such that $f(z) = l(T(z);x)$. In other words, $p_d[f]$ has a cross section iff f has an (E,l)-lift (see the remarks after the NFT).

b) Let $f \in \mathcal{C}(\mathbb{T}^d, E)$ take values only in $l(SO(3);x)$ and let us pick a function $T : \mathbb{T}^d \to SO(3)$ such that $f(z) = l(T(z);x)$. Then the bijection $\hat{\Upsilon}_d[T]$ is onto $\hat{\mathcal{E}}_d[f]$. Also T is continuous iff $\hat{\Upsilon}_d[T]$ is a homeomorphism, i.e., $\hat{\Upsilon}[T] \in \mathrm{Homeo}(\mathbb{T}^d \times H, \hat{\mathcal{E}}_d[f])$.

c) Let $x \in E$ and $(j, A) \in \mathcal{SOS}(d, j)$. Let f be an invariant (E,l)-field of (j, A) which takes values only in $l(SO(3);x)$. Then $p_d[f]$ has a cross section iff a T in $\mathcal{TF}_H(j, A)$ exists such that $f(z) = l(T(z);x)$. Moreover if $p_d[f]$ has a cross section then $(j, A) \in \mathcal{CB}_H(d, j)$.

Proof of Theorem 8.19a: "$\Rightarrow$": Let σ be a cross section of $p_d[f]$. Since $\sigma \in \mathcal{C}(\mathbb{T}^d, \hat{\mathcal{E}}_d[f])$ and $\hat{\mathcal{E}}_d[f] \subset \mathcal{E}_d$ we have $\sigma(z) = (\tau(z), T(z))$ where $\tau \in \mathcal{C}(\mathbb{T}^d, \mathbb{T}^d)$ and $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$. We compute $z = id_{\mathbb{T}^d}(z) = p_d[f](\sigma(z)) = p_d[f](\tau(z), T(z)) = \tau(z)$ whence $\sigma(z) = (z, T(z)) \in \hat{\mathcal{E}}_d[f]$ so that, by Definition 8.14, $f(z) = l(T(z);x)$.

"$\Leftarrow$": Let $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ such that $f(z) = l(T(z);x)$. Thus, by (8.173), $(z, T(z)) \in \hat{\mathcal{E}}_d[f]$ whence with the function $\sigma \in \mathcal{C}(\mathbb{T}^d, \hat{\mathcal{E}}_d[f])$ defined by $\sigma(z) := (z, T(z))$, we see that $p_d[f](\sigma(z)) = z$. Therefore $p_d[f] \circ \sigma = id_{\mathbb{T}^d}$ so that σ is a cross section of $p_d[f]$. $\Box$
Proof of Theorem 8.19b: It follows from (8.184) and Definition 8.14 that

$$\hat{\Upsilon}_d[E,l,x,T](\mathbb{T}^d \times H) = \bigcup_{z \in \mathbb{T}^d} \Big(\{z\} \times \mathcal{R}[E,l,x,l(T(z);x)] \Big)$$
$$= \bigcup_{z \in \mathbb{T}^d} \Big(\{z\} \times \mathcal{R}[E,l,x,f(z)] \Big) = \hat{\mathcal{E}}_d[E,l,x,f] \,,$$

whence $\hat{\Upsilon}_d[T]$ is onto $\hat{\mathcal{E}}_d[f]$. To prove the second claim, first of all, let T be continuous. Thus, by the remarks after Definition 8.18, $\Upsilon_d[T] \in \mathrm{Homeo}(\mathcal{E}_d)$. Then, recalling from the remarks after Definition 8.18, that $\hat{\Upsilon}_d[T]$ is a bijection and a restriction of $\Upsilon_d[T]$ we conclude that $\hat{\Upsilon}_d[T] \in \mathrm{Homeo}(\mathbb{T}^d \times H, \hat{\mathcal{E}}_d[f])$. Secondly let $\hat{\Upsilon}_d[T] \in \mathrm{Homeo}(\mathbb{T}^d \times H, \hat{\mathcal{E}}_d[f])$. It follows from (8.183) that $(z, T(z)) = \hat{\Upsilon}_d[T](z, I_{3\times3})$ whence, since $\hat{\Upsilon}_d[T]$ is continuous, so is T. $\Box$
Proof of Theorem 8.19c: Let first of all $p_d[f]$ have a cross section. Thus, by Theorem 8.19a, a $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ exists such that $f(z) = l(T(z);x)$. Since f is an invariant (E,l)-field of (j, A) we thus conclude from the NFT, Theorem 8.1, that $T \in \mathcal{TF}_H(j, A)$. Let secondly $T \in \mathcal{TF}_H(j, A)$ such that $f(z) = l(T(z);x)$. Thus T is continuous whence, by Theorem 8.19a, $p_d[f]$ has a cross section. This completes the proof of the first claim. It follows from

the first claim that if $p_d[f]$ has a cross section then $\mathcal{TF}_H(j,A)$ is nonempty whence, by Definition 5.1, $(j,A) \in \mathcal{CB}_H(d,j)$. $\square$

The following remark reconsiders IFF's in terms of the CST:

Remark:

(30) Let $(j,A) \in \mathcal{SOS}(d,j)$ have an invariant polarization field f which takes values only in $l_v(SO(3);S_\lambda) = \mathbb{S}^2_\lambda$ where $\lambda > 0$, i.e., $|f| = \lambda > 0$. Then, by Theorem 8.19c, $p_d[\mathbb{R}^3, l_v, S_\lambda, f]$ has a cross section iff a T in $\mathcal{TF}_H(j,A)$ exists such that $f(z) = l_v(T(z);S_\lambda)$, i.e., iff $f(z) = \lambda T(z)(0,0,1)^t$ where $H := Iso(\mathbb{R}^3, l_v, S_\lambda) = SO(2)$. Thus, by Definition 5.3, $p_d[\mathbb{R}^3, l_v, S_\lambda, f]$ has a cross section iff (j,A) has an IFF whose third column is f/λ. $\square$

A similar remark could be made about SOR by using $(E,l) = (SO(3), l_{SOR})$. For the bundle aspect of the CST see Section 8.8. By the bundle aspect of the CST the above remarks indicate that the concepts of IFF and SOR are rather deep.

As mentioned after Theorem 8.17, a key motivation of using $\check{\mathcal{E}}_d[z,y]$ for the existence problem of invariant (E,l)-fields is our belief that the topological spaces $\overline{\check{\mathcal{E}}_d[z,y]}$ are in general different from the $\hat{\mathcal{E}}_d[f]$ unless (8.178) holds. The CST sheds further light on this issue since, by Theorem 8.19a-b, $\hat{\mathcal{E}}_d[f]$ is homeomorphic to $\mathbb{T}^d \times Iso(E,l;x)$ if $p_d[f]$ has a cross section whence the key idea is to compare the topological spaces $\overline{\check{\mathcal{E}}_d[z,y]}$ with $\mathbb{T}^d \times Iso(E,l;x)$. Note that in the important case of the existence problem of the ISF we have $Iso(\mathbb{R}^3, l_v; (0,0,1)^t) = SO(2)$ whence $\hat{\mathcal{E}}_d[f]$ in this case is homeomorphic to $\mathbb{T}^{d+1}$ (note that $SO(2)$ is homeomorphic to $\mathbb{T}^1$).

The following corollary to the CST is a partial converse of Theorem 8.17 giving us further insight into the relation between $\hat{\mathcal{E}}_d[f]$ and $\check{\mathcal{E}}_d[z,y]$.

Theorem 8.20 *Let (E,l) be an $SO(3)$-space where E is Hausdorff. Let also $(j,A) \in \mathcal{SOS}(d,j)$ have an invariant (E,l)-field f and let j be topologically transitive, i.e., let a $z_0 \in \mathbb{T}^d$ exist such that $\overline{\{j^n(z_0) : n \in \mathbb{Z}\}} = \mathbb{T}^d$. By Lemma 8.4 an $x \in E$ exists such that f takes values only in $l(SO(3);x)$. If $p_d[f]$ has a cross section then*

$$\hat{\mathcal{E}}_d[f] = \overline{\check{\mathcal{E}}_d[z_0, f(z_0)]} \,. \tag{8.185}$$

Proof of Theorem 8.20: Our strategy is to prove that $\hat{\Upsilon}_d[T]^{-1}(\hat{\mathcal{E}}_d[f]) = \hat{\Upsilon}_d[T]^{-1}(\overline{\check{\mathcal{E}}_d[z_0, f(z_0)]})$. Since f is an invariant (E,l)-field of (j,A) it follows from (8.12) that, for every $n \in \mathbb{Z}$, $f(j^n(z_0)) = l(\Psi[j,A](n;z_0); f(z_0))$ whence

$$\begin{aligned}\check{\mathcal{E}}_d[z_0, f(z_0)] &= \bigcup_{n\in\mathbb{Z}} \Big(\{j^n(z_0)\} \times \{r \in SO(3) : l(r;x) = l(\Psi[j,A](n;z_0); f(z_0))\} \Big) \\ &= \bigcup_{n\in\mathbb{Z}} \Big(\{j^n(z_0)\} \times \{r \in SO(3) : l(r;x) = f(j^n(z_0))\} \Big) \subset \hat{\mathcal{E}}_d[f] \,, \end{aligned} \tag{8.186}$$

where the first equality in (8.186) follows from (8.177). Of course the inclusion in (8.186) follows from the definition of $\hat{\mathcal{E}}_d[f]$. By Theorem 8.19a, a $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ exists such that

$f(z) = l(T(z); x)$ whence, by Theorem 8.19b, $\hat{\Upsilon}_d[T] \in \text{Homeo}(\mathbb{T}^d \times H, \hat{\mathcal{E}}_d[f])$ where $H := Iso(E, l; x)$. Because of (8.186), $\hat{\Upsilon}_d[T]^{-1}(\check{\mathcal{E}}_d[z_0, f(z_0)])$ is well-defined and so we compute

$$\begin{aligned}\overline{\check{\mathcal{E}}_d[z_0, f(z_0)]} &= \overline{\hat{\Upsilon}_d[T](\hat{\Upsilon}_d[T]^{-1}(\check{\mathcal{E}}_d[z_0, f(z_0)]))} \\ &= \hat{\Upsilon}_d[T](\overline{\hat{\Upsilon}_d[T]^{-1}(\check{\mathcal{E}}_d[z_0, f(z_0)])}) \ ,\end{aligned} \tag{8.187}$$

where in the second equality we used [Du, Section XII.2] and the fact that $\hat{\Upsilon}_d[T]$ is a homeomorphism. Using (8.182),(8.186) and the remarks after Definition 8.18, we compute

$$\begin{aligned}&\hat{\Upsilon}_d[T]^{-1}(\check{\mathcal{E}}_d[z_0, f(z_0)]) = \Upsilon_d[T^t](\check{\mathcal{E}}_d[z_0, f(z_0)]) \\ &= \Upsilon_d[T^t](\bigcup_{n\in\mathbb{Z}}\Big(\{j^n(z_0)\} \times \{r \in SO(3) : l(r; x) = f(j^n(z_0))\}\Big)) \\ &= \Upsilon_d[T^t](\bigcup_{n\in\mathbb{Z}}\Big(\{j^n(z_0)\} \times \{r \in SO(3) : l(r; x) = l(T(j^n(z_0)); x)\}\Big)) \\ &= \bigcup_{n\in\mathbb{Z}}\Big(\{j^n(z_0)\} \times \{T^t(j^n(z_0))r : r \in SO(3), l(r; x) = l(T(j^n(z_0)); x)\}\Big)\Big) \\ &= \bigcup_{n\in\mathbb{Z}}\Big(\{j^n(z_0)\} \times \{r' \in SO(3) : l(T(j^n(z_0))r'; x) = l(T(j^n(z_0)); x)\}\Big)\Big) \\ &= \bigcup_{n\in\mathbb{Z}}\Big(\{j^n(z_0)\} \times \{r' \in SO(3) : l(r'; x) = x\}\Big)\Big) = \Big(\bigcup_{n\in\mathbb{Z}}\{j^n(z_0)\}\Big) \times H \\ &= B \times H \ ,\end{aligned}$$

where $B := \{j^n(z_0) : n \in \mathbb{Z}\}$ which implies

$$\hat{\Upsilon}_d[T]^{-1}(\check{\mathcal{E}}_d[z_0, f(z_0)]) = B \times H \ . \tag{8.188}$$

To prove (8.185), we conclude from (8.188) that

$$\overline{\hat{\Upsilon}_d[T]^{-1}(\check{\mathcal{E}}_d[z_0, f(z_0)])} = \overline{B \times H} = \bar{B} \times \bar{H} = \mathbb{T}^d \times \bar{H} = \mathbb{T}^d \times H \ , \tag{8.189}$$

where in the second equality we used [Du, Section IV.1] and in the fourth equality we used that H is closed (the latter follows from Remark 17 and the fact that E is Hausdorff). Inserting (8.189) into (8.187) results in $\overline{\check{\mathcal{E}}_d[z_0, f(z_0)]} = \hat{\Upsilon}_d[T](\mathbb{T}^d \times H) = \hat{\mathcal{E}}_d[f]$ where in the second equality we used from Theorem 8.19b that $\hat{\Upsilon}_d[T]$ is onto $\hat{\mathcal{E}}_d[f]$. □

The following remark discusses Theorem 8.20 in the special case of the ISF.

Remark:

(31) Let $(j, A) \in \mathcal{SOS}(d, j)$ where j is topologically transitive and so we have a $z_0 \in \mathbb{T}^d$ such that the set $\{j^n(z_0) : n \in \mathbb{Z}\}$ is dense in $\mathbb{T}^d$. Also let $(E, l) = (\mathbb{R}^3, l_v)$ and f be an invariant polarization field of (j, A) such that f is not the zero field. By Lemma 8.4 and Remark 13 we can pick a $\lambda \in (0, \infty)$ such that f takes values only in the sphere

$l(SO(3); S_\lambda) = \mathbb{S}^2_\lambda$. Let also $p_d[\mathbb{R}^3, l_v, S_\lambda, f]$ have a cross section, i.e., let by Remark 30 (j, A) have an IFF whose third column is f/λ. Then, by Theorem 8.20,

$$\hat{\mathcal{E}}_d[\mathbb{R}^3, l_v, S_\lambda, f] = \overline{\check{\mathcal{E}}_d[\mathbb{R}^3, l_v, j, A, z_0, S_\lambda, f(z_0)]} \,. \tag{8.190}$$

Note that in the special case $d = 1$ the condition, that $p_d[\mathbb{R}^3, l_v, S_\lambda, f]$ has a cross section, is redundant (see the remarks at the beginning of this section). Thus for addressing the ISF conjecture, one should perhaps start with $d = 1$. □

8.8 Underlying bundle theory

While bundle aspects were not needed in the present work it is worthwhile to mention them since they are the origin of the ToA (see [Fe, He2, Zi2] and Chapter 9 in [HK1]) and therefore supply a steady flow of ideas, many of which not even mentioned here (e.g., algebraic hull, characteristic class, rigidity). The "unreduced" principal bundle underlying our formalism is a product principal $SO(3)$-bundle with base space $\mathbb{T}^d$, i.e., it can be written as the 4-tuple $(\mathcal{E}_d, p_d, \mathbb{T}^d, L_d)$ where $\mathcal{E}_d = \mathbb{T}^d \times SO(3)$ is the bundle space, $p_d \in \mathcal{C}(\mathcal{E}_d, \mathbb{T}^d)$ the bundle projection, i.e., $p_d(z, r) := z$, and $(\mathcal{E}_d, L_d)$ the underlying $SO(3)$-space where $L_d : SO(3) \times \mathcal{E}_d \to \mathcal{E}_d$ defined by $L_d(r; z, r') := (z, r' r^t)$. For every (j, A), bundle theory gives us, via the so-called automorphism group of the unreduced principal bundle, a natural particle-spin map on $\mathcal{E}_d$ which turns out to be $\hat{\mathcal{P}}[j, A]$ in (8.174). Note that here the cocycle property of the spin transfer matrix function is crucial. The reductions are those principal H-bundles which are subbundles of the unreduced bundle such that their bundle space is a closed subset of $\mathcal{E}_d$ and such that H is a closed subgroup of $SO(3)$. By the well-known Reduction Theorem [Fe, Chapter 6], [Hus, Chapter 6], every $(\hat{\mathcal{E}}_d[f], p_d[E, l, x, f], \mathbb{T}^d, L)$, for which E is Hausdorff, is a reduction where L is the restriction of L_d to $H \times \hat{\mathcal{E}}_d[f]$ and conversely, every reduction is of this form. By bundle theory, the natural particle-spin map on $\hat{\mathcal{E}}_d[f]$ for a given (j, A) is that bijection on $\mathcal{E}_d[f]$ which is a restriction of $\hat{\mathcal{P}}[j, A]$. Clearly this function is a bijection iff $\hat{\mathcal{E}}_d[f]$ is $\hat{\mathcal{P}}[j, A]$-invariant and then the reduction is called "invariant under (j, A)". Thus indeed the IRT deals with invariant reductions and it states that a reduction is invariant under (j, A) iff f is an invariant field.

The bundle-theoretic aspect of the CST follows from the simple fact that the cross sections of $p_d[E, l, x, f]$ are the bundle-theoretic cross sections of the reduction. Thus, by bundle theory, $p_d[E, l, x, f]$ has a cross section iff the principal bundle $(\hat{\mathcal{E}}_d[f], p_d[f], \mathbb{T}^d, L)$ is trivial, i.e., is isomorphic to a product principal bundle by the isomorphism $\hat{\Upsilon}_d[T]$ from Section 8.7.2.

The First ToA Transformation Rule has its bundle counterpart in a transformation rule under the $SO(3)$ gauge transformation group [Hus, Chapter 9] of the unreduced principal bundle.

Every (E, l) in the formalism uniquely determines an "associated bundle" (relative to the unreduced bundle) which, up to bundle isomorphism, is of the form $(\mathbb{T}^d \times E, p, \mathbb{T}^d)$ where $p(z, x) := z$. Thereby the fields are just the nontrivial data of the cross sections of p. Moreover the automorphism group of the unreduced principal bundle acts naturally on the cross sections of p and it is this action which induces the field map $\tilde{\mathcal{P}}[E, l, j, A]$ of (8.2). Thus invariant (E, l)-fields are the nontrivial data of invariant cross sections of associated bundles. This is similar to the situation in gauge field theories where the matter fields carry the data of cross sections of associated bundles. Note also that the reduced principal bundles correspond to a certain subclass of the associated bundles hence the cross sections in the CST correspond to a subclass of the cross sections of the associated bundles.

As a side aspect, the above mentioned reductions reveal a relation to Yang-Mills Theory via the principal connections. For example, via Remark 30, in the presence of an IFF we have an invariant $SO(2)$ reduction which has a cross section and describes planar spin motion. Since this reduction is a smooth principal bundle, it has a well-defined class of principal connections leading via path lifting to parallel transport motions which, remarkably, reproduce the form of the well-known T–BMT equation, and thus in discrete time give us $\mathcal{P}[j, A]$. These aspects will be extended to nonplanar spin motion in future work.

9 Summary and Outlook

In this work we have studied the discrete-time spin motion in storage rings in terms of the ToA. We thus generalized the notions of invariant polarization field and invariant frame field and reconsidered the notions of spin tune and spin-orbit resonance within this framework. We demonstrated its convenience in many ways, among them the ability to unify the description of spin-1/2 and spin-1 particles by exhibiting common properties of the spin vector motion resp. spin-tensor motion.

For future work there are several natural avenues. One obvious avenue is the study of the existence and uniqueness problems of invariant polarization fields and invariant polarization-tensor fields in terms of the IRT and the closely related notions of "algebraic hull" and "rigidity" [Fe, Section 6],[HK1, Section 9]. The algebraic hull of (j, A) is, roughly speaking, the "smallest" subgroup H of $SO(3)$ for which (j, A) has an H-normal form. The study of invariant fields in terms of the IRT will be focused on the comparison of the topological spaces $\hat{\mathcal{E}}_d[f]$ and $\overline{\check{\mathcal{E}}_d[z, y]}$ introduced in Section 8.7.1.

Rigidity of (j, A) occurs if, roughly speaking, the behaviour of (j, A) does not change under the extension of the time group $\mathbb{Z}$ to a larger time group. Also the underlying principal bundle invites the study of the path lifting of its principal connections [Na] and the study the so-called characteristic classes of its reduced principal bundles. Characteristic classes occur if one studies a principal bundle in terms of so-called universal bundles [Hus]. Note that characteristic classes, the so-called Chern classes, play a key role in Yang-Mills theory. Another avenue is the study, in the case $j = \mathcal{P}_\omega$, of the spin trajectories $x(\cdot)$ in terms of a Fourier Analysis since then the equations of motion are characterized by quasiperiodic functions in time. In particular a perturbation analysis via averaging techniques seems feasible. One could also weaken the condition that A, j, l etc. are continuous functions to the condition of Borel measurability. Moreover the ToA can be easily modified from our $SO(3)$ formalism to the quaternion formalism and the spinor formalism where the group $SU(2)$ will take over the role of $SO(3)$.

We now give a summary of Chapter 8 including the relevant material from Chapters 2-7. A spin-orbit system is a pair (j, A) where $j \in \text{Homeo}(\mathbb{T}^d)$ is the particle 1-turn map and $A \in \mathcal{C}(\mathbb{T}^d, SO(3))$ with the torus $\mathbb{T}^d$ introduced in Section 2.2. In the special case $j = \mathcal{P}_\omega$, ω is the orbital tune and $\mathcal{P}_\omega$ is the corresponding translation on the torus after one turn. The ToA is defined in Section 8.2 and it considers arbitrary $SO(3)$-spaces (E, l), defined in Section 2.4. Various other group-theoretical notions are defined in Section 2.4. For every $SO(3)$-space (E, l) and every spin-orbit system (j, A) in $\mathcal{SOS}(d, j)$ a 1-turn particle-spin map $\mathcal{P}[E, l, j, A] \in \text{Homeo}(\mathbb{T}^d \times E)$ is defined by (8.1), i.e., $\mathcal{P}[E, l, j, A](z, x) := (j(z), l(A(z); x))$. Also a 1-turn field map $\tilde{\mathcal{P}}[E, l, j, A]$ is a bijection on $\mathcal{C}(\mathbb{T}^d, E)$ defined by (8.2), i.e., $\tilde{\mathcal{P}}[E, l, j, A](f) := l\Big(A \circ j^{-1}; f \circ j^{-1}\Big)$. The special case $(E, l) = (\mathbb{R}^3, l_v)$ where x is the spin vector S is studied in Chapters 2-7 and here $\mathcal{P}[\mathbb{R}^3, l_v, j, A] = \mathcal{P}[j, A]$ is given by (2.23) and $\tilde{\mathcal{P}}[\mathbb{R}^3, l_v, j, A] = \tilde{\mathcal{P}}[j, A]$ is given by (3.3), i.e.,

$$\tilde{\mathcal{P}}[j, A](f) := (Af) \circ j^{-1} .$$

We note also that the particle-spin maps are just characteristic maps of the field maps. If $f \in \mathcal{C}(\mathbb{T}^d, E)$ satisfies $\tilde{\mathcal{P}}[E, l, j, A](f) = f$ then f is called an invariant (E, l)-field of (j, A). In the special case $(E, l) = (\mathbb{R}^3, l_v)$ an invariant $(\mathbb{R}^3, l_v)$-field is also called an invariant polarization field and in the subcase $|f| = 1$ it is called an invariant spin field. A $j \in \text{Homeo}(\mathbb{T}^d)$ is called topologically transitive if a $z_0 \in \mathbb{T}^d$ exists such that the topological closure $\overline{\{j^n(z_0) : n \in \mathbb{Z}\}}$ of $\{j^n(z_0) : n \in \mathbb{Z}\}$ equals $\mathbb{T}^d$. The ISF-conjecture states that a spin-orbit system (j, A) has an ISF if j is topologically transitive.

Note that a special case of this conjecture is: If a spin-orbit system $(\mathcal{P}_\omega, A)$ is off orbital resonance, then it has an ISF.

If $(j, A) \in \mathcal{SOS}(d, j)$ and $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ then $(j, A') \in \mathcal{SOS}(d, j)$ is called the transform of (j, A) under T where A' is defined by (4.1), i.e., $A'(z) := T^t(j(z))A(z)T(z)$. The First ToA Transformation Rule in Section 8.2.3 transforms $\tilde{\mathcal{P}}[E, l, j, A]$ into $\tilde{\mathcal{P}}[E, l, j, A']$ and one has $\tilde{\mathcal{P}}[E, l, j, A'] = \tilde{\mathcal{P}}[E, l, id_{\mathbb{T}^d}, T]^{-1} \circ \tilde{\mathcal{P}}[E, l, j, A] \circ \tilde{\mathcal{P}}[E, l, id_{\mathbb{T}^d}, T]$. If H is a subgroup of $SO(3)$ and $(j, A) \in \mathcal{SOS}(d, j)$ then (j, A') in $\mathcal{SOS}(d, j)$ is an H-normal form of (j, A) if A' is H-valued and (j, A') is a transform of (j, A). If H and H' are subgroups of $SO(3)$ then we write $H \trianglelefteq H'$ if an $r \in SO(3)$ exists such that $rHr^t \subset H'$. If $H \trianglelefteq H'$ and (j, A) has an H-normal form then it also has an H'-normal form whence spin-orbit tori are ordered in terms of their normal forms. The NFT in Section 8.2.4 states that if $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ and $f \in \mathcal{C}(\mathbb{T}^d, E)$ are related by $f(z) = l(T(z); x)$ then f is an invariant (E, l)-field of (j, A) iff $T^t(j(z))A(z)T(z) \in H(x) = Iso(E, l; x) = \{r \in SO(3) : l(r; x) = x\}$. Thus invariant fields can be studied in terms of isotropy groups via the notion of normal form. In particular the "smaller" a subgroup H of $SO(3)$ the less likely it is for (j, A) to have an H-normal form. Following Chapter 6, a spin-orbit system (j, A) has a spin tune $\nu \in [0, 1)$ if (j, A') with $A'(z) = \exp(2\pi\nu\mathcal{J})$ is a transform of (j, A). We say that $(\mathcal{P}_\omega, A)$ is on spin-orbit resonance if it has spin tunes and if for every spin tune ν we can find $m \in \mathbb{Z}^d, n \in \mathbb{Z}$ such that $\nu = m \cdot \omega + n$. The Uniqueness Theorem, Theorem 7.1b, states that, if $(\mathcal{P}_\omega, A)$ has spin tunes and is off orbital resonance and off spin-orbit resonance, then it has only two ISF's and they differ only by a sign. The polarization of a bunch is defined in terms of the density matrix function in Section 8.6.1 and its size is estimated in Section 7.1. The decomposition method in Section 8.2.5 decomposes each $SO(3)$-space (E, l) into transitive $SO(3)$-spaces and it predicts that for topologically transitive j an invariant (E, l)-field takes only values in one (E, l)-orbit. This allows us to classify, via the DT, invariant fields in terms of isotropy groups. In Sections 8.4-8.6 we apply the ToA to the $SO(3)$-spaces $(\mathbb{R}^3, l_v)$ and (E_t, l_t) to study spin-1/2 and spin-1 particles. In Section 8.7 we study the existence problem of invariant (E, l)-fields in terms of the topological spaces $\hat{\mathcal{E}}_d[E, l, x, f]$ and in Section 8.8 we discuss the bundle-theoretic aspects of the present work. We revisit some old theorems and prove several theorems which we believe to be new. Among the former we mention Theorems 3.2, 5.4, 6.4 and 7.1 and among the latter Lemmas 8.4 and 8.8 and Theorems 6.2, 8.1, 8.9, 8.10, 8.11, 8.12, 8.13, 8.15, 8.17, 8.19 and 8.20.

10 Table of Notation

$\mathcal{A}, A_{CT}[\omega, \mathcal{A}], A, A_{d,\nu}$	(2.2), (2.18), (2.21), (6.3)
$\mathcal{ACB}(d, j)$	Definition 6.1
$B(E, l, E', l'; x, x'), B(E, l; x, x')$	Section 8.3.1
$\hat{\beta}[E, l, E', l'; x, x', r], \hat{\beta}[E, l; x, x', r]$	Section 8.3.1
$\mathcal{C}(X, Y)$ set of continuous functions from X to Y	Appendix A.4
$\mathcal{CB}_H$	Definition 5.1
$(E, L) -$ lift	Section 8.2.4
(E, L) $(G -$ set$)$	Definition 2.3
(E, L) $(G -$ space$)$	Definition 2.6
$(E, L) -$ orbit, E/L	Definition 2.4, Definition 2.6
$(E_t, l_t), R_t$	Section 8.4.1
$(E_{v\times t}, l_{v\times t}), (E^{1/2}_{dens}, l^{1/2}_{dens})$ and (E^1_{dens}, l^1_{dens})	Section 8.6
$\mathcal{E}_d, \hat{\mathcal{E}}_d[E, l, x, f], \check{\mathcal{E}}_d[E, l, j, A, z, x, y]$	Section 8.7.1
Equivalent spin $-$ orbit systems, $\overline{(j, A)}$	Definition 4.2
Group, conjugate subgroups, topological group	Definition 2.2, Definition 2.5
H $-$ normal Form	Definition 5.1
Homeo(X, Y) (set of homeomorphisms from X to Y)	Appendix A.4
Invariant frame field (IFF)	Definition 5.3
Uniform invariant frame field	Remark 5 in Chapter 6
Invariant spin field (ISF), invariant polarization field	Definition 3.1
Invariant $(E, l) -$ field, invariant $n -$ turn $(E, l) -$ field	Section 8.2.1
$Iso(E, L; x)$ (Isotropy group of $G -$ space (E, L) at x)	Definition 2.6
$j, \mathcal{J}$	Section 2.3, (5.6)
$L[j], L[j, A], \tilde{L}[j, A]$	(2.32), (2.37), (3.7)
$L[E, l, j, A], \tilde{L}[E, l, j, A]$	(8.9), (8.13)
$l_{dec}[x]$	(8.37)
Λ_j	Section 8.4.1
$N(H, H'), \trianglelefteq$	(8.54), Section 5.1
$\Xi(j, A)$	(6.5)
$\mathcal{P}_\omega$	(2.17)
$\mathcal{P}_{CT}[\omega, \mathcal{A}]$	(2.19)
$\mathcal{P}[j, A], \tilde{\mathcal{P}}[j, A]$	(2.23), (3.3)
$\mathcal{P}[E, l, j, A], \tilde{\mathcal{P}}[E, l, j, A], \hat{\mathcal{P}}[j, A]$	(8.1), (8.2), (8.174)
$p_d, p_d[f]$	Section 8.7
π_d	(2.13)
Resonant, nonresonant, orbital resonance	Section 2.3
$SO(3)$	Section 2.1
$SO(2)$	(5.5)
$SO(2) \rtimes \mathbb{Z}_2, SO_{diag}(3)$	(8.66), (8.88)
$S_\lambda, \mathbb{S}^2_\lambda$	Sections 8.2.4, 8.2.5

Appendix

A Conventions and terminology

A.1 Function, image, inverse image

A "function" $f : X \to Y$ is determined by its graph and its codomain. The "graph" of f is the set $\{(x, f(x)) : x \in X\}$ and the "codomain" of f is Y. The "domain" of f is X and the "range" of f is the set $f(X) := \{f(x) : x \in X\}$. More generally, if M is a subset of X then the "image" of M under f is the set $f(M) := \{f(x) : x \in M\}$. If M is a subset of Y then the "inverse image" of M under f is the set $f^{-1}(M) := \{x \in X : f(x) \in M\}$.

One calls f a "surjection" or "onto" if its range and codomain are equal. One calls f "one-one" or an "injection" if $f(x) = f(x')$ implies that $x = x'$. One calls f a "bijection" if it is one-one and a surjection.

If $f : X \to Y$ and $g : Y \to Z$ are functions then $g \circ f$ is the function $g \circ f : X \to Z$ defined by $(g \circ f)(x) := g(f(x))$. One calls the operation $\circ$ the "composition" of functions. If X is a set then the function $id_X : X \to X$ is defined by $id_X(x) := x$ and is called the "identity function" on X. If $f : X \to Y$ is a bijection then a unique function $f^{-1} : Y \to X$ exists such that $f^{-1} \circ f = id_X, f \circ f^{-1} = id_Y$ and it is called the "inverse" of f. Clearly f is a bijection iff it has an inverse.

Note that if $f : X \to Y$ is a bijection then f^{-1} can either mean the inverse function or the inverse image operation. However it will always be clear from the context what the meaning is.

If $f : X' \to Y$ is a function and $X \subset X'$ then we define the function $f|X : X \to Y$ as a restriction of f to X, i.e., by $(f|X)(x) := f(x)$. If $f : X \times Y \to Z$ is a function and $x \in X, y \in Y$ then the restriction $f|(\{x\} \times Y)$ is denoted also by $f(x, \cdot)$ and the restriction $f|(X \times \{y\})$ is denoted also by $f(\cdot, y)$.

If $f : X \to X$ is a function then $x \in X$ is called a "fixed point" of f if $f(x) = x$.

Note finally that according to our, very common, definition of a function two functions with the same graph are different iff they have different codomains. Thus the alternative, and equally common, way to define a function in terms of its graph (i.e., without invoking the codomain) is different from our definition.

A.2 Partition, representing set, equivalence relation

If X is a set and if P is a set whose elements are disjoint nonempty subsets of X whose union is X then one calls P a "partition of X". If P is a partition of X then a subset X' of X is called a "representing set of P" if every element of P contains exactly one element of X'. Note that partitions and their representing sets are used throughout this work. When needed, we will always find a representing set. From a more general view point, one knows that a representing set always exists if one assumes the Axiom of Choice [Dud].

If X is a set and B a subset of $X \times X$ then B is called a "relation" on X. The relation B is called "symmetric" if $(x, y) \in B$ implies that $(y, x) \in B$. The relation B is called "reflexive" if $(x, x) \in B$ for all $x \in X$. The relation B is called "transitive" if $(x, y) \in B$ and $(y, z) \in B$ implies that $(x, z) \in B$.

A relation on X is called an "equivalence relation on X" if it is symmetric,reflexive, and transitive. If B is an equivalence relation on X and $x \in X$ then the set $\{y \in Y : (x, y) \in B\}$ is called the "equivalence class of x under the equivalence relation B". We also write $x \sim y$ if $(x, y) \in B$.

The equivalence classes of B form a partition of X as follows. Clearly the equivalence classes of B are nonempty sets and overlap X. Moreover by, transitivity, if two equivalence classes of B have a common element then they are equal.

A.3 Topology, topological space, open set, closed set, closure

A collection, τ, of subsets of a set X is called a "topology on X" if τ is closed under arbitrary unions and finite intersections and if $X, \emptyset \in \tau$. Any pair (X, τ) is called a "topological space (over X)". The elements of τ are called the "open" sets of (X, τ).

The "closed" sets of (X, τ) are the complements of the open sets (X, τ). If M is a subset of X then its "closure" $\bar{M}$ is defined as the intersection of all closed supsets of M.

If (X, τ) is a topological space and if X' is a subset of X then the "subspace topology" τ' of X' from X is the collection $\{X' \cap M : M \in \tau\}$ and the topological space (X', τ') is called a "topological subspace" of (X, τ).

Since the topology τ is always clear from the context we write X instead of (X, τ). For example the topology of $\mathbb{R}^d$ is obtained from the Euclidean norm and the topology of $\mathbb{Z}^d$ is discrete, i.e., every subset of $\mathbb{Z}^d$ is open.

A.4 Continuous function, homeomorphism, cross section

Let (X, τ) and (X', τ') be topological spaces. Then a function $f : X \to X'$ is called "continuous w.r.t. (X, τ) and (X', τ')" if for every $M \in \tau'$ the inverse image of M under f belongs to τ, i.e., $f^{-1}(M) \in \tau$. We denote the collection of continuous functions by $\mathcal{C}(X, X')$. A function $f \in \mathcal{C}(X, X')$ is called a "homeomorphism" and X, X' are called a "homeomorphic" if f is a bijection and if its inverse is continuous. We denote the collection of those homeomorphisms by $\mathrm{Homeo}(X, X')$ and we also define $\mathrm{Homeo}(X) := \mathrm{Homeo}(X, X)$. The topological spaces X and X' are called "homeomorphic" if $\mathrm{Homeo}(X, X')$ is nonempty.

If $f \in \mathcal{C}(Z, Z')$ with Z' being a subspace of $X \times Y$ then we denote the two components of f by f_1, f_2, i.e., $f_1 \in \mathcal{C}(Z, X)$ and $f_2 \in \mathcal{C}(Z, Y)$ where $f(z) = (f_1(z), f_2(z))$. If $f \in \mathcal{C}(X, Y)$ then the inverse image $f^{-1}(\{y\})$ is called the "fibre of f over y". If $f \in \mathcal{C}(X, Y)$ then a function $\sigma \in \mathcal{C}(Y, X)$ is called a "cross section of f" if $f \circ \sigma = id_Y$. Note that a cross section is often called a "section".

A.5 Product topology, Hausdorff space, compact space, path-connected space

If X and Y are topological spaces then the product topology on $X \times Y$ is defined such that sets $M \times N$ are open if M and N are open and such that every open set of $X \times Y$ is a union of those $M \times N$. For example the topology of $\mathcal{E}_d = \mathbb{T}^d \times SO(3)$ is the product topology where the topology of $SO(3)$ is the subspace topology from $\mathbb{R}^{3\times3}$.

A topological space X is called "Hausdorff" if for every pair of distinct elements x, x' of X open sets M, M' exists such that $x \in M, x' \in M'$ and $M \cap M' = \emptyset$. A topological space X is called "compact" if for any union of X by open sets of X already the union of finitely many of those open sets equals X.

If X is a topological space and then a subset A of X is called "compact" if A is, as a topological subspace of X, compact.

A topological space X is called "path-connected" if for elements $x, x' \in X$ a continuous function $f : [0, 1] \to X$ exists such that $f(0) = x$ and $f(1) = x'$. A subset A of X is called "path-connected" if A is, as a topological subspace of X, path-connected. One has the following intermediate-value theorem: If X, Y are topological spaces such that X is path-connected and if $g : X \to Y$ is a continuous function then the range of g is a path-connected subset of Y.

A.6 Co-induced topology, identifying function

Let X be a topological space and let $p : X \to Y$ be a surjection where Y is a set. A natural topology on Y is defined such that a subset $B \subset Y$ is open iff the inverse image $p^{-1}(B) \subset X$ is open. One calls the topology on Y "co-induced by p" [wiki]. Using older terminology, one also says that the topology on Y is the "identification topology" w.r.t. p and that p is "identifying" [Du, Hu]. Of course an identifying function is continuous (but not vice versa). Time and again we will use co-induced topologies and we will often use the following lemma to prove the continuity of a function:

(Continuity Lemma)
Let X be a topological space and let $p : X \to Y$ be a surjection where Y is a set. Let the topology on Y be co-induced by p. If Z is a topological space and $f \in \mathcal{C}(X, Z)$ and $F : Y \to Z$ are functions such that $F \circ p = f$ then F is continuous. If in addition f is an

identification map then F is an identification map, too.

Proof of the Continuity Lemma: To see that F is continuous we need to show that the inverse image $F^{-1}(V)$ is open for all open subsets V of Z. In fact since the topology on Y is co-induced by p we get $p^{-1}(F^{-1}(V)) = (F \circ p)^{-1}(V) = f^{-1}(V)$, thus if V is open, $f^{-1}(V) = p^{-1}(F^{-1}(V))$ is open. Thus indeed F is continuous. The second claim is shown in the same vein (see also Section VI.3 in [Du]). □

B Various Proofs

This appendix contains those proofs which are too long for the main text.

B.1 Proof of Lemma 8.8

Proof of Lemma 8.8a: By Definitions 5.2 and 8.7 of $\trianglelefteq$ and $N(Iso(E,l;x), Iso(E',l';x'))$ it is clear that the first claim follows from (8.57). To show (8.57) we first prove the inclusion (B.3) so let $\beta \in B(E,l,E',l';x,x')$, i.e., by Definition 8.7, β is a topological $SO(3)$-map from $(l(SO(3);x), l_{dec}[x])$ to $(l'(SO(3),x'), l'_{dec}[x'])$. Clearly we can pick an $r_0 \in SO(3)$ such that

$$\beta(x) = l'(r_0^t; x') \ . \tag{B.1}$$

By (8.37) and (B.1) we have, for all $r \in SO(3)$,

$$\beta(l(r;x)) = \beta(l_{dec}[x](r;x)) = l'_{dec}[x'](r;\beta(x)) = l'(r;\beta(x)) = l'(rr_0^t;x') \ , \tag{B.2}$$

where in the second equality we used that β is a topological $SO(3)$-map from $(l(SO(3);x), l_{dec}[x])$ to $(l'(SO(3),x'), l'_{dec}[x'])$. On the other hand if $r_1 \in Iso(E,l;x)$ then, for all $r \in SO(3)$,

$$\beta(l(r;x)) = \beta(l(r;l(r_1;x))) = \beta(l(rr_1;x)) = l'(rr_1r_0^t;x') \ ,$$

where in the third equality we used (B.2) whence, by using again (B.2), $l'(rr_0^t;x') = l'(rr_1r_0^t;x')$ so that $r_0r_1r_0^t \in Iso(E',l';x')$ which implies that $r_0 Iso(E,l;x) r_0^t \subset Iso(E',l';x')$, i.e., $r_0 \in N(Iso(E,l;x), Iso(E',l';x'))$. Thus $\hat\beta[E,l,E',l';x,x',r_0]$ is well defined and, by (8.55) and (B.2), $\beta = \hat\beta[E,l,E',l';x,x',r_0]$ whence β belongs to the set on the rhs of (8.57) so that we have shown that

$$B(E,l,E',l';x,x') \subset \{\hat\beta[E,l,E',l';x,x',r_0] : r_0 \in N(Iso(E,l;x), Iso(E',l';x'))\} \ . \tag{B.3}$$

To show the reverse inclusion let $r_0 \in N(Iso(E,l;x), Iso(E',l';x'))$ so we have to show that $\hat\beta[E,l,E',l';x,x',r_0]$ belongs to $B(E,l,E',l';x,x')$. We first show that $\hat\beta[E,l,E',l';x,x',r_0]$ is an $SO(3)$-map. In fact it follows from (8.37) and (8.55) that

$$\begin{aligned}
&\hat\beta[E,l,E',l';x,x',r_0](l_{dec}[x](r_0;l(r_1;x))) = \hat\beta[E,l,E',l';x,x',r_0](l(r_0;l(r_1;x))) \\
&\qquad = \hat\beta[E,l,E',l';x,x',r_0](l(r_0r_1;x)) = l'(r_0r_1r_0^t;x') \ , \\
&l'_{dec}[x'](r_0;\hat\beta[E,l,E',l';x,x',r_0](l(r_1;x))) = l'(r_0;\hat\beta[E,l,E',l';x,x',r_0](l(r_1;x))) \\
&\qquad = l'(r_0;l'(r_1r_0^t;x')) = l'(r_0r_1r_0^t;x') \ ,
\end{aligned}$$

whence, for every $y \in l(SO(3);x)$, we have $\hat{\beta}[E,l,E',l';x,x',r_0](l_{dec}[x](r_0;y)) = l'_{dec}[x'](r_0;\hat{\beta}[E,l,E',l';x,x',r_0](y))$ so that indeed $\hat{\beta}[E,l,E',l';x,x',r_0]$ is an $SO(3)$-map. To show that $\hat{\beta}[E,l,E',l';x,x',r_0]$ is continuous we first note that the function $l_{dec}[x](\cdot;x) : SO(3) \to l(SO(3);x)$, defined by $(l_{dec}[x](\cdot;x))(r) := l_{dec}[x](r;x) = l(r;x)$, is continuous since l is continuous. Moreover since E is Hausdorff, its subspace $l(SO(3);x)$ is Hausdorff, too. Thus $l_{dec}[x](\cdot;x)$ is a continuous function from the compact space $SO(3)$ onto the Hausdorff space $l(SO(3);x)$ whence, by the Closed Map Lemma [Du, Section XI.2], $l_{dec}[x](\cdot;x)$ is a closed map so that it is an identification map, i.e., the topology on $l(SO(3);x)$ is co-induced by $l_{dec}[x](\cdot;x)$ (for the notions of co-induced and identification map see Appendix A.6). On the other hand, by (8.37) and (8.55),

$$\hat{\beta}[E,l,E',l';x,x',r_0](l_{dec}[x](r;x)) = \hat{\beta}[E,l,E',l';x,x',r_0](l(r;x))$$
$$= l'(rr_0^t;x') = l'(r;l'(r_0^t;x')) = l'(r;x'') = l'_{dec}[x'](r;x'') \ ,$$

i.e.,

$$\hat{\beta}[E,l,E',l';x,x',r_0] \circ l_{dec}[x](\cdot;x) = l'_{dec}[x'](\cdot;x'') \ , \tag{B.4}$$

where $x'' := l'(r_0^t;x')$. Since by the above argument, $l_{dec}[x](\cdot;x)$ and $l'_{dec}[x'](\cdot;x'')$ are identification maps, it follows from (B.4) and the Continuity Lemma in Appendix A.6 that the surjection $\hat{\beta}[E,l,E',l';x,x',r_0]$ is an identification map (whence it is continuous). We conclude that $\hat{\beta}[E,l,E',l';x,x',r_0]$ is a continuous $SO(3)$-map whence it is a topological $SO(3)$-map from $(l(SO(3);x), l_{dec}[x])$ to $(l'(SO(3),x'), l'_{dec}[x'])$, i.e., $\hat{\beta}[E,l,E',l';x,x',r_0]$ belongs to $B(E,l,E',l';x,x')$ so that indeed

$$B(E,l,E',l';x,x') \supset \{\hat{\beta}[E,l,E',l';x,x',r_0] : r_0 \in N(Iso(E,l;x), Iso(E',l';x'))\} \ ,$$

whence (8.57) follows from (B.3). □

Proof of Lemma 8.8b: Let $Iso(E,l;x) \trianglelefteq Iso(E',l';x')$ and let $r_0 \in N(Iso(E,l;x), Iso(E',l';x'))$. Let also $r_1, r_2 \in SO(3)$. Then, by (8.32),

$$Iso(E,l;l(r_1;x)) = r_1 Iso(E,l;x) r_1^t \ , \quad Iso(E',l';l'(r_2;x')) = r_2 Iso(E',l';x') r_2^t \ , \tag{B.5}$$

whence, since $r_0 \in N(Iso(E,l;x), Iso(E',l';x'))$,

$$r_2 r_0 Iso(E,l;x) r_0^t r_2^t \subset r_2 Iso(E',l';x') r_2^t = Iso(E',l';l'(r_2;x')) \ , \tag{B.6}$$

so that, by (B.5),

$$r_2 r_0 r_1^t Iso(E,l;l(r_1;x)) r_1 r_0^t r_2^t = r_2 r_0 Iso(E,l;x) r_0^t r_2^t \subset Iso(E',l';l'(r_2;x')) \ , \tag{B.7}$$

which implies that $(r_2 r_0 r_1^t) \in N(Iso(E,l;l(r_1;x)), Iso(E',l';l'(r_2;x')))$, i.e., $Iso(E,l;l(r_1;x)) \trianglelefteq Iso(E',l';l'(r_2;x'))$. Thus $\hat{\beta}[E,l,E',l';y,y',r_2r_0r_1^t]$ is well defined where $y := l(r_1;x)$ and $y' := l'(r_2;x')$ and we compute, by (8.55),

$$\hat{\beta}[E,l,E',l';y,y',r_2r_0r_1^t](l(rr_1;x)) = \hat{\beta}[E,l,E',l';y,y',r_2r_0r_1^t](l(r;l(r_1;x)))$$
$$= \hat{\beta}[E,l,E',l';y,y',r_2r_0r_1^t](l(r;y)) = l'(r(r_2r_0r_1^t)^t;y')$$

$$= l'(rr_1 r_0^t r_2^t; l'(r_2; x')) = l'(rr_1 r_0^t; x') = \hat\beta[E,l,E',l';x,x',r_0](l(rr_1;x)) \ ,$$

i.e., $\hat\beta[E,l,E',l';y,y',r_2 r_0 r_1^t] = \hat\beta[E,l,E',l';x,x',r_0]$.

Choosing $r_1, r_2 \in SO(3)$ such that $r_2 r_0 r_1^t = I_{3\times 3}$ (e.g., $r_1 := r_0, r_2 := I_{3\times 3}$) and since $(r_2 r_0 r_1^t) \in N(Iso(E,l;l(r_1;x)), Iso(E',l';l'(r_2;x')))$ one gets
$I_{3\times 3} \in N(Iso(E,l;y), Iso(E',l';y'))$ whence $Iso(E,l;y) \subset Iso(E',l';y')$. □

Proof of Lemma 8.8c: We first prove the first claim.
"⇒": Let $Iso(E,l;x), Iso(E',l';x')$ be conjugate, i.e., let $r_0 \in SO(3)$ exist such that $r_0 Iso(E,l;x) r_0^t = Iso(E',l';x')$. Thus $r_0 \in N(Iso(E,l;x), Iso(E',l';x'))$ whence, by Lemma 8.8a, $\hat\beta[E,l,E',l';x,x',r_0] \in B(E,l,E',l';x,x',r_0)$. To show that $\hat\beta[E,l,E',l';x,x',r_0]$ is an isomorphism from $(l(SO(3);x), l_{dec}[x])$ to $(l'(SO(3),x'), l'_{dec}[x'])$ we only have to show that $\hat\beta[E,l,E',l';x,x',r_0]$ is a homeomorphism. From the proof of Lemma 8.8a we know that $\hat\beta[E,l,E',l';x,x',r_0]$ is an identification map whence, $\hat\beta[E,l,E',l';x,x',r_0]$ is a homeomorphism iff it is one-one. To show that $\hat\beta[E,l,E',l';x,x',r_0]$ is one-one let $r_1, r_2 \in SO(3)$ such that $\hat\beta[E,l,E',l';x,x',r_0](l(r_1;x)) = \hat\beta[E,l,E',l';x,x',r_0](l(r_2;x))$ whence, by (8.55),

$$l'(r_1 r_0^t; x') = \hat\beta[E,l,E',l';x,x',r_0](l(r_1;x))) = \hat\beta[E,l,E',l';x,x',r_0](l(r_2;x))) = l'(r_2 r_0^t; x') \ ,$$

so that $l'(r_0 r_1^t r_2 r_0^t; x') = x'$ which implies, by (2.45), that $r_0 r_1^t r_2 r_0^t \in Iso(E',l';x')$. Since $r_0 Iso(E,l;x) r_0^t = Iso(E',l';x')$ this implies that $r_0 r_1^t r_2 r_0^t \in r_0 Iso(E,l;x) r_0^t$ whence $r_1^t r_2 \in Iso(E,l;x)$ so that $l(r_1^t r_2; x) = x$, i.e., $l(r_1;x) = l(r_2;x)$. Thus $\hat\beta[E,l,E',l';x,x',r_0]$ is one-one as was to be shown.

"⇐": We first prove the useful formula (B.8). In fact if in addition (E'',l'') is an $SO(3)$-space, E'' is Hausdorff, $x'' \in E''$ and if $r_0 \in N\Big(Iso(E,l;x), Iso(E',l';x')\Big)$ and $r_1 \in N\Big(Iso(E',l';x'), Iso(E'',l'';x'')\Big)$ then, by (8.54), $(r_1 r_0) \in N\Big(Iso(E,l;x), Iso(E'',l'';x'')\Big)$ whence, by (8.55),

$$\hat\beta[E,l,E'',l'';x,x'',r_1 r_0] = \hat\beta[E',l',E'',l'';x'',x',r_1] \circ \hat\beta[E,l,E',l';x,x',r_0] \ . \qquad \text{(B.8)}$$

Let $(l(SO(3);x), l_{dec}[x])$, $(l'(SO(3),x'), l'_{dec}[x'])$ be isomorphic. Thus an isomorphism, say β, exists from $(l(SO(3);x), l_{dec}[x])$ to $(l'(SO(3),x'), l'_{dec}[x'])$ whence, by Lemma 8.8a, we can pick an r_0 in $N(Iso(E,l;x), Iso(E',l';x'))$ such that $\beta = \hat\beta[E,l,E',l';x,x',r_0]$ and we have, by (8.54),

$$r_0 Iso(E,l;x) r_0^t \subset Iso(E',l';x') \ . \qquad \text{(B.9)}$$

On the other hand, by (B.8), $\hat\beta[E,l,E',l';x,x',r_0^t]$ is the inverse, say β^{-1}, of β. Since β^{-1} is an isomorphism from $(l'(SO(3),x'), l'_{dec}[x'])$ to $(l(SO(3);x), l_{dec}[x])$, the above argument, which used β to give us (B.9), can now be repeated for $\beta^{-1} = \hat\beta[E,l,E',l';x,x',r_0^t]$ giving us in analogy to (B.9)

$$r_0^t Iso(E',l';x') r_0 \subset Iso(E,l;x) \ , \qquad \text{(B.10)}$$

whence $Iso(E',l';x') \subset r_0 Iso(E',l';x') r_0^t$ so that, by (B.9),
$r_0 Iso(E,l;x) r_0^t \subset Iso(E',l';x') \subset r_0 Iso(E,l;x) r_0^t$ which implies that $r_0 Iso(E,l;x) r_0^t =$

$Iso(E',l';x')$, i.e., $Iso(E,l;x)$ and
$Iso(E',l';x')$ are conjugate. This completes the proof of the first claim. At the same time we have proven the second claim, i.e., that, for every $r_0 \in SO(3)$ such that $r_0 Iso(E,l;x) r_0^t = Iso(E',l';x')$, $\hat{\beta}[E,l,E',l';x,x',r_0]$ is an isomorphism. To prove the third claim, let $Iso(E,l;x)$ and $Iso(E',l';x')$ be conjugate, i.e., let $r \in SO(3)$ exist such that $rIso(E,l;x)r^t = Iso(E',l';x')$. Defining $y := x, y' := l'(r^t;x')$ we conclude from Remark 7 that $Iso(E',l';y') = Iso(E',l';l'(r^t;x')) = r^t Iso(E',l';x')r = r^t r Iso(E,l;x) r^t r = Iso(E,l;x) = Iso(E,l;y)$ as was to be shown. □

B.2 Proof of Theorem 8.9

Proof of Theorem 8.9a: Using the notation of Section 8.2.5 we define $f_x \in \mathcal{C}(\mathbb{T}^d, l(SO(3);x))$ by $f_x(z) := f(z)$ and $g_{x'} \in \mathcal{C}(\mathbb{T}^d, l'(SO(3);x'))$ by $g_{x'}(z) := g(z)$. It follows from Theorem 8.5 that $(g_{x'})' = \hat{\beta}[E,l,E',l';x,x',r_0] \circ (f_x)'$ whence, by (8.40), $(g')_{x'} = \hat{\beta}[E,l,E',l';x,x',r_0] \circ (f')_x$ so that $g'(z) = \hat{\beta}[E,l,E',l';x,x',r_0](f'(z))$. □
Proof of Theorem 8.9b: Let f be an invariant (E,l)-field of (j,A). Then, by Theorem 8.9a, g is an invariant (E',l')-field of (j,A). Let g be an invariant (E',l')-field of (j,A). Also by, Lemma 8.8c, $\hat{\beta}[E,l,E',l';x,x',r_0]^{-1}$ is a topological $SO(3)$-map from $(l'(SO(3),x'), l'_{dec}[x'])$ to $(l(SO(3);x), l_{dec}[x])$. Thus, by Theorem 8.9a, we conclude that f is an invariant (E,l)-field of (j,A). □

References

[AP] D.K. Arrowsmith, C.M. Place, *An Introduction to Dynamical Systems*, Cambridge University Press (1990).

[Ba] D.P. Barber, *Pre-existing betatron motion and spin flipping with RF fields in storage rings*, Proc.19th Int. Spin Physics Symposium, Juelich, Germany, October 2010.

[BEH00] D.P. Barber, J.A. Ellison and K. Heinemann, *Quasi-periodicity of spin motion in storage rings - a new look at spin tune*, Proc.14th Int. Spin Physics Symposium, Osaka, Japan, October 2000, AIP proceedings 570, (2001).

[BEH] D.P. Barber, J.A. Ellison and K.Heinemann, *Quasiperiodic spin-orbit motion and spin tunes in storage rings*, Phys. Rev. ST Accel. Beams **7** (2004) 124002.

[BH] K. Heinemann, D.P. Barber, *Spin transport, spin diffusion and Bloch equations in electron storage rings*, Nucl. Instr. Meth. **A463** (2001), p.62. Erratum-ibid.A469:294,2001.

[BHHV] D.P. Barber, K.Heinemann, G.H. Hoffstaetter and M. Vogt, five articles in proceedings of ICFA workshop “Quantum Aspects of Beam Physics”, Monterey, U.S.A., 1998, edited by P. Chen, World Scientific (1999). Also in extended form as DESY Technical Report 98-096 (1998) and e–print archive: physics/9901038, 9901041, 9901042, 9901043, 9901044.

[BHV98] D.P. Barber, G.H. Hoffstaetter and M. Vogt, *The amplitude dependent spin tune and the invariant spin field in high energy proton accelerators*, Proc.13th Int.Symp. High Energy Spin Physics, Protvino, Russia, September 1998, World Scientific (1999).

[BHV00] D.P. Barber, G.H. Hoffstaetter and M. Vogt, *Using the amplitude dependent spin tune to study high order spin-orbit resonances in storage rings*, Proc.14th Int. Spin Physics Symposium, Osaka, Japan, October 2000, AIP proceedings 570, (2001).

[BV1] D.P. Barber, M. Vogt, *Spin motion at and near orbital resonance in storage rings with Siberian Snakes I: at orbital resonance*, New Journal of Physics **8** (2006) 296.

[BV2] D.P. Barber, M. Vogt, *The Invariant Polarisation-Tensor Field for Spin-1 Particles in Storage Rings*, Proc.18th Int. Spin Physics Symposium, Charlotteville (Virginia), USA, October 2008, AIP proceedings 1149, (2008).

[Be] M. Berz, Proc. Computational Accelerator Physics 96, Williamsburg (Virginia), USA, September 1996, AIP proceedings 391, (1997).

[Cr] J. Cronin, *Differential Equations: Introduction and Qualitative Theory*, Dekker, New York, 1980.

[DK73] Ya.S. Derbenev, A.M. Kondratenko, *Polarization kinetics of particles in storage rings*, Sov. Phys. JETP **37**, p.968 (1973).

[Dud] R.M. Dudley, "Real Analysis And Probability", Cambridge Studies in Advanced Mathematics (2002).

[Du] J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966.

[EH] J.A. Ellison, K. Heinemann, "Polarization Fields and Phase Space Densities in Storage Rings: Stroboscopic Averaging and the Ergodic Theorem", Physica D **234** (2007), p.131.

[Fe] R. Feres, *Dynamical systems and semisimple groups: an introduction*, Cambridge University Press, Cambridge, 1998.

[Fo] E. Forest, KEK Preprint 2010-39, (2010).

[HK1] *Handbook of dynamical systems Vol. 1A*. Edited by B. Hasselblatt and A. Katok, North-Holland, Amsterdam, 2002.

[HK2] B. Hasselblatt, A. Katok, *Introduction to the modern theory of dynamical systems*, Cambridge University Press, New York, 1995.

[He1] K. Heinemann, unpublished notes.

[He2] K. Heinemann, Doctoral Thesis, University of New Mexico, May 2010.

[HBEV1] K. Heinemann, J.A. Ellison, D.P. Barber and M. Vogt, *New and Unifying Formalism for Study of Particle-Spin Dynamics Using Tools Distilled From Theory of Bundles*, THPRO061 in *Proceedings of the 2014 International Particle Accelerator Conference, Dresden, Germany, 15-20 June 2014*, JACOW, 2014.

[HBEV2] K. Heinemann, D.P. Barber, J.A. Ellison and M.Vogt, arXiv:1409.4373v2 (physics.acc-ph,math-ph math.MP), 2014.

[HH] K. Heinemann and G.H. Hoffstaetter, *Tracking algorithm for the stable spin polarization field in storage rings using stroboscopic averaging*, Phys.Rev. **E 54** (4), 4240 (1996).

[Ho] G.H. Hoffstaetter, *"High Energy Polarised Proton Beams: a Modern View"*, Springer Tract in Modern Physics, Vol 218 (2006).

[HV] G.H. Hoffstaetter, M. Vogt, *"Strength of higher-order spin-orbit resonances*, Phys.Rev. **E 70** (2004) 056501.

[Hu] S.-T. Hu, *Introduction to general topology*, Holden-Day, San Francisco, 1966.

[Hus] D. Husemoller, *Fibre Bundles*, Third edition, Springer, New York, 1994.

[Ja] J.D. Jackson, *Classical Electrodynamics*, 3rd edition, Wiley (1998).

[KR] A. Katok, E.A. Robinson, *Cocycles, cohomology and combinatorial constructions in ergodic theory*. In: *Seattle 1999, Summer Research Institute on Smooth Ergodic Theory*, Proceedings of Symposia in Pure Mathematics 69 (2001) 107.

[Ka] K. Kawakubo, *The theory of transformation groups*, Oxford University Press, Oxford, 1991.

[Ma] S.R. Mane: Phys. Rev. A36, p.105, p.120 (1987).

[Mu] J. Munkres, "Topology", 2nd Edition, Prentice Hall, (2000).

[Na] G.L. Naber, *Topology, Geometry, and Gauge Fields: Interactions*, New York, Springer, (2000).

[Vo] M. Vogt, Doctoral Thesis, University of Hamburg, June 2000, Report DESY-THESIS-2000-054 (2000).

[wiki] Wikipedia page on "Induced topology".

[Yo1] K. Yokoya, *The action-angle variables of classical spin motion in circular accelerators*, DESY-86-57 (1986).

[Yo2] K. Yokoya, *Calculation of the equilibrium polarization of stored electron beams using lie algebra*, Nucl. Instr. Meth., **A258**, p.149 (1987).

[Yo3] K. Yokoya, *An algorithm for calculating the spin tune in circular accelerators*. DESY-99-006 (1999).

[Zi1] R.J. Zimmer, *Ergodic Theory and Semisimple Groups*, Birkhaeuser, Boston, 1984.

[Zi2] R.J. Zimmer, Ergodic theory and the automorphism group of a G-structure. In: *Berkeley 1984, Proceedings of a conference in honor of George W. Mackey, Mathematical Sciences Research Institute Publications, Springer, (1987) 247.

[Zi3] M. Lipkin, "The Zimmer Program", The Chicago Maroon, March 2010. See also: http://chicagomaroon.com/2010/03/05/the-zimmer-program/